MINIMAL FINITE ELEMENT SPACES FOR $2m$-TH ORDER PARTIAL DIFFERENTIAL EQUATIONS IN $\mathbf{R}^n$

MING WANG AND JINCHAO XU

ABSTRACT. This paper is devoted to a canonical construction of a family of piecewise polynomials with the minimal degree that provide a consistent approximation of Sobolev spaces H^m in R^n (with $n \geq m \geq 1$) and also a convergent (nonconforming) finite element space for $2m$-th order elliptic boundary value problems in R^n. This class of spaces, denoted by M_h^m, are given by piecewise polynomials with degree not greater than m, namely the space P_m. Degrees of freedom for M_h^m in each element are given in terms of integral averages of normal derivatives of order $m-k$ on all subsimplexes of dimension $n-k$ for $1 \leq k \leq m$. The total number of these degrees of freedom in each element amounts to C_{n+m}^m which is precisely the dimension of P_m. One remarkable property of these sequence of spaces M_h^m is that $\partial_i M_h^m \subset M_h^{m-1}$ and, furthermore, $\text{span}(\partial_1 M_h^m, \partial_2 M_h^m, \ldots, \partial_n M_h^m) = M_h^{m-1}$.

The finite element spaces M_h^m constructed in this paper is the only class of finite element spaces that are known and proved to be convergent for approximation of any $2m$-th order elliptic problems in any R^n such that $n \geq m \geq 1$. It recovers the non-conforming linear elements for the Poisson equations ($m = 1$) and the well-known Morley element for biharmonic equations ($m = 2$).

In order to analyze the convergence of the new class of finite element method, a general convergence theory based on a simple weak continuity assumption is also developed in this paper for nonconforming finite element methods. This new theory can be applied directly to all the simplicial and tensor-product nonconforming finite elements that are known to the authors, including the new finite element spaces proposed in this paper.

For both theoretical and practical considerations, a procedure of constructing nodal basis functions of the new finite element spaces is also presented in the paper.

1. INTRODUCTION

In the study of qualitative and numerical analysis of partial differential equations and, in general, of approximation theory, we are often interested in the approximation of functions in Sobolev spaces by piecewise polynomials (such as finite element

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spaces) defined on a partition of the domain by, say, a number of simplexes (as shown in Figure 1 in a two dimensional domain).

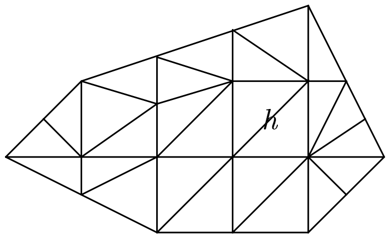

FIGURE 1. Partition $\mathcal{T}_h$

Conforming subspaces. One of the commonly studied Sobolev space is H^1 that consists of square integral functions whose first order derivatives are also square integrable. For this space, it is easy to construct approximation subspaces consisting of piecewise polynomial subspaces of any degree (that are defined on simplicial partitions of underlying domain). Such type of subspaces can be used as conforming finite element discretization for second order elliptic boundary value problems (see Ciarlet [10]). Here, by "conforming", we mean that the approximate spaces are proper *subspaces* of H^1.

The next commonly studied Sobolev space is H^2 that consists of square integral functions whose first and second order derivatives are all square integrable. It turns out that it is much more difficult to construct conforming finite element spaces, namely piecewise polynomial subspaces, of H^2. The difficulty increases as the spatial dimension, denoted by n, increases. The minimal degree of conforming elements is 5 for $n = 2$ (the well-known Argyris elements, see [10]) and 9 for $n = 3$ (see [45]). We do not know any results for $n \geq 4$. Such subspaces can be used for finite element discretization for 4-th order elliptic and parabolic partial differential equations, such as the Kirchhoff plate model for $n = 2$ (see [10]) and Cahn-Hillard equations for $n = 3$ (see [8, 7, 12, 15, 35]).

For a general Sobolev space H^m that consists of square integrable function whose all up to m-th order derivatives are square integral, construction of piecewise polynomial subspaces become increasingly more difficult as the differential order m and/or spatial dimension n increase. In fact, we do not know any such piecewise polynomial subspaces on a general partition of simplexes when $m \geq 3$ and $n \geq 2$ (or $m \geq 2$ and $n \geq 4$).

Nonconforming spaces: consistent approximation. Although for H^2, 5-th order subspaces (for $n = 2$) and 9-th order subspaces (for $n = 3$) can be constructed, piecewise polynomials of such a high order are difficult to apply in practice. As an alternative, the so-called nonconforming finite element spaces, namely piecewise polynomial spaces that are not necessarily subspaces of H^2, have been constructed and used in practice. Obviously not all piecewise polynomial spaces will be convergent finite element spaces and certain "continuity" or consistency conditions need to be imposed. Such conditions have been widely studied in the literature, cf. [10, 11, 13, 14], [17] – [19], [22], [24] – [34], [37, 47]. An example of these condition is the "consistent approximation" condition which we shall now briefly describe.

Let $\{V_h\}$ denote piecewise polynomial spaces defined on a sequence of partition, denoted by $\mathcal{T}_h$, of simplexes $\{T\}$, whose diameters are all bounded by h. For an n dimensional multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, define

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

For $v_h \in V_h$, we denote $\partial_h^\alpha v_h$ the partial derivatives of v_h taken piecewise with respect to the partition $\mathcal{T}_h$. We say that $\{V_h\}$ is a consistent approximation to the Sobolev space H^m if it satisfies the following two properties:

1. **Approximation property:** for any $v \in H^m$,

$$\lim_{h \to 0} \inf_{v_h \in V_h} \sum_{|\alpha| \le m} \|\partial^\alpha v - \partial_h^\alpha v_h\|_{L^2} = 0;$$

2. **Consistent property:** for any infinite sequence $\{v_{h_k}\}$ with $v_{h_k} \in V_{h_k}$ and $h_k \to 0$ as $k \to \infty$ such that $\{\partial_{h_k}^\alpha v_{h_k}\}$ is weakly convergent, in L^2, to v^α for each multi-index α satisfying $|\alpha| \le m$, it is always true that $v^\alpha = \partial^\alpha v^0$ for all $|\alpha| \le m$.

Consistent approximation spaces provide approximation to functions in H^m by functions outside of the space H^m, therefore consistent approximation is a kind of "outer" approximation space. Furthermore, all the (weak) limiting functions from the consistent approximation spaces V_h belong to H^m. This can be viewed as a "closedness" property. In fact, the *consistent approximation* property was originally called *weakly compact* when it was first introduced by Stummel [32]. The most important property of consistent approximation space is as follows:

> **Theorem** (Stummel [32]). *$\{V_h\}$ is a convergent family of finite element spaces for a general $2m$-th order elliptic boundary value problems if and only if $\{V_h\}$ is a consistent approximation of H^m.*

Therefore, consistent approximation spaces for Sobolev spaces H^m are of both theoretical and practical interests.

Let us now review what are the known consistent approximation spaces for Sobolev spaces. Obviously all conforming finite element subspaces of H^m are consistent approximation spaces. When $m \ge 2$ and $n \ge 2$, as mentioned before, conforming finite element spaces are rare for H^m and the purpose of this paper is to construct nonconforming but consistent approximation spaces in these cases.

Minimal degree. From both theoretical and practical view points, we are particularly interested in consistent approximation spaces for H^m consisting of piecewise polynomials with the smallest possible degree, denoted by $d_{\min}(m,n)$, in R^n.

It is easy to see that

$$d_{\min}(m,n) \ge m, \quad \forall m \ge 1, n \ge 1.$$

Since it is well-known that convergent linear simplicial finite elements can be easily constructed for second order elliptic boundary value problems in any dimension, we have

$$d_{\min}(1,n) = 1, \quad \forall n \geq 1.$$

In fact we have two families of finite element spaces that have minimal degree for $m = 1$ and $n \geq 2$, namely the conforming linear and the nonconforming linear.

The answer for $m = 2$ is less obvious, but it is also known for $n \geq 2$:

$$d_{\min}(2,n) = 2, \quad \forall n \geq 2.$$

The best-known example of this family is the classic Morley element [21] for biharmonic equations for $n = 2$. This unusual finite element space has also been extended to higher dimensions, see Ruas [23] and Wang and Xu [38]. The two extended Morley families of nonconforming quadratic elements in R^n in [23] and [38] coincide for $n = 3$ but differ considerably for $n \geq 4$. Thus for $n = 2, 3$, we only know one family of finite element spaces that have minimal degree, but for $n \geq 4$, we know two such families.

With the new class of consistent approximation space to be constructed in this paper, we can conclude in general that.

$$d_{\min}(m,n) = m, \quad \forall m \geq 1, n \geq m.$$

New consistent approximation spaces for general H^m. A universal construction will be given in this paper for consistent approximation spaces for H^m in R^n (with $n \geq m$) consisting of piecewise polynomials of degree m. This space can be used as finite element spaces for the discretization of $2m$-th order elliptic boundary value problems.

This class of spaces, denoted by M_h^m, are given by piecewise polynomials with degree not greater than m, namely the space P_m. Degrees of freedom for M_h^m in each element are given in terms of integral averages of normal derivatives of order $m-k$ on all subsimplexes of dimension $n-k$ for $1 \leq k \leq m$. The total number of these degrees of freedom in each element amounts to

$$\sum_{k=1}^{m} C_{n+1}^{n-k+1} C_{m-1}^{m-k} = C_{n+m}^{m}$$

which is precisely the dimension of P_m. One remarkable property of these sequence of spaces M_h^m is that $\partial_i M_h^m \subset M_h^{m-1}$ and, furthermore,

$$\text{span } (\partial_1 M_h^m, \partial_2 M_h^m, \ldots, \partial_n M_h^m) = M_h^{m-1}.$$

The degrees of freedom are just the ones of the nonconforming linear element when $m = 1$, while they are the ones of the Morley element [38] when $m = 2$. That is, we recover these two nonconforming elements in a canonical fashion.

Applications. While the construction in this paper is mainly motivated by theoretical considerations, the new element can also be applied to practical problems. The modelling for plates in linear elasticity is a classic area that 4-th order partial differential equations find their applications in two spatial dimensions. In recent years, the modelling in material science makes use of 4-th order equations (see [8, 7, 12, 15, 35]) and also 6-th order equations ([4, 42, 43]) in three dimensions. Elliptic or parabolic equations of 8-th or higher order are rare for practical applications, but in theory of differential geometry (see [9]), elliptic equations of order $m = n/2$ in any dimension n has been used.

While encountering a high order partial differential equations, one often tries to transform them into a system of lower order equations. Such a practice is attributed to the fact that higher order partial differential equations are often thought to be too difficult to be efficiently discretized by finite element or finite difference methods. One strong message this paper sends is that a direct discretization of high order partial differential equations is also practical. For example, a 6-th order partial differential equations in 3 dimensions can be discretized by only piecewise cubic polynomials, which has 20 degrees of freedom on each element and is not very difficult to carry out in practice. For practical as well as theoretical considerations, nodal basis functions for our new elements will be explicitly constructed in this paper (see Section 4).

We would like to point out that there is an important reason that one probably should use our new elements for directly discretizing high order partial differential equations rather than transforming it into a system of lower order equations. One classic folklore in the finite element community is the example of the simply supported plate model on a *polygonal* domain Ω that can be reduced to the following boundary problem for the biharmonic equation

$$\Delta^2 u = f \text{ in } \Omega, \qquad u = \Delta u = 0 \text{ on } \partial\Omega. \tag{1.1}$$

One naturally attempts to introduce an intermediate variable $v = -\Delta u$ and to transform the above problem into two decoupled systems of second order equations as follows

$$-\Delta v = f \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega; \tag{1.2}$$

and

$$-\Delta u = v \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

But it is easy to see that (1.1) is not always equivalent to (1.2) and (1.3). For example, when Ω is a concave polygon, for most f, the solution u of (1.1) (for which $\Delta u \notin H^1(\Omega)$, see [6]) will be different from the solution u from (1.2) and (1.3).

Nodal basis functions. It is both theoretically and (apparently) practically interesting to see if a set of nodal basis functions can be explicitly constructed for our new finite element spaces. As shown in Section 4, we are indeed able to do so. In particular, we have given all the details for $m = 1, 2$ and 3.

Philosophic comments. It is theoretically pleasing that the degrees of freedom in our construction just fit so perfectly well in the general case $n \geq m$. When $n < m$, the situation all a sudden becomes more complicated and it is not clear how a general construction is possible. One might wonder if such an extraordinary "fitting" is related to some deeper or more general mathematical structure.

The fact that nonconforming finite element methods can be constructed for any order of partial differential equations in such a generality and elegancy may also lead one to argue that nonconforming finite elements may be, at least sometimes, more "natural" than conforming finite element methods when they are used for discretization partial differential equations. Indeed, conforming finite element spaces, as mentioned earlier, can not be constructed easily in general cases. The "natural" property such as $\partial_h M_h^m \subset M_h^{m-1}$ that holds for our new nonconforming elements can not be expected for conforming elements. Another interesting property is that the degrees of freedom that define our new finite element spaces for H^m (and also the corresponding interpolation operators) are all well defined for all functions in H^m, but this is not the case for any conforming elements in multiple dimensions. In fact, except for one spatial dimension, for all the known conforming element spaces of H^m ($m \geq 1$), the degrees of freedom that define these spaces (and the corresponding interpolation operators) are not well-defined for functions in H^m (extra smoothness is required).

Discontinuous Galerkin methods. In addition to conforming and nonconforming finite element methods, discontinuous Galerkin methods (see [3]), which have received considerable research interests in recent years, represents another type of discretization methods for $2m$-th order partial differential equations. The discontinuous Galerkin method uses discontinuous piecewise polynomial spaces and it imposes the consistency of these spaces by introducing certain penalty terms on the element interfaces in the discrete variational forms. Thus the study of this type methods focus exclusively on the construction and the analysis of appropriate discrete variational forms. While there have been a lot of studies of these methods for both 2nd order and 4th order partial differential equations, a general and canonical construction for any $2m$-th order equations is lacking.

Outline of the paper. The rest of the paper is organized as follows. Section 2 gives a detailed description of our family of minimal degree finite element spaces. Section 3 discusses a general convergence theory for nonconforming finite element methods of $2m$-th order elliptic partial differential operator, applies it to our new nonconforming elements. Section 4 considers the construction of the nodal basis functions of our finite elements. The last section contains some brief concluding remarks.

2. Nonconforming finite element spaces of minimal degree

In this section, we will construct a minimal piecewise polynomial approximations to $H^m(\Omega)$ for $\Omega \subset R^n$ with $n \geq m \geq 1$.

We first introduce some basic notation. Given a nonnegative integer k and a bounded domain $G \subset R^n$ with boundary ∂G, let $H^k(G)$, $H_0^k(G)$, $(\cdot,\cdot)_{k,G}$, $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$ denote the usual Sobolev spaces, inner product, norm and semi-norm respectively.

For $k \geq 1$, let A_k be the set consisting of all k dimensional multi-indexes. We will use α, α' to denote the multi-indexes in A_n, $\gamma, \gamma', \gamma''$ the multi-indexes in A_{n+1} and β, β' the others. Let e_i denote the corresponding dimensional multi-indexes with the i-th component 1 and the others 0. Define

$$\mathcal{C}_k = \{\ \sigma = (\sigma_1, \sigma_2, \cdots, \sigma_k)\ :\ 1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq n+1\}.$$

Following the description in [10], a finite element can be represented by a triple (T, P_T, D_T), with T the geometric shape of the element, P_T the shape function space and D_T the set of the degrees of freedom, such that D_T is P_T-unisolvent.

Let Ω be a bounded polyhedron domain of R^n. Assume that $\{h\}$ is a sequence of positive number and $h \to 0$. For each h, let $\mathcal{T}_h$ be a partition of Ω corresponding to a finite element (T, P_T, D_T), and let h be the mesh size, i.e., the maximal diameter of the elements in $\mathcal{T}_h$.

For any element $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest ball containing T and ρ_T be the diameter of the largest ball contained in T. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ is quasi-uniform, namely it satisfies that

$$h_T \leq h \leq \eta \rho_T, \quad \forall T \in \mathcal{T}_h, \tag{2.1}$$

with η being a positive constant independent of h.

For a subset $B \subset R^n$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials defined on B with degree not greater than r, and $Q_r(B)$ the space of all polynomials with degree in each variable not greater than r. Define

$$P_{r,h} = \{\, v \in L^2(\Omega)\ :\ v|_T \in P_r(T),\ \forall T \in \mathcal{T}_h\}. \tag{2.2}$$

We will give the description of (T, P_T, D_T) for our new finite element first. Then we will show P_T-unisolvent property and give the construction and the error estimate of the corresponding interpolation operator. Moreover, we will define the global finite element spaces and show their basic properties, such as the approximation property and the inclusion property.

2.1. **The local degrees of freedom.** For our new element (T, P_T, D_T), T is a simplex and $P_T = P_m(T)$. The set of degrees of freedom, denoted by D_T^m, will be given in the following.

Given an n-simplex T with vertices a_i, $1 \leq i \leq n+1$, let $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ be the barycentric coordinates of T. For $1 \leq k \leq n$ and $\sigma \in \mathcal{C}_k$, let F_σ be the $(n-k)$-dimensional subsimplex of T without $a_{\sigma_1}, \cdots, a_{\sigma_k}$ as its vertexes. For any $(n-k)$-dimensional subsimplex F of T, let $|F|$ denote its measure, and let $\nu_{F,1}, \cdots, \nu_{F,k}$ be its unit outer normals which are linearly independent.

For $1 \le k \le m$, any $(n-k)$-dimensional subsimplex F of T and $\beta \in A_k$ with $|\beta| = m-k$, we define

$$d_{T,F,\beta}(v) = \frac{1}{|F|}\int_F \frac{\partial^{|\beta|} v}{\partial \nu_{F,1}^{\beta_1}\cdots \nu_{F,k}^{\beta_k}}, \quad \forall v \in H^m(T). \tag{2.3}$$

By the Sobolev embedding theorems [1], $d_{T,F,\beta}$ is a continuous linear functional on $H^m(T)$. Then the set of the degrees of freedom is given by

$$D_T^m = \{\ d_{T,F_\sigma,\beta}\ :\ \beta \in A_k \text{ with } |\beta| = m-k,\ \sigma \in \mathcal{C}_k,\ 1 \le k \le m\}. \tag{2.4}$$

That is, the degrees of freedom are the integral averages of normal derivatives of order $m-k$ on all subsimplexes of dimension $n-k$ for $1 \le k \le m$.

For each $1 \le k \le m$, T has C_{n+1}^{n-k+1} subsimplexes of $(n-k)$-dimension. For each $(n-k)$-dimensional subsomplex F, the number of all $(m-k)$-th order direction derivatives, with respect to $\nu_{F,1}, \cdots, \nu_{F,k}$, is C_{m-1}^{m-k}. Therefore, by the well-known Vandermonde combinatorial identity, the number of the total degrees of freedom is given by

$$\sum_{k=1}^{m} C_{n+1}^{n-k+1} C_{m-1}^{m-k} = C_{n+m}^{m}$$

which is precisely the dimension of $P_m(T)$.

Let $J = C_{n+m}^m$. We also number all the degrees of freedom by

$$d_{T,1}(v), d_{T,2}(v), \cdots, d_{T,J}(v).$$

Then $D_T^m = \{d_{T,1}(v), d_{T,2}(v), \cdots, d_{T,J}(v)\}$.

For $1 \le k \le m$ and an $(n-k)$-dimensional subsimplex F, say $F = F_\sigma$ with $\sigma \in \mathcal{C}_k$, different choices (for $k > 1$) of $\nu_{F,1}, \cdots, \nu_{F,k}$ will lead to equivalent degrees of freedom. The particular and convenient choice of normal directions are as follows

$$\nu_{F,i} = -\frac{\nabla \lambda_{\sigma_i}}{\|\nabla \lambda_{\sigma_i}\|}, \quad 1 \le i \le k, \tag{2.5}$$

see Section 4 for related discussions.

Some special cases: $1 \le m \le 3$**.** We now give some brief discussions for all the corresponding spaces for three lowest indices $1 \le m \le 3$. The degrees of freedom in these cases are depicted in Table 1 for $m \le n \le 3$. For $m = 1$ and $n = 1$, we obtain the well-known conforming linear elements. This is the only conforming element in this family of elements. For $m = 1$ and $n \ge 2$, we obtain the well-known nonconforming linear elements. For $m = 2$, we recover the well-known Morley element for $n = 2$ and its generalization to $n \ge 2$ (see Wang and Xu [38]). For $m = 3$ and $n = 3$, we obtain a new cubic element on a simplex that has 20 degrees of freedom.

2.2. **Unisolvent property and canonical nodal interpolation.** We need to show the P_T-unisolvent property of our new finite element. First, we show a crucial property.

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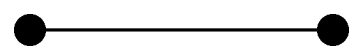
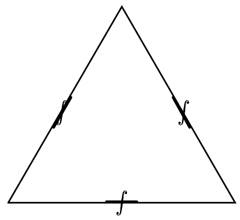
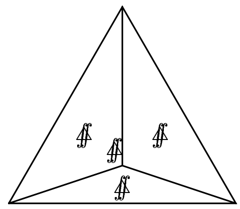
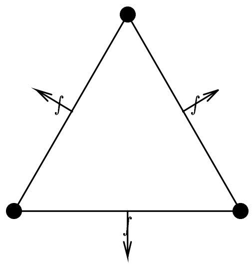
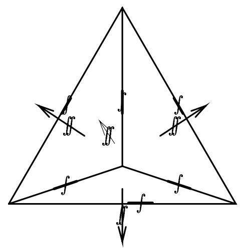
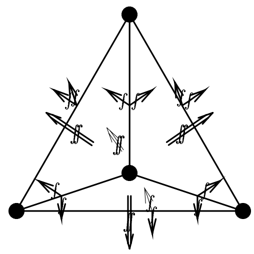
1			
2			
3			

TABLE 1. Degrees of Freedom

Lemma 2.1. *Let $1 \leq k \leq m$ and F be an $(n-k)$-dimensional subsimplex of T. Then for any $v \in H^m(T)$, the integrals of its all $(m-k)$-th order derivatives on F*

$$\int_F \partial^\alpha v, \quad |\alpha| = m-k,$$

are uniquely determined by all $d_{T,F',\beta}(v)$ given in (2.3) with $k \leq r \leq m$, F' $(n-r)$-dimensional subsimplex of F, $\beta \in A_r$ and $|\beta| = m-r$.

Proof. Let $v \in H^m(T)$. We prove the lemma by induction. When $k = m$,

$$\frac{1}{|F|}\int_F v = d_{T,F,0}(v).$$

The lemma is obviously true.

Assume that the lemma is true for all $k \in \{i+1, \cdots, m\}$ with $1 \leq i < m$. We consider the case that $k = i$.

Denote all $(n-k-1)$-dimensional subsimplexes of the $(n-k)$-simplex F by $S_1, S_2, \cdots, S_{n-k+1}$, and the unit out normal of S_j by $n^{(j)}$, viewed as the boundary of an $(n-k)$-simplex in $(n-k)$-dimensional space. Choose orthogonal unit vectors $\tau_{F,k+1},\cdots,\tau_{F,n}$ that are tangent to F. Then

$$\nu_{F,1},\cdots,\nu_{F,k},\tau_{F,k+1},\cdots,\tau_{F,n}$$

form a basis of R^n.

Now let $|\alpha|=m-k$. If $\alpha_{k+1}=\cdots=\alpha_n=0$, then

$$\frac{1}{|F|}\int_F \frac{\partial^{m-k}v}{\partial\nu_{F,1}^{\alpha_1}\cdots\nu_{F,k}^{\alpha_k}\partial\tau_{F,k+1}^{\alpha_{k+1}}\cdots\partial\tau_{F,n}^{\alpha_n}} = d_{T,F,\beta}(v)$$

with $\beta\in A_k$ and $\beta_j=\alpha_j$, $1\le j\le k$. Otherwise, without loss of generality, let $\alpha_{k+1}>0$. Green's formula gives

$$\begin{aligned}&\int_F \frac{\partial^{m-k}v}{\partial\nu_{F,1}^{\alpha_1}\cdots\nu_{F,k}^{\alpha_k}\partial\tau_{F,k+1}^{\alpha_{k+1}}\cdots\partial\tau_{F,n}^{\alpha_n}}\\ &\quad=\sum_{j=1}^{n-k+1} n^{(j)}\cdot\tau_{F,k+1}\int_{S_j}\frac{\partial^{m-k-1}v}{\partial\nu_{F,1}^{\alpha_1}\cdots\nu_{F,k}^{\alpha_k}\partial\tau_{F,k+1}^{\alpha_{k+1}-1}\partial\tau_{F,k+2}^{\alpha_{k+2}}\cdots\partial\tau_{F,n}^{\alpha_n}}.\end{aligned}$$

By the assumption of induction, the right hand of the above identity can be expressed in terms of all $d_{T,F',\beta}(v)$ with $k<r\le m$, F' $(n-r)$-dimensional subsimplex of F, $\beta\in A_r$ and $|\beta|=m-r$. Consequently, the lemma is true for $k=i$. □

Lemma 2.2. *For $1\le i\le J$, there exists a unique polynomial $p_i\in P_m(T)$ such that*

$$d_{T,j}(p_i)=\delta_{ij},\quad 1\le j\le J, \tag{2.6}$$

where δ_{ij} is the Kronecker delta.

Proof. As the dimension of $P_m(T)$ is also J, it is sufficient to show that if $p\in P_m(T)$ and

$$d_{T,F_\sigma,\beta}(p)=0,\quad \beta\in A_k \text{ with } |\beta|=m-k,\ \sigma\in\mathcal{C}_k,\ 1\le k\le m, \tag{2.7}$$

then $p\equiv 0$.

By Lemma 2.1 and its proof, we deduce that

$$\int_{F_\sigma}\partial^\alpha p=0,\quad |\alpha|=m-k,\ \sigma\in\mathcal{C}_k,\ 1\le k\le m. \tag{2.8}$$

By Green's formula and (2.8) we have for all $1\le k_1\le k_2\le\cdots\le k_m\le n$,

$$\begin{aligned}\frac{\partial^m p}{\partial x_{k_1}\cdots\partial x_{k_m}} &= \frac{1}{|T|}\int_T \frac{\partial^m p}{\partial x_{k_1}\cdots\partial x_{k_m}}\\ &=\frac{1}{|T|}\sum_{\sigma\in\mathcal{C}_1}\int_{F_\sigma}\frac{\partial^{m-1}p}{\partial x_{k_2}\cdots\partial x_{k_m}}(\nu_{F_\sigma,1})_{k_1}=0.\end{aligned}$$

where $|T|$ is the measure of T. That is, $p\in P_{m-1}(T)$.

When $1\le k<m$ and $p\in P_{m-k}(T)$, by (2.8), all $(m-k)$-th order derivatives of p are zero and p is in $P_{m-k-1}(T)$. Thus p is a constant. By (2.8) we obtain that $p\equiv 0$. □

Lemma 2.2 shows the P_T-unisolvent property of our new elements, namely a polynomial $p \in P_m(T)$ is uniquely determined by $d_{T,j}(p)$, $1 \le j \le J$. The polynomials p_i given by (2.6) is called the basis function corresponding to degree of freedom $d_{T,i}$. We will give the construction of the basis functions in Section 4. Based on Lemma 2.2, we can define the interpolation operator $\Pi_T : H^m(T) \to P_m(T)$ by

$$\Pi_T v = \sum_{i=1}^{J} p_i d_{T,i}(v), \quad \forall v \in H^m(T). \tag{2.9}$$

We would like emphasis here that operator Π_T is well-defined for all functions in $H^m(T)$.

By the interpolation theory [10], we obtain the following error estimate of the interpolation operator.

Lemma 2.3. *For $s = 0, 1$,*

$$|v - \Pi_T v|_{k,T} \le C(\eta) h_T^{m+s-k} |v|_{m+s,T}, \ 0 \le k \le m+s, \ \forall v \in H^{m+s}(T) \tag{2.10}$$

for all n-simplex T with $h_T \le \eta \rho_T$. Here $C(\eta)$ is a constant that only depends on η.

2.3. **Global finite element spaces.** We define our piecewise polynomial spaces M_h^m and M_{h0}^m as follows.

(1) M_h^m consists of all functions v_h in $P_{m,h}$ such that for any $k \in \{1, \cdots, m\}$, any $(n-k)$-dimensional subsimplex F of any $T \in \mathcal{T}_h$ and any $\beta \in A_k$ with $|\beta| = m-k$, $d_{T,F,\beta}(v_h)$ is continuous through F.

(2) M_{h0}^m consists of all functions v_h in M_h^m such that for any $k \in \{1, \cdots, m\}$, any $(n-k)$-dimensional subsimplex F of any $T \in \mathcal{T}_h$ and any $\beta \in A_k$ with $|\beta| = m-k$, if $F \subset \partial\Omega$ then $d_{T,F,\beta}(v_h) = 0$.

Define an operator Π_h on $H^m(\Omega)$ as follows:

$$(\Pi_h v)|_T = \Pi_T(v|_T), \quad \forall T \in \mathcal{T}_h, \ \forall v \in H^m(\Omega). \tag{2.11}$$

By the definition, $\Pi_h v \in M_h^m$ for any $v \in H^m(\Omega)$ and $\Pi_h v \in M_{h0}^m$ for any $v \in H_0^m(\Omega)$.

For convenience, following [44], the symbols $\lesssim$, $\gtrsim$ and $\eqsim$ will be used in the rest of this paper. That $X_1 \lesssim Y_1$ and $X_2 \gtrsim Y_2$, mean that $X_1 \le c_1 Y_1$ and $c_2 X_2 \ge Y_2$ for some positive constants c_1 and c_2 that are independent of mesh size h. That $X_3 \eqsim Y_3$ means that $X_3 \lesssim Y_3$ and $X_3 \gtrsim Y_3$.

We define, for $w \in L^2(\Omega)$ with $w|_T \in H^m(T)$, $\forall T \in \mathcal{T}_h$,

$$\|w\|_{m,h}^2 = \sum_{T \in \mathcal{T}_h} \|w\|_{m,T}^2, \quad |w|_{m,h}^2 = \sum_{T \in \mathcal{T}_h} |w|_{m,T}^2.$$

Now we consider the approximate property of M_h^m and M_{h0}^m.

Theorem 2.1. *For any $v \in H^{m+1}(\Omega)$,*

$$\|v - \Pi_h v\|_{m,h} \lesssim h|v|_{m+1,\Omega}, \tag{2.12}$$

and for any $v \in H^m(\Omega)$,

$$\lim_{h \to 0} \|v - \Pi_h v\|_{m,h} = 0. \tag{2.13}$$

Proof. First, let $v \in H^{m+1}(\Omega)$. By Lemma 2.3, we obtain (2.12) directly.

Now let $w \in H^m(\Omega)$. Since $H^{m+1}(\Omega)$ is dense in $H^m(\Omega)$, for any $\varepsilon > 0$ there exists $\phi \in H^{m+1}(\Omega)$ such that

$$\|w - \phi\|_{m,\Omega} < \varepsilon.$$

By (2.12), there exists $\tilde{h} > 0$ such that

$$\|\phi - \Pi_h \phi\|_{m,h} < \varepsilon$$

when $h < \tilde{h}$. Therefore by (2.10)

$$\|w - \Pi_h w\|_{m,h} \leq \|w - \phi\|_{m,h} + \|\Pi_h(w - \phi)\|_{m,h} + \|\phi - \Pi_h \phi\|_{m,h} \lesssim \varepsilon$$

when $h < \tilde{h}$. This leads to (2.13). □

When $n > 1$, M_h^m is not a subspace of $H^m(\Omega)$ and, as shown at the end of Section 4, M_h^m is not even a subspace of $C^0(\bar{\Omega})$. Although functions in M_h^m are not continuous on whole Ω in general, they have certain weak continuity. By the definitions of M_h^m and M_{h0}^m, Lemma 2.1 and its proof, the following lemma can be obtained directly.

Lemma 2.4. *Let $k \in \{1, \cdots, m\}$ and F be an $(n-k)$-dimensional subsimplex of $T \in \mathcal{T}_h$. Then for any $v_h \in M_h^m$ and any $T' \in \mathcal{T}_h$ with $F \subset T'$,*

$$\int_F \partial^\alpha (v_h|_{T'}) = \int_F \partial^\alpha (v_h|_T), \quad |\alpha| = m - k. \tag{2.14}$$

If $F \subset \partial\Omega$, then for any $v_h \in M_{h0}^m$,

$$\int_F \partial^\alpha (v_h|_T) = 0, \quad |\alpha| = m - k. \tag{2.15}$$

An equivalent definition. By Lemma 2.4, we can give an equivalent definition of M_h^m and M_{h0}^m: M_h^m consists of all functions v_h in $P_{m,h}$ such that for any $k \in \{1, \cdots, m\}$, any $(n-k)$-dimensional subsimplex F of any $T \in \mathcal{T}_h$ and any α with $|\alpha| = m - k$, the integral of $\partial_h^\alpha v_h$ over F is continuous through F; M_{h0}^m consists of all functions v_h in M_h^m such that for any $k \in \{1, \cdots, m\}$, any $(n-k)$-dimensional subsimplex F of any $T \in \mathcal{T}_h$ and any α with $|\alpha| = m - k$, if $F \subset \partial\Omega$ then the integral of $\partial_h^\alpha v_h$ over F vanishes.

Lemma 2.5. *Let $|\alpha| < m$ and F be an $(n-1)$-dimensional subsimplex of $T \in \mathcal{T}_h$. Then for any $v_h \in M_h^m$, $\partial_h^\alpha v_h$ is continuous at a point on F at least. If $F \subset \partial\Omega$ and $v_h \in M_{h0}^m$ then $\partial_h^\alpha v_h$ vanishes at a point on F at least.*

Proof. Let $v_h \in M_h^m$ and $T' \in \mathcal{T}_h$ with $F \subset T'$. By Lemma 2.4, there is an $(n - m + |\alpha|)$-dimensional subsimplex F' of F such that

$$\int_{F'} \partial^\alpha (v_h|_{T'}) = \int_{F'} \partial^\alpha (v_h|_T).$$

Then $\partial_h^\alpha v_h$ is continuous at a point on F' at least.

If $F \subset \partial\Omega$ and $v_h \in M_{h0}^m$ then there is an $(n-m+|\alpha|)$-dimensional subsimplex F' of F by Lemma 2.4, such that

$$\int_{F'} \partial^\alpha(v_h|_T) = 0.$$

Thus $\partial_h^\alpha v_h$ vanishes at a point on F' at least. □

2.4. **Inclusion properties.** We now discuss two simple inclusion properties of our finite element spaces. First, we have the following observation.

Lemma 2.6. *Given any $n > m \geq 1$ and a simplex T, the set of subsimplexes of T that are used to define for D_T^m is a subset of that for D_T^{m+1}. More precisely, the degrees of freedom for D_T^{m+1} can be obtained by taking the integral of one order higher normal derivatives of functions on the same subsimplexes used for D_T^m, plus the integral average of function over all the additional $(n-m-1)$-subsimplexes.*

To obtain a more interesting inclusion property, we define

$$\partial M_h^m = \text{span}\{\partial^{e_1} M_h^m, \partial^{e_2} M_h^m, \cdots, \partial^{e_n} M_h^m\}$$

and

$$\partial M_{h0}^m = \text{span}\{\partial^{e_1} M_{h0}^m, \partial^{e_2} M_{h0}^m, \cdots, \partial^{e_n} M_{h0}^m\}.$$

Theorem 2.2. *Let $n \geq m > 1$, then*

$$\partial M_h^m = M_h^{m-1}, \quad \partial M_{h0}^m = M_{h0}^{m-1}. \tag{2.16}$$

Proof. By the equivalent definition of M_h^m and M_{h0}^m, we obtain directly that

$$\partial M_h^m \subset M_h^{m-1}, \quad \partial M_{h0}^m \subset M_{h0}^{m-1}.$$

For any $k \in \{1, 2, \cdots, m-1\}$, any $T \in \mathcal{T}_h$, any $(n-k)$-dimensional subsimplex F of T and any $\beta \in A_k$ with $|\beta| = m-1-k$, let w be the global basis function of M_h^{m-1} corresponding to degree of freedom $d_{T,F,\beta}$, and let v be the global basis function of M_h^m corresponding to degree of freedom $d_{T,F,\beta'}$ with $\beta' = \beta + (1, 0, \cdots, 0)$. By the definitions, $w = \nu_{F,1}^\top \nabla v$. Then the theorem follows. □

A note on the case $m = 0$. In the above construction, we made the assumption that $m \geq 1$. But we may slightly enlarge this construction to include the trivial case $m = 0$, namely $L^2(\Omega)$ space. Technically, we can just replace the constraint $1 \leq k \leq m$ by $\min(1, m) \leq k \leq m$. In this case, the shape function space is again $P_m(T) = P_0(T)$, namely the constant, and the corresponding degree of freedom is just the volume integral on each simplex (see Figure 2). This trivial case of finite element space, denoted by M_h^0, may be viewed as a close relative to M_h^m $(m \geq 1)$, but not a direct family member in view of the properties stated in Lemma 2.6.

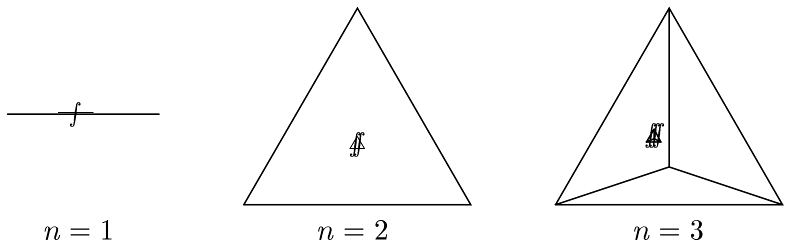


FIGURE 2. $m = 0$

3. A general convergence analysis for nonconforming elements with application to new element

In this section, we will present a general convergence theory for nonconforming finite element methods based on some easily verifiable sufficient conditions. This new theory applies to all the nonconforming elements defined on simplexes and cubes that are known to us, including, in particular, the new class of finite element method introduced in this paper.

Let b_α be nonnegative constants and $b_\alpha > 0$ when $|\alpha| = m$. Define

$$a(v, w) = \int_\Omega \Big(\sum_{|\alpha| \le m} b_\alpha \partial^\alpha v \, \partial^\alpha w \Big), \quad \forall v, w \in H^m(\Omega). \tag{3.1}$$

Let W be $H_0^m(\Omega)$ or $H^m(\Omega)$, and let $f_\alpha \in L^2(\Omega)$, $|\alpha| \le m$. We consider the following variational problem: find $u \in W$ such that

$$a(u, v) = \sum_{|\alpha| \le m} (f_\alpha, \partial^\alpha v), \quad \forall v \in W. \tag{3.2}$$

We assume that problem (3.2) has unique solution for any $f_\alpha \in L^2(\Omega)$ with $|\alpha| \le m$.

The above variational problem corresponds to the following $2m$-th order partial differential equation:

$$\sum_{|\alpha| \le m} (-1)^{|\alpha|} \partial^\alpha (b_\alpha \partial^\alpha u) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \partial^\alpha f_\alpha, \quad \text{in } \Omega. \tag{3.3}$$

When $W = H_0^m(\Omega)$, the variational problem (3.2) corresponds to the homogeneous Dirichlet boundary problem of partial equation (3.3) with boundary conditions:

$$\frac{\partial^k u}{\partial \nu^k}\Big|_{\partial\Omega} = 0, \quad 0 \le k \le m - 1, \tag{3.4}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^\top$ is the unit outer normal to $\partial\Omega$.

When $W = H^m(\Omega)$, problem (3.2) corresponds to the boundary problem of (3.3) with some natural boundary conditions.

For $v, w \in L^2(\Omega)$ that $v|_T, w|_T \in H^m(T)$, $\forall T \in \mathcal{T}_h$, we define

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T \Big(\sum_{|\alpha|\le m} b_\alpha \partial^\alpha v \, \partial^\alpha w \Big). \tag{3.5}$$

Let V_h be a nonconforming finite element space to approximate $H^m(\Omega)$ corresponding to $\mathcal{T}_h$, and V_{h0} be the corresponding subspace of V_h to approximate $H_0^m(\Omega)$. When $W = H^m(\Omega)$ let W_h be V_h otherwise let $W_h = V_{h0}$. The nonconforming finite element method for problem (3.2) corresponding to W_h is: to find $u_h \in W_h$ such that

$$a_h(u_h, v_h) = \sum_{|\alpha|\le m} (f_\alpha, \partial_h^\alpha v_h), \quad \forall v_h \in W_h. \tag{3.6}$$

We will discuss the convergent property of solution u_h of problem (3.6).

3.1. **Consistent approximation.** As mentioned in the introduction, the first condition guaranteeing the convergent property is the approximation condition. We say that $\{W_h, W\}$ satisfies the approximation condition if

$$\lim_{h\to 0} \inf_{v_h\in W_h} \|v - v_h\|_{m,h} = 0, \quad \forall v \in W. \tag{3.7}$$

By means of the interpolation theory (see [10]), the approximation condition is easy to be handled.

By the approximation theory $\{P_{r,h}, H^m(\Omega)\}$ satisfies the approximation condition when $r \ge m$, while $\{P_{m-1,h}, H^m(\Omega)\}$ fails. Then among the piecewise polynomial approximations to $H^m(\Omega)$, the m-th degree is the least.

We say that $\{W_h, W\}$ satisfies the consistent condition if for any infinite sequence $\{v_{h_k}\}$ with $v_{h_k} \in W_{h_k}$ and $h_k \to 0$ as $k \to \infty$ such that $\{\partial_{h_k}^\alpha v_{h_k}\}$ is weakly convergent, in $L^2(\Omega)$, to v^α for each multi-index α satisfying $|\alpha| \le m$, it is always true that $v^0 \in W$ and $v^\alpha = \partial^\alpha v^0$ for all $|\alpha| \le m$.

We say that $\{W_h\}$ is a consistent approximation of W if $\{W_h, W\}$ satisfies both the approximation condition and the consistent condition.

The bilinear form $a_h(\cdot,\cdot)$ is called to be uniformly W_h-elliptic if

$$\|v_h\|_{m,h}^2 \lesssim a_h(v_h, v_h), \quad \forall v_h \in W_h. \tag{3.8}$$

The following theorem was shown in [32].

Theorem 3.1. *Assume that $a_h(\cdot,\cdot)$ is uniformly W_h-elliptic. Then for any $f_\alpha \in L^2(\Omega)$ with $|\alpha| \le m$, the solution u_h of problem (3.6) converges to the solution u of problem (3.2):*

$$\lim_{h\to 0} \|u - u_h\|_{m,h} = 0$$

if and only if $\{W_h\}$ is a consistent approximation of W.

Proof. Let us first prove the "if" part of the result. By (3.8) and definition of u_h, it is easy to see that $\{u_h\}$ is bounded in the sense:

$$\|u_h\|_{m,h} \lesssim \sum_{|\alpha|\le m} \|f_\alpha\|_{0,\Omega}.$$

Thus, by the consistent assumption, there is a subsequence $\{u_{h_k}\}$ together with $u' \in W$ such that $\{\partial^\alpha_{h_k} u_{h_k}\}$ is weakly convergent to $\partial^\alpha u'$ in $L^2(\Omega)$ for all $|\alpha| \le m$.

Given any $v \in W$, there is a sequence $v_h \in W_h$ such that $\|v - v_h\|_{m,h} \to 0$. Thus, we have

$$\begin{aligned}|a_{h_k}(u_{h_k}, v_{h_k}) - a(u', v)| &\le |a_{h_k}(u_{h_k}, v_{h_k} - v)| + |a_{h_k}(u_{h_k} - u', v)| \\ &\lesssim \|u_{h_k}\|_{m,h_k}\|v_{h_k} - v\|_{m,h_k} + |a_{h_k}(u_{h_k} - u', v)| \to 0 \text{ as } h_k \to 0\end{aligned}$$

and

$$\sum_{|\alpha|\le m} (f_\alpha, \partial^\alpha_{h_k} v_{h_k}) \to \sum_{|\alpha|\le m} (f_\alpha, \partial^\alpha v).$$

Consequently

$$a(u', v) = \sum_{|\alpha|\le m} (f_\alpha, \partial^\alpha v).$$

Thus u' is the solution of problem (3.2). Since the solution is unique, $u' = u$ and any subsequent of $\{\partial^\alpha_h u_h\}$ with $|\alpha| \le m$ will have a subsequence that converges to the same $\partial^\alpha u$ weakly. Thus the whole sequence $\{\partial^\alpha_h u_h\}$ will have to converge to $\partial^\alpha u$ weakly. To prove strong convergence, by the approximation condition, we can take a sequence $u'_h \in W_h$ such that $\|u - u'_h\|_{m,h} \to 0$. Then

$$\begin{aligned}\|u - u_h\|^2_{m,h} &\lesssim \|u - u'_h\|^2_{m,h} + a_h(u'_h - u_h, u'_h - u_h) \\ &\lesssim \|u - u'_h\|^2_{m,h} + a(u,u) - 2a_h(u, u_h) + a_h(u_h, u_h) \\ &= \|u - u'_h\|^2_{m,h} + a(u,u) - 2a_h(u, u_h) + \sum_{|\alpha|\le m} (f_\alpha, \partial^\alpha_h u_h) \\ &\to a(u,u) - 2a(u,u) + \sum_{|\alpha|\le m} (f_\alpha, \partial^\alpha u) = 0 \text{ as } h \to 0.\end{aligned}$$

Now we consider the "only if" part. Given $w \in W$, let $f_\alpha = b_\alpha \partial^\alpha w$, $|\alpha| \le m$. Then $u = w$ and

$$\lim_{h\to 0} \inf_{w_h \in W_h} \|w - w_h\|_{m,h} \le \lim_{h \to 0} \|u - u_h\|_{m,h} = 0.$$

Thus, $\{W_h, W\}$ satisfies the approximation condition.

If $\{W_h, W\}$ is not consistent, then there exists an infinite sequence $\{w_{h_k}\}$ with $w_{h_k} \in W_{h_k}$ and $h_k \to 0$ as $k \to \infty$, such that $\{\partial^\alpha_{h_k} w_{h_k}\}$ weakly converges to w^α

in $L^2(\Omega)$ for all $|\alpha| \le m$, but there exist α' and function ϕ with $0 < |\alpha'| \le m$, $\phi \in C_0^\infty(R^n)$ when $W_h = V_{h0}$ and $\phi \in C_0^\infty(\Omega)$ when $W_h = V_h$, such that

$$\int_\Omega \left(\partial^{\alpha'}\phi w^0 + (-1)^{|\alpha'|+1}\phi w^{\alpha'}\right) \neq 0.$$

Let f_α be given by

$$f_\alpha = \begin{cases} \partial^{\alpha'}\phi, & \alpha = 0 \\ (-1)^{|\alpha'|+1}\phi, & \alpha = \alpha' \\ 0, & \text{otherwise.} \end{cases}$$

Then for f_α given above we have

$$\sum_{|\alpha|\le m} (f_\alpha, \partial^\alpha v) = 0, \quad \forall v \in W.$$

Consequently, the corresponding solution u of problem (3.2) is zero and

$$0 = \lim_{k\to\infty} a_{h_k}(u_{h_k}, w_{h_k}) = \lim_{k\to\infty}\int_\Omega \left(\partial^{\alpha'}\phi w_{h_k} + (-1)^{|\alpha'|+1}\phi\partial_{h_k}^{\alpha'} w_{h_k}\right) \neq 0.$$

This is impossible. □

3.2. **Weak Continuity.** To check the consistent condition, one can use the generalized patch test proposed in [32]. Other easier and sufficient conditions can also be used, such as the patch test (see [5, 16, 36, 37]), the weak patch test [37], F-E-M test [29] and IPT test [46]. Here we give a sufficient condition based on the so called "weak continuity".

We say that V_h has the weak continuity (or the weak discontinuity) if for any v_h in V_h, any $(n-1)$-dimensional face F of $T \in \mathcal{T}_h$ and any $|\alpha| < m$, $\partial_h^\alpha v_h$ is continuous at a point on F at least. Correspondingly, we say that V_{h0} satisfies the weak zero-boundary condition if for any v_h in V_{h0}, any $(n-1)$-dimensional face F of $T \in \mathcal{T}_h$ with $F \subset \partial\Omega$ and any $|\alpha| < m$, $\partial_h^\alpha v_h$ vanishes at a point on F at least.

By Lemma 2.5, we know that M_h^m has the weak continuity and M_{h0}^m satisfies the weak zero-boundary condition.

With the weak continuity, we obtain that v_h is a single polynomial of degree less than m on whole Ω if $v_h \in V_h$ and $|v_h|_{m,h} = 0$. Moreover, $v_h \equiv 0$ when $v_h \in V_{h0}$ and the weak zero-boundary condition is satisfied, that is, $|\cdot|_{m,h}$ is a norm of V_{h0}. In this sense, the weak continuity and the weak zero-boundary condition are viewed as necessary conditions.

We assume in the rest of the section that there exists a nonnegative integer t such that $V_h \subset P_{t,h}$ for all h, and that $\mathcal{T}_h$ is a partition consisting of n-simplexes or consisting of n-cubes with their sides parallel to some coordinate axes respectively. Following the way used in [37], we have the following lemma.

Lemma 3.1. *Let V_h have the weak continuity and V_{h0} satisfy the weak zero-boundary condition. Then for any $v_h \in V_h$ and any $|\alpha| < m$ there exists a piecewise polynomial $v_\alpha \in H^1(\Omega)$ such that*

$$|\partial_h^\alpha v_h - v_\alpha|_{j,h} \lesssim h^{m-|\alpha|-j}|v_h|_{m,h}, \quad 0 \le j \le m - |\alpha|, \tag{3.9}$$

and v_α can be chosen in $H_0^1(\Omega)$ when $v_h \in V_{h0}$.

Proof. For set $B \subset R^n$, let $\mathcal{T}_h(B) = \{T \in \mathcal{T}_h : B \cap T \neq \emptyset\}$ and $N_h(B)$ be the number of the elements in $\mathcal{T}_h(B)$.

Let $v_h \in V_h$, $|\alpha| < m$. For $T \in \mathcal{T}_h$, denote by v_h^T the continuous extension of v_h from the interior of T to T. Given any $(n-1)$-dimensional face F of T, let us define the jump of $\partial_h^\alpha v_h$ across F as follows: $[\partial_h^\alpha v_h]_F = \partial^\alpha v_h^T|_F - \partial^\alpha v_h^{T'}|_F$ if $F = T \cap T'$ for some other $T' \in \mathcal{T}_h$ and $[\partial_h^\alpha v_h]_F = \partial^\alpha v_h^T|_F$ if $F = T \cap \partial\Omega$.

First, we show that if $F \not\subset \partial\Omega$ or $v_h \in V_{h0}$ then

$$[\partial_h^\alpha v_h]_F^2 \lesssim h^{2(m-|\alpha|)-n} \sum_{T' \in \mathcal{T}_h, F \subset T'} |v_h|_{m,T'}^2. \tag{3.10}$$

By the weak continuity and the weak zero-boundary condition there exists $x \in F$ such that $[\partial_h^\alpha v_h]_F$ vanishes at x, this leads to

$$[\partial_h^\alpha v_h]_F^2 \leq h^2 \max_{y \in F} \left[\frac{\partial}{\partial \tau}\partial_h^\alpha v_h\right]_F^2 (y) \lesssim h^2 \sum_{|\alpha'|=|\alpha|+1} \max_{y \in F} \left[\partial_h^{\alpha'} v_h\right]_F^2 (y)$$

where τ is a unit tangent of F. Repeating the same argument, we have

$$[\partial_h^\alpha v_h]_F^2 \lesssim h^{2(m-|\alpha|)} \sum_{|\alpha'|=m} \max_{y \in F} \left[\partial_h^{\alpha'} v_h\right]_F^2 (y).$$

By the inverse inequality, we obtain (3.10).

Let $l = m - |\alpha|$ and $0 \leq j \leq l$. If T is an n-simplex then we take $S_{l,T} = P_l(T)$ and $\Pi_{l,T}$ the interpolating operator corresponding to the element of n-simplex of type (l), otherwise take $S_{l,T} = Q_l(T)$ and $\Pi_{l,T}$ the interpolating operator corresponding to the element of n-cube of type (l) (see [10], p. 48,57). Let $\Xi_{l,T}$ be the set of nodal points of $\Pi_{l,T}$.

Now we define $v_\alpha \in H^1(\Omega)$ as follows: for all $T \in \mathcal{T}_h$, $v_\alpha|_T \in S_{l,T}$ and for each $x \in \Xi_{l,T}$ if $x \in \partial\Omega$ and $v_h \in V_{h0}$ then $v_\alpha(x) = 0$ otherwise

$$v_\alpha(x) = \frac{1}{N_h(x)} \sum_{T' \in \mathcal{T}_h(x)} \partial^\alpha v_h^{T'}(x). \tag{3.11}$$

Then v_α is well-defined, and $v_\alpha \in H_0^1(\Omega)$ when $v_h \in V_{h0}$.

By the interpolating theory,

$$|\partial_h^\alpha v_h - \Pi_{l,T}\partial_h^\alpha v_h|_{j,T} \lesssim h^{m-|\alpha|-j} |v_h|_{m,T}. \tag{3.12}$$

Using the affine argument, we can show the following inequality

$$|p|_{j,T}^2 \lesssim h^{n-2j} \sum_{x \in \Xi_{l,T}} |p(x)|^2, \quad \forall p \in S_{l,T}. \tag{3.13}$$

Since $\Pi_{l,T}\partial_h^\alpha v_h - v_\alpha|_T \in S_{l,T}$,

$$|\Pi_{l,T}\partial_h^\alpha v_h - v_\alpha|_{j,T}^2 \lesssim h^{n-2j} \sum_{x \in \Xi_{l,T}} |\Pi_{l,T}\partial^\alpha v_h^T(x) - v_\alpha(x)|^2. \tag{3.14}$$

If $x \in \Xi_{l,T} \cap \Omega$ or $v_h \notin V_{h0}$, then by (3.11) we have

$$|\Pi_{l,T}\partial^\alpha v_h^T(x) - v_\alpha(x)|^2 = \Big|\frac{1}{N_h(x)}\sum_{T'\in\mathcal{T}_h(x)}\Big(\partial^\alpha v_h^T(x) - \partial^\alpha v_h^{T'}(x)\Big)\Big|^2.$$

For $T' \in \mathcal{T}_h(x)$ and $T' \neq T$, there exist $T_1, \cdots, T_L \in \mathcal{T}_h(x)$ such that $T_1 = T$, $T_L = T'$ and $\tilde{F}_j = T_j \cap T_{j+1}$ is common $(n-1)$-dimensional face of T_j and T_{j+1}, $1 \le j < L$. By (3.10) and the fact that $N_h(x)$ is bounded, we obtain

$$\Big|\partial^\alpha v_h^T(x) - \partial^\alpha v_h^{T'}(x)\Big|^2 \lesssim \sum_{j=1}^{L-1}\max_{y\in F_j}[\partial_h^\alpha v_h]_{F_j}^2(y) \lesssim h^{2(m-|\alpha|)-n}\sum_{T'\in\mathcal{T}_h(x)}|v_h|_{m,T'}^2.$$

If $x \in \Xi_{l,T} \cap \partial\Omega$ and $v_h \in V_{h0}$, then we have by definition of v_α

$$|\Pi_{l,T}\partial^\alpha v_h^T(x) - v_\alpha(x)|^2 = |\partial^\alpha v_h^T(x)|^2 \lesssim h^{2(m-|\alpha|)-n}\sum_{T'\in\mathcal{T}_h(x)}|v_h|_{m,T'}^2.$$

From (3.14) we derive that

$$|\Pi_{l,T}\partial_h^\alpha v_h - v_\alpha|_{j,T}^2 \lesssim h^{2(m-|\alpha|-j)}\sum_{T'\in\mathcal{T}_h(T)}|v_h|_{m,T'}^2. \tag{3.15}$$

By (3.12), (3.15) and the triangle inequality, we get

$$|\partial_h^\alpha v_h - v_\alpha|_{j,h}^2 \lesssim h^{2(m-|\alpha|-j)}\sum_{T\in\mathcal{T}_h}\sum_{T'\in\mathcal{T}_h(T)}|v_h|_{m,T'}^2. \tag{3.16}$$

Then (3.9) follows. □

We can generalize the sufficient conditions of the consistent property for the second and forth order problems in [30, 46] to the $2m$-th order problem. For example, we have

Condition SPT. There exist a nonnegative integer s and linear operator Π_h^α : $\partial^\alpha V_h \to P_{s,h}$ for any $|\alpha| = m-1$, such that, for any $v_h \in V_h$,

(1) for any $(n-1)$-dimensional face F of $T \in \mathcal{T}_h$, the integral value of $\Pi_h^\alpha\partial_h^\alpha v_h$ over F is continuous and vanishes when $F \subset \partial\Omega$ and $v_h \in V_{h0}$;

(2) for any $T \in \mathcal{T}_h$,

$$\int_{\partial T}\partial_h^\alpha v_h\nu = \int_{\partial T}\Pi_h^\alpha\partial_h^\alpha v_h\nu, \tag{3.17}$$

and

$$|\Pi_h^\alpha\partial_h^\alpha v_h|_{1,T} \lesssim |v_h|_{m,T}, \tag{3.18}$$

where ν is the unit outer normal of ∂T.

Theorem 3.2. *Let V_h have the weak continuity, V_{h0} satisfy the weak zero-boundary condition and Condition SPT be satisfied. Then $\{V_h, H^m(\Omega)\}$ and $\{V_{h0}, H_0^m(\Omega)\}$ are all consistent.*

Proof. Let $\varphi \in C_0^\infty(\Omega)$ (or $C_0^\infty(R^n)$) and $\{v_{h_k}\}$ be an infinite sequence with $v_{h_k} \in V_{h_k}$ (or $V_{h_k 0}$) and $h_k \to 0$ as $k \to \infty$ such that $\{\partial_{h_k}^\alpha v_{h_k}\}$ is weakly convergent, in $L^2(\Omega)$, to v^α for each multi-index α satisfying $|\alpha| \le m$.

Now let $1 \le i \le n$ and $|\alpha| < m$. By Lemma 3.1 we have that for each k, there exists a piecewise polynomial $v_{\alpha k} \in H^1(\Omega)$ (or $H_0^1(\Omega)$) such that

$$|\partial_{h_k}^\alpha v_{h_k} - v_{\alpha k}|_{j,h_k} \lesssim h_k^{m-|\alpha|-j} |v_{h_k}|_{m,h_k}, \quad 0 \le j \le m - |\alpha|. \tag{3.19}$$

We obtain from (3.19), Green's formula and the Schwarz inequality that

$$\begin{aligned}
&\Big| \int_\Omega (\varphi \partial_{h_k}^{\alpha+e_i} v_{h_k} + \partial^{e_i}\varphi \partial_{h_k}^\alpha v_{h_k}) \Big| \\
&\qquad = \Big| \int_\Omega \Big(\varphi \partial_{h_k}^{e_i} (\partial_{h_k}^\alpha v_{h_k} - v_{\alpha k}) + \partial^{e_i}\varphi (\partial_{h_k}^\alpha v_{h_k} - v_{\alpha k}) \Big) \Big| \\
&\qquad \lesssim h_k^{m-|\alpha|-1} \|\varphi\|_{1,\Omega} |v_{h_k}|_{m,h_k},
\end{aligned}$$

and this leads to that

$$\int_\Omega (\varphi v^{\alpha+e_i} + \partial^{e_i}\varphi v^\alpha) = 0, \tag{3.20}$$

when $|\alpha| < m-1$.

Given $T \in \mathcal{T}_h$ and an $(n-1)$-dimensional face F of T, let $P_{\partial T}^0 : L^2(\partial T) \to P_0(\partial T)$ and $P_F^0 : L^2(F) \to P_0(F)$ be the orthogonal projections. When $|\alpha| = m-1$, set

$$v_{h_k}^\alpha = \Pi_{h_k}^\alpha \partial_{h_k}^\alpha v_{h_k}, \quad v_{h_k e}^\alpha = \partial_{h_k}^\alpha v_{h_k} - \Pi_{h_k}^\alpha \partial_{h_k}^\alpha v_{h_k}.$$

By the assumption and Green's formula, we have

$$\begin{aligned}
&\int_\Omega (\varphi \partial_{h_k}^{\alpha+e_i} v_{h_k} + \partial^{e_i}\varphi \partial_{h_k}^\alpha v_{h_k}) \\
&\qquad = \sum_{T\in\mathcal{T}_{h_k}} \sum_{F\subset\partial T} \int_F \varphi v_{h_k}^\alpha \nu_i + \sum_{T\in\mathcal{T}_{h_k}} \int_{\partial T} \varphi v_{h_k e}^\alpha \nu_i \\
&\qquad = \sum_{T\in\mathcal{T}_{h_k}} \sum_{F\subset\partial T} \int_F (\varphi - P_F^0\varphi)(v_{h_k}^\alpha - P_F^0 v_{h_k}^\alpha)\nu_i \\
&\qquad\quad + \sum_{T\in\mathcal{T}_{h_k}} \int_{\partial T} (\varphi - P_{\partial T}^0\varphi)(v_{h_k e}^\alpha - P_{\partial T}^0 v_{h_k e}^\alpha)\nu_i.
\end{aligned}$$

By the Schwarz inequality, the interpolation theory in [10] and (3.18), we obtain that

$$\begin{aligned}
&\Big| \int_\Omega (\varphi \partial_{h_k}^{\alpha+e_i} v_{h_k} + \partial^{e_i}\varphi \partial_{h_k}^\alpha v_{h_k}) \Big| \\
&\qquad \le \sum_{T\in\mathcal{T}_{h_k}} \sum_{F\subset\partial T} \|\varphi - P_F^0\varphi\|_{0,F} \|v_{h_k}^\alpha - P_F^0 v_{h_k}^\alpha\|_{0,F}
\end{aligned}$$

$$
+\sum_{T\in\mathcal{T}_{h_k}} \|\varphi - P^0_{\partial T}\varphi\|_{0,\partial T}\|v^\alpha_{h_k e} - P^0_{\partial T} v^\alpha_{h_k e}\|_{0,\partial T}
$$

$$
\lesssim h_k \sum_{T\in\mathcal{T}_{h_k}} |\varphi|_{1,T}|v_{h_k}|_{m,T} \lesssim h_k|\varphi|_{1,\Omega}|v_{h_k}|_{m,h_k}.
$$

Thus (3.20) is also true when $|\alpha| = m-1$.

Consequently, $v^\alpha = \partial^\alpha v^0$ for all $|\alpha|\le m$ and $v^0 \in H^m(\Omega)$ (or $H^m_0(\Omega)$). □

By Lemma 2.4, Lemma 2.5 and Theorem 3.2, we obtain the following corollary directly.

Corollary 3.1. *Both of* $\{M^m_h, H^m(\Omega)\}$ *and* $\{M^m_{h0}, H^m_0(\Omega)\}$ *are consistent.*

By Corollary 3.1 and Theorem 2.1 we know that M^m_h is a consistent approximation of $H^m(\Omega)$ and M^m_{h0} is a consistent approximation of $H^m_0(\Omega)$.

To applying Theorem 3.1, we need the uniform W_h-elliptic property. For this purpose the following theorem will be useful.

Theorem 3.3. *Let* V_h *have the weak continuity and* V_{h0} *satisfy the weak zero-boundary condition, then the generalized inequality of Poincare-Friedrichs*

$$
\|v_h\|_{m,h} \lesssim |v_h|_{m,h}, \quad \forall v_h \in V_{h0}, \tag{3.21}
$$

and the generalized Poincare inequality

$$
\|v_h\|^2_{m,h} \lesssim |v_h|^2_{m,h} + \sum_{|\alpha|<m}\Big(\int_\Omega \partial^\alpha_h v_h\Big)^2, \quad \forall v_h \in V_h, \tag{3.22}
$$

are true.

Proof. The following inequalities are true,

$$
\|v\|_{1,\Omega} \lesssim |v|_{1,\Omega}, \quad \forall v \in H^1_0(\Omega). \tag{3.23}
$$

$$
\|v\|^2_{1,\Omega} \lesssim |v|^2_{1,\Omega} + \Big(\int_\Omega v\Big)^2, \quad \forall v \in H^1(\Omega). \tag{3.24}
$$

For $v_h \in V_{h0}$, $|\alpha|<m$, let $v_\alpha \in H^1_0(\Omega)$ be as in (3.9). Then from (3.23) and (3.9),

$$
\|\partial^\alpha_h v_h\|^2_{0,\Omega} \lesssim \|\partial^\alpha_h v_h - v_\alpha\|^2_{0,\Omega} + \|v_\alpha\|^2_{0,\Omega}
$$

$$
\lesssim |v_h|^2_{m,h} + |v_\alpha|^2_{1,\Omega} \lesssim |v_h|^2_{m,h} + |v_h|^2_{|\alpha|+1,h}.
$$

Consequently,

$$
|v_h|_{k,h} \lesssim |v_h|_{m,h} + |v_h|_{k+1,h}, \quad 0\le k<m. \tag{3.25}
$$

This leads to the (3.21).

By (3.24) and same argument we obtain (3.22). □

By Lemma 2.5 and Theorem 3.3, we have

Corollary 3.2. *The following inequalities are true:*

$$\|v_h\|_{m,h} \lesssim |v_h|_{m,h}, \quad \forall v_h \in M_{h0}^m, \tag{3.26}$$

$$\|v_h\|_{m,h}^2 \lesssim |v_h|_{m,h}^2 + \sum_{|\alpha|<m} \Big(\int_\Omega \partial_h^\alpha v_h\Big)^2, \quad \forall v_h \in M_h^m. \tag{3.27}$$

The proof of Theorem 3.1 was given in [32] and the proof of Lemma 3.1 is very similar to the one used in [37]. We write them here for self-completeness.

Theorem 3.2 gives a sufficient condition for the consistent property. A lot of nonconforming finite elements satisfy the condition, such as, the Crouzeix-Raviart element [11] and the Wilson element (see [41, 10]) for the second order problem, the Morley element [21, 23, 38] and the rectangle Morley element (see [47, 40]), the Veubake elements [36], the Adini element [2, 20, 40], the three or higher dimensional Bogner-Fox-Schmit element [40], the 12 and 15-parameter plate bending elements (see [47]), the cubic element and incomplete cubic element given in [39], for the forth order problem, our new element for $2m$-th order problem given in this paper. Among the elements mentioned above, the corresponding operators Π_h^α are identity operator for all simplical elements and the rectangle Morley element. There are some (and rare) cases for which Π_h^α are not identity operator. For example, in the situation of the two dimensional Adini element, $\Pi_h^{e_i}$ ($i = 1, 2$) is the interpolation operator of the conforming bilinear element for the second order problem.

3.3. **Error estimate.** Now we discuss the error estimate of the nonconforming finite element solution of problem (3.6) when $W = H_0^m(\Omega)$ and $W_h = V_{h0}$. Let u be the solution of problems (3.2) and u_h be the one of problem (3.6).

Lemma 3.2. *Assume that V_h has the weak continuity, V_{h0} satisfies the weak zero-boundary condition and Condition SPT is satisfied. Let $r_m = \max\{1, m-1\}$, $r'_m = 1$, $r_i = \max\{0, i-1\}$ and $r'_i = 0$, $0 \le i < m$, and $r = m + r_m$. If $u \in H^r(\Omega)$ and $f_\alpha \in H^{r_{|\alpha|}}(\Omega)$ for all $|\alpha| \le m$, then*

$$\begin{aligned}
&\sup_{v_h \in V_{h0}, \|v_h\|_{m,h}=1} \Big| a_h(u, v_h) - \sum_{|\alpha|\le m} (f_\alpha, \partial_h^\alpha v_h) \Big| \\
&\lesssim \sum_{k=1}^{r-m} h^k |u|_{m+k,\Omega} + \sum_{|\alpha|<m} \sum_{k=r'_{|\alpha|}}^{r_{|\alpha|}} h^{m-|\alpha|+k} |b_\alpha u|_{|\alpha|+k,\Omega} \\
&\quad + \sum_{|\alpha|\le m} \sum_{k=r'_{|\alpha|}}^{r_{|\alpha|}} h^{m-|\alpha|+k} |f_\alpha|_{k,\Omega} \\
&\lesssim h\Big(\|u\|_{r,\Omega} + \sum_{|\alpha|\le m} \|f_\alpha\|_{r_{|\alpha|},\Omega} \Big).
\end{aligned} \tag{3.28}$$

Proof. Let $v_h \in V_{h0}$. For $|\alpha| < m$, let $v_\alpha \in H_0^1(\Omega)$ be the piecewise polynomial as in (3.9). Define for all $|\alpha| \le m$,

$$E_\alpha = \int_\Omega \Big(\partial^\alpha u\, \partial_h^\alpha v_h - (-1)^{|\alpha|}\partial^{2\alpha} u\, v_0\Big), \tag{3.29}$$

$$\bar{E}_\alpha = \int_\Omega \Big(f_\alpha \partial_h^\alpha v_h - (-1)^{|\alpha|}\partial^\alpha f_\alpha v_0\Big). \tag{3.30}$$

It can be verified that

$$a_h(u, v_h) - \sum_{|\alpha|\le m} (f_\alpha, \partial_h^\alpha v_h) = \sum_{|\alpha|\le m} (b_\alpha E_\alpha - \bar{E}_\alpha). \tag{3.31}$$

Given $|\alpha| \le m$, it can be written as $\alpha = \sum_{i=1}^{|\alpha|} e_{j_i}$. Set

$$\alpha_{(k)} = \sum_{i=1}^{k} e_{j_i}, \quad \alpha'_{(k)} = \alpha - \sum_{i=1}^{k} e_{j_i}, \quad 0 \le k \le |\alpha|.$$

Define for $|\alpha| = m$,

$$E'_\alpha = \int_\Omega \Big(\partial^\alpha u\, \partial_h^\alpha v_h + \partial^{\alpha_{(1)}}\partial^\alpha u\, \partial_h^{\alpha'_{(1)}} v_h\Big),$$

$$E''_\alpha = \int_\Omega \partial^{\alpha_{(1)}}\partial^\alpha u\, (v_{\alpha'_{(1)}} - \partial_h^{\alpha'_{(1)}} v_h),$$

$$E'''_\alpha = \sum_{k=1}^{m-1} (-1)^k \int_\Omega \partial^{\alpha_{(k)}}\partial^\alpha u\, (v_{\alpha'_{(k)}} - \partial^{e_{j_{k+1}}} v_{\alpha'_{(k+1)}}).$$

Then

$$E_\alpha = E'_\alpha + E''_\alpha + E'''_\alpha.$$

We write

$$w_h = \Pi_h^{\alpha'_{(1)}} \partial_h^{\alpha'_{(1)}} v_h, \quad w_{he} = \partial_h^{\alpha'_{(1)}} v_h - \Pi_h^{\alpha'_{(1)}} \partial_h^{\alpha'_{(1)}} v_h.$$

Then by the assumption and Green's formula, we have

$$\begin{aligned}
E'_\alpha &= \sum_{T\in\mathcal{T}_h} \int_{\partial T} \partial^\alpha u (w_h + w_{he}) \nu_{j_1} \\
&= \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \int_F \Big(\partial^\alpha u - P_F^0 \partial^\alpha u\Big)\Big(w_h - P_F^0 w_h\Big)\nu_{j_1} \\
&\quad + \sum_{T\in\mathcal{T}_h} \int_{\partial T} \Big(\partial^\alpha u - P_{\partial T}^0 \partial^\alpha u\Big)\Big(w_{he} - P_{\partial T}^0 w_{he}\Big)\nu_{j_1}.
\end{aligned}$$

Using the Schwarz inequality, the interpolation theory and (3.9) we obtain

$$|E'_\alpha| \le \sum_{T\in\mathcal{T}_h} \sum_{F\subset\partial T} \|\partial^\alpha u - P_F^0 \partial^\alpha u\|_{0,F} \|w_h - P_F^0 w_h\|_{0,F}$$

$$+\sum_{T\in\mathcal{T}_h}\|\partial^\alpha u - P^0_{\partial T}\partial^\alpha u\|_{0,\partial T}\|w_{he} - P^0_{\partial T}w_{he}\|_{0,\partial T}$$

$$\lesssim \sum_{T\in\mathcal{T}_h} h|u|_{m+1,T}|v_h|_{m,T} \lesssim h|u|_{m+1,\Omega}|v_h|_{m,h}.$$

On the other hand, we have by the Schwarz inequality, the triangle inequality and (3.9),

$$|E''_\alpha| + |E'''_\alpha| \lesssim \Big(h|u|_{m+1,\Omega} + \sum_{k=2}^{m-1} h^k|u|_{m+k,\Omega}\Big)|v_h|_{m,h}.$$

Consequently,

$$|E_\alpha| \lesssim \Big(h|u|_{m+1,\Omega} + \sum_{k=2}^{m-1} h^k|u|_{m+k,\Omega}\Big)|v_h|_{m,h}, \quad |\alpha| = m. \tag{3.32}$$

By same argument, we obtain

$$|\bar{E}_\alpha| \lesssim \Big(h|f_\alpha|_{1,\Omega} + \sum_{k=2}^{m-1} h^k|f_\alpha|_{k,\Omega}\Big)|v_h|_{m,h}, \quad |\alpha| = m. \tag{3.33}$$

When $|\alpha| < m$, we can write $\bar{E}_\alpha$ as

$$\bar{E}_\alpha = \int_\Omega f_\alpha(\partial^\alpha_h v_h - v_\alpha) + \sum_{k=1}^{|\alpha|}(-1)^{|\alpha|-k}\int_\Omega \partial^{\alpha'_{(k)}} f_\alpha(v_{\alpha_{(k)}} - \partial^{e_{j_k}} v_{\alpha_{(k-1)}}).$$

Then by the Schwarz inequality and (3.9) we obtain

$$|\bar{E}_\alpha| \lesssim \Big(h^{m-|\alpha|}|f_\alpha|_{0,\Omega} + \sum_{k=1}^{|\alpha|-1} h^{m-|\alpha|+k}|f_\alpha|_{k,\Omega}\Big)|v_h|_{m,h},\ |\alpha| < m. \tag{3.34}$$

Similarly,

$$|E_\alpha| \lesssim \Big(h^{m-|\alpha|}|u|_{|\alpha|,\Omega} + \sum_{k=1}^{|\alpha|-1} h^{m-|\alpha|+k}|u|_{k+|\alpha|,\Omega}\Big)|v_h|_{m,h}, \quad |\alpha| < m. \tag{3.35}$$

By (3.31), (3.32), (3.33), (3.34) and (3.35), we obtain the desired estimation. □

From Theorem 3.3, Lemma 3.2 and the well-known Strang Lemma (see [31] or [10]), we obtain the following theorem.

Theorem 3.4. *Assume that V_h has the weak continuity, V_{h0} satisfies the weak zero-boundary condition and Condition SPT is satisfied. Let $r_m = \max\{1, m-1\}$, $r'_m = 1$, $r_i = \max\{0, i-1\}$ and $r'_i = 0$, $0 \le i < m$, and $r = m + r_m$. If there exists an integer $s \ge 1$ such that*

$$\inf_{v_h\in V_{h0}}\|v - v_h\|_{m,h} \lesssim h^s|v|_{m+s,\Omega}, \quad \forall v \in H^{m+s}(\Omega)\cap H^m_0(\Omega), \tag{3.36}$$

then for $f_\alpha \in H^{r_{|\alpha|}}(\Omega)$, $|\alpha| \le m$,

$$
\begin{aligned}
\|u-u_h\|_{m,h} \lesssim\ & h^s|u|_{m+s,\Omega} + \sum_{k=1}^{r-m} h^k|u|_{m+k,\Omega} \\
& + \sum_{|\alpha|<m} \sum_{k=r'_{|\alpha|}}^{r_{|\alpha|}} h^{m-|\alpha|+k}|b_\alpha u|_{|\alpha|+k,\Omega} \\
& + \sum_{|\alpha|\le m} \sum_{k=r'_{|\alpha|}}^{r_{|\alpha|}} h^{m-|\alpha|+k}|f_\alpha|_{k,\Omega} \\
\lesssim\ & h\Big(\|u\|_{r,\Omega} + \sum_{|\alpha|\le m} \|f_\alpha\|_{r_{|\alpha|},\Omega}\Big).
\end{aligned}
\tag{3.37}
$$

when $u \in H^r(\Omega) \cap H^{m+s}(\Omega)$.

Using Lemma 2.4, Theorem 2.1, Corollary 3.1, Corollary 3.2, Theorem 3.1 and Theorem 3.4, we obtain

Corollary 3.3. *Let u be the solution of problem (3.2) with $W = H_0^m(\Omega)$ and u_h be the one of problem (3.6) with $W_h = M_{h0}^m$. Then for any $f_\alpha \in L^2(\Omega)$, $|\alpha| \le m$,*

$$
\lim_{h\to 0} \|u-u_h\|_{m,h} = 0. \tag{3.38}
$$

Let $r_m = \max\{1, m-1\}$, $r'_m = 1$, $r_i = \max\{0, i-1\}$ *and* $r'_i = 0$, $0 \le i < m$, *and* $r = m + r_m$, *then for* $f_\alpha \in H^{r_{|\alpha|}}(\Omega)$, $|\alpha| \le m$,

$$
\begin{aligned}
\|u-u_h\|_{m,h} \lesssim\ & \sum_{k=1}^{r-m} h^k|u|_{m+k,\Omega} + \sum_{|\alpha|<m} \sum_{k=r'_{|\alpha|}}^{r_{|\alpha|}} h^{m-|\alpha|+k}|b_\alpha u|_{|\alpha|+k,\Omega} \\
& + \sum_{|\alpha|\le m} \sum_{k=r'_{|\alpha|}}^{r_{|\alpha|}} h^{m-|\alpha|+k}|f_\alpha|_{k,\Omega} \\
\lesssim\ & h\Big(\|u\|_{r,\Omega} + \sum_{|\alpha|\le m} \|f_\alpha\|_{r_{|\alpha|},\Omega}\Big).
\end{aligned}
\tag{3.39}
$$

when $u \in H^r(\Omega)$.

4. Construction of nodal basis functions

In this section, we will describe a procedure to construct a set of nodal basis for our new finite element spaces. Such a construction is both of theoretical and especially of practical interests.

More specifically, let $m \le n$ and T be an n-simplex. We will now study the construction of the nodal basis functions of $P_m(T)$ with respect to the degrees of freedom described in Section 2.1. For $(n+1)$ dimensional multi-index γ, define

$$\lambda^\gamma = \lambda_1^{\gamma_1}\lambda_2^{\gamma_2}\cdots\lambda_{n+1}^{\gamma_{n+1}}.$$

Let $1 \le k \le m$. For $\sigma \in \mathcal{C}_k$ and $0 \le i \le k$, define

$$\begin{aligned}\Gamma_\sigma &= \{\ \gamma \in A_{n+1}\ :\ \gamma_j = 0,\ j \ne \sigma_l,\ 1 \le l \le k\},\\ \Gamma_{\sigma,i} &= \{\ \gamma \in \Gamma_\sigma\ :\ |\gamma| = i,\ \gamma_{\sigma_j} \le 1,\ 1 \le j \le k\}.\end{aligned}$$

For $\sigma \in \mathcal{C}_k$ and $\gamma \in \Gamma_\sigma$, define

$$q_{\sigma,\gamma} = \sum_{i=0}^{k}(-1)^i\frac{(n-k+i)!}{(n-k)!}\sum_{\gamma'\in\Gamma_{\sigma,i}}\frac{\lambda^{\gamma+\gamma'}}{(\gamma+\gamma')!}. \tag{4.1}$$

Given $\sigma \in \mathcal{C}_k$, $-\nabla\lambda_{\sigma_1}, -\nabla\lambda_{\sigma_2}, \cdots, -\nabla\lambda_{\sigma_k}$ are linearly independent and are outer normal vectors of F_σ. Then for $1 \le j \le k$, $\nu_{F_\sigma,j}$ can be written as

$$\nu_{F_\sigma,j} = \sum_{l=1}^{k} c_{jl}\nabla\lambda_{\sigma_l}, \tag{4.2}$$

where c_{jl} are constants. For $\beta \in A_k$ and $|\beta| = m-k$, define

$$Q_{\sigma,\beta} = \prod_{j=1}^{k}\Big(\sum_{l=1}^{k} c_{jl}\lambda_{\sigma_l}\Big)^{\beta_j}. \tag{4.3}$$

We write $Q_{\sigma,\beta}$ as

$$Q_{\sigma,\beta} = \sum_{\substack{\gamma\in\Gamma_\sigma\\ |\gamma|=m-k}} \bar{c}_{\beta\gamma}\lambda^\gamma, \tag{4.4}$$

where $\bar{c}_{\beta\gamma}$ are constants, and define

$$\bar{p}_{\sigma,\beta} = \frac{1}{\beta!}\sum_{\substack{\gamma\in\Gamma_\sigma\\ |\gamma|=m-k}} \gamma!\,\bar{c}_{\beta\gamma}\,q_{\sigma,\gamma}. \tag{4.5}$$

For $1 \le k \le m$, $\sigma \in \mathcal{C}_k$, $\beta \in A_k$ and $|\beta| = m-k$, define

$$p_{\sigma,\beta} = \begin{cases} \bar{p}_{\sigma,\beta}, & k = 1,\\ \bar{p}_{\sigma,\beta} - \displaystyle\sum_{j=1}^{k-1}\sum_{\sigma'\in\mathcal{C}_j}\sum_{\substack{\beta'\in A_j\\ |\beta'|=m-j}} d_{T,F_{\sigma'},\beta'}(\bar{p}_{\sigma,\beta})p_{\sigma',\beta'}, & k > 1.\end{cases} \tag{4.6}$$

It is will be shown that $p_{\sigma,\beta}$ is the basis function corresponding to degree of freedom $d_{T,F_\sigma,\beta}$.

If we choose, for $1 \le k \le m$ and $\sigma \in \mathcal{C}_k$ that

$$\nu_{F_\sigma,j} = -\frac{\nabla\lambda_{\sigma_j}}{\|\nabla\lambda_{\sigma_j}\|}, \quad 1 \le j \le k, \tag{4.7}$$

then for $\beta \in A_k$ with $|\beta| = m - k$, function $\bar{p}_{\sigma,\beta}$ can be written as

$$\bar{p}_{\sigma,\beta} = \frac{1}{B_{\sigma,\beta}} \sum_{i=0}^{k} (-1)^{m-k+i} \frac{(n-k+i)!}{(n-k)!} \sum_{\gamma \in \Gamma_{\sigma,i}} \frac{\lambda^{\gamma_{\sigma,\beta}+\gamma}}{(\gamma_{\sigma,\beta}+\gamma)!}, \tag{4.8}$$

where

$$B_{\sigma,\beta} = \prod_{j=1}^{k} \|\nabla \lambda_{\sigma_j}\|^{\beta_j}, \quad (\gamma_{\sigma,\beta})_i = \begin{cases} \beta_j, & i = \sigma_j \\ 0, & \text{otherwise.} \end{cases} \tag{4.9}$$

When $m = 1$, it follows from (4.6) and (4.8) that

$$p_{\sigma,0} = 1 - n\lambda_{\sigma_1}, \quad \sigma \in \mathcal{C}_1. \tag{4.10}$$

Then we recover the basis functions of nonconforming linear element.

When $m = 2$, by (4.6) and (4.8) we have

$$\begin{cases} p_{\sigma,e_1} = \dfrac{1}{2\|\nabla \lambda_{\sigma_1}\|} \lambda_{\sigma_1}(n\lambda_{\sigma_1} - 2), & \sigma \in \mathcal{C}_1, \\ p_{\sigma,0} = 1 - (n-1)(\lambda_{\sigma_1} + \lambda_{\sigma_2}) + n(n-1)\lambda_{\sigma_1}\lambda_{\sigma_2} & \\ \qquad -(n-1)\nabla\lambda_{\sigma_1}^{\top}\nabla\lambda_{\sigma_2} \displaystyle\sum_{l=1,2} \frac{\lambda_{\sigma_l}(n\lambda_{\sigma_l} - 2)}{2\|\nabla\lambda_{\sigma_l}\|^2}, & \sigma \in \mathcal{C}_2. \end{cases} \tag{4.11}$$

Then we recover the basis functions given in [38].

We have for $m = 3$,

$$p_{\sigma,2e_1} = \frac{1}{2\|\nabla \lambda_{\sigma_1}\|^2} \lambda_{\sigma_1}^2 \Big(1 - \frac{n}{3}\lambda_{\sigma_1}\Big), \quad \sigma \in \mathcal{C}_1, \tag{4.12}$$

$$\begin{cases} \bar{p}_{\sigma,e_1} = \dfrac{-\lambda_{\sigma_1}}{\|\nabla\lambda_{\sigma_1}\|} \Big(1 - \dfrac{n-1}{2}(\lambda_{\sigma_1} + 2\lambda_{\sigma_2}) + \dfrac{n(n-1)}{2}\lambda_{\sigma_1}\lambda_{\sigma_2}\Big), \\ \bar{p}_{\sigma,e_2} = \dfrac{-\lambda_{\sigma_2}}{\|\nabla\lambda_{\sigma_2}\|} \Big(1 - \dfrac{n-1}{2}(2\lambda_{\sigma_1} + \lambda_{\sigma_2}) + \dfrac{n(n-1)}{2}\lambda_{\sigma_1}\lambda_{\sigma_2}\Big), \end{cases} \sigma \in \mathcal{C}_2, \tag{4.13}$$

and

$$\begin{aligned} \bar{p}_{\sigma,0} = {} & 1 - (n-2)\sum_{1 \le i \le 3} \lambda_{\sigma_i} + (n-1)(n-2) \sum_{1 \le i < j \le 3} \lambda_{\sigma_i}\lambda_{\sigma_j} \\ & -n(n-1)(n-2)\lambda_{\sigma_1}\lambda_{\sigma_2}\lambda_{\sigma_3}, \qquad \sigma \in \mathcal{C}_3. \end{aligned} \tag{4.14}$$

Now we turn to showing that $p_{\sigma,\beta}$ is the basis function.

Lemma 4.1. *Let $1 \le k \le m$, $\sigma \in \mathcal{C}_k$, $\gamma \in \Gamma_\sigma$ and $|\gamma| = m - k$. Then for $\sigma' \in \mathcal{C}_k$ and $|\alpha| = m - k$*

$$\frac{1}{|F_{\sigma'}|} \int_{F_{\sigma'}} \partial^\alpha q_{\sigma,\gamma} = \frac{\partial^\alpha \lambda^\gamma}{\gamma!} \prod_{j=1}^{k} \delta_{\sigma_j,\sigma'_j}, \tag{4.15}$$

and for $k < r \le m$, $|\alpha| = m - r$ and any $(n-r)$-dimensional subsimplex F of T,

$$\int_F \partial^\alpha q_{\sigma,\gamma} = 0. \tag{4.16}$$

Proof. Let $k \le r \le m$ and F be an $(n-r)$-dimensional subsimplex of T. For multi-index γ', define

$$\partial_\lambda^{\gamma'} = \frac{\partial^{|\gamma'|}}{\partial\lambda_1^{\gamma'_1}\partial\lambda_2^{\gamma'_2}\cdots\partial\lambda_{n+1}^{\gamma'_{n+1}}}.$$

Then

$$\partial_\lambda^{\gamma'} q_{\sigma,\gamma} = \frac{1}{(n-k)!}\sum_{i=0}^{k}(-1)^i(n-k+i)! \sum_{\substack{\gamma''\in\Gamma_{\sigma,i}\\ \gamma+\gamma''\ge\gamma'}} \frac{\lambda^{\gamma+\gamma''-\gamma'}}{(\gamma+\gamma''-\gamma')!},$$

where $\gamma+\gamma''\ge\gamma'$ means that $\gamma_j+\gamma''_j\ge\gamma'_j$, $1\le j\le n+1$. When $\gamma+\gamma''\ge\gamma'$ and $\lambda^{\gamma+\gamma''-\gamma'}|_F \not\equiv 0$, it can be computed that

$$\frac{1}{|F|}\int_F \lambda^{\gamma+\gamma''-\gamma'} = \frac{(n-r)!(\gamma+\gamma''-\gamma')!}{(n-r+|\gamma|+|\gamma''|-|\gamma'|)!}.$$

Thus

$$\frac{1}{|F|}\int_F \partial_\lambda^{\gamma'} q_{\sigma,\gamma} = \frac{(n-r)!}{(n-k)!}\sum_{i=0}^{k}(-1)^i \sum_{\substack{\gamma''\in\Gamma_{\sigma,i},\ \gamma+\gamma''\ge\gamma'\\ \lambda^{\gamma+\gamma''-\gamma'}|_F\not\equiv 0}} 1, \quad |\gamma'| = m-r. \tag{4.17}$$

From (4.17) we can prove that for $|\gamma'| = m-r$

$$\frac{1}{|F|}\int_F \partial_\lambda^{\gamma'} q_{\sigma,\gamma} = \begin{cases} \dfrac{1}{\gamma!}\,\partial_\lambda^\gamma \lambda^\gamma, & F = F_\sigma \text{ and } \gamma' = \gamma,\\ 0, & \text{otherwise.}\end{cases} \tag{4.18}$$

Given $|\alpha| = m-r$, it can written as

$$\partial^\alpha = \sum_{|\gamma'|=m-r} g_{\alpha\gamma'}\partial_\lambda^{\gamma'} \tag{4.19}$$

where $g_{\alpha\gamma'}$ are constants. Then the lemma follows from (4.19) and (4.18).

Now we try to show (4.18).

a) Case of $\gamma'\notin\Gamma_\sigma$. For $0\le i\le k$ there is no $\gamma''\in\Gamma_{\sigma,i}$ with $\gamma+\gamma''\ge\gamma'$. Consequently,

$$\frac{1}{|F|}\int_F \partial_\lambda^{\gamma'} q_{\sigma,\gamma} = 0. \tag{4.20}$$

b) Case of $r = k$ and $\gamma' = \gamma$. In this case, for all $0\le i\le k$ and $\gamma''\in\Gamma_{\sigma,i}$ we have that $\gamma+\gamma''\ge\gamma'$. When $F = F_\sigma$ and $i > 0$, it is true that $\lambda^{\gamma+\gamma''-\gamma'}|_F \equiv 0$, $\gamma''\in\Gamma_{\sigma,i}$. Thus

$$\frac{1}{|F_\sigma|}\int_{F_\sigma} \partial_\lambda^{\gamma'} q_{\sigma,\gamma} = 1 = \frac{1}{\gamma!}\,\partial_\lambda^\gamma\lambda^\gamma.$$

When $F \neq F_\sigma$, let k' be the number of $\lambda_{\sigma_j}|_F \not\equiv 0$, $1 \leq j \leq k$. Then

$$\frac{1}{|F|}\int_F \partial_\lambda^\gamma q_{\sigma,\gamma} = \frac{(n-r)!}{(n-k)!}\sum_{i=0}^{k'}(-1)^i C_{k'}^i = 0. \tag{4.21}$$

c) Case of $\gamma \geq \gamma'$ and $\gamma \neq \gamma'$. In this case, for all $0 \leq i \leq k$ and $\gamma'' \in \Gamma_{\sigma,i}$ we have that $\gamma + \gamma'' \geq \gamma'$. Let k' be the number of $\lambda_{\sigma_j}|_F \not\equiv 0$, $1 \leq j \leq k$. On F, either $\lambda^{\gamma-\gamma'}|_F \equiv 0$, which makes (4.20) true, or $k' > 0$, which makes (4.21) true.

d) Case of $\gamma' \in \Gamma_\sigma$, $|\gamma'| = m - r$ and $\gamma \not\geq \gamma'$. If $\gamma + \gamma'' \not\geq \gamma'$ for any $0 \leq i \leq k$ and any $\gamma'' \in \Gamma_{\sigma,i}$ then (4.20) is true. Otherwise, let i_1 be the least number that $\gamma + \gamma'' \geq \gamma'$ for some $\gamma'' \in \Gamma_{\sigma,i_1}$. Obviously, $i_1 > 0$. It is can be shown that $i_1 < k$. In contradiction to this, we assume $i_1 = k$. Because $\gamma, \gamma' \in \Gamma_\sigma$, we have that

$$\gamma'_j = \gamma_j + 1, \quad j = \sigma_1, \sigma_2, \cdots, \sigma_k.$$

Then we obtain a contradict result that $|\gamma'| = m > |\gamma'|$. There is only one γ^{i_1} in Γ_{σ,i_1} with $\gamma + \gamma^{i_1} \geq \gamma'$. For $i_1 \leq i \leq k$, if $\gamma'' \in \Gamma_{\sigma,i}$ and $\gamma + \gamma'' \geq \gamma'$ then $\gamma'' \geq \gamma^{i_1}$. Let $\tilde{\gamma} \in A_{n+1}$ be given by

$$\tilde{\gamma}_j = \max\{0, \gamma_j - \gamma'_j\}, \quad 1 \leq j \leq n+1.$$

Let k' be the number of $\lambda_{\sigma_j}|_F \not\equiv 0$ and $\gamma_{\sigma_j}^{i_1} = 0$, $1 \leq j \leq k$. On F, either $\lambda^{\tilde{\gamma}}|_F \equiv 0$, which makes (4.20) true, or $k' > 0$, which makes (4.21) true.

Summing the above discussion, we obtain (4.18). $\square$

Lemma 4.2. *Let $1 \leq k \leq m$, $\sigma \in \mathcal{C}_k$, $\beta \in A_k$ and $|\beta| = m - k$. Then for $\sigma' \in \mathcal{C}_k$, $\beta' \in A_k$ and $|\beta'| = m - k$*

$$d_{T,F_{\sigma'},\beta'}(\bar{p}_{\sigma,\beta}) = \begin{cases} 1, & \beta' = \beta \text{ and } \sigma' = \sigma, \\ 0, & \text{otherwsie}, \end{cases} \tag{4.22}$$

and for $k < r \leq m$, $|\alpha| = m - r$ and any $(n-r)$-dimensional subsimplex F of T,

$$\int_F \partial^\alpha \bar{p}_{\sigma,\beta} = 0. \tag{4.23}$$

Proof. Form Lemma 4.1 and (4.5), we only need to show that

$$d_{T,F_\sigma,\beta}(\bar{p}_{\sigma,\beta}) = 1. \tag{4.24}$$

There exist constants $\tilde{c}_{\beta\alpha}$, $|\alpha| = m - k$, such that

$$\frac{\partial^{m-k}}{\partial\nu_{F_\sigma,1}^{\beta_1}\cdots\nu_{F_\sigma,k}^{\beta_k}} = \sum_{|\alpha|=m-k}\tilde{c}_{\beta\alpha}\partial^\alpha.$$

By (4.5), (4.4), (4.15) and (2.3), we obtain that

$$d_{T,F_\sigma,\beta}(\bar{p}_{\sigma,\beta}) = \frac{1}{\beta!}\,\frac{\partial^{m-k}Q_{\sigma,\beta}}{\partial\nu_{F_\sigma,1}^{\beta_1}\cdots\partial\nu_{F_\sigma,k}^{\beta_k}}.$$

Then (4.24) follows from (4.2) and (4.3). $\square$

From Lemma 4.1, Lemma 4.2 and (4.6) we obtain the following result.

Theorem 4.1. *Let* $1 \le k \le m$, $\sigma \in \mathcal{C}_k$, $\beta \in A_k$ *and* $|\beta| = m - k$. *Then* $p_{\sigma,\beta}$ *is the basis function corresponding to degree of freedom* $d_{T,F_\sigma,\beta}$.

By the basis functions given above, we have the following corollary.

Corollary 4.1. *Let* $n \ge 2$. M_h^m *is not a subspace of* $C^0(\bar{\Omega})$ *if there exist two simplexes in* $\mathcal{T}_h$ *with common* $(n-1)$*-dimensional subsimplex.*

Proof. Let $T, T' \in \mathcal{T}_h$ with common $(n-1)$-dimensional subsimplex F, and denote by a_{n+1} the vertex of T opposite to F. Let $\sigma = (1) \in \mathcal{C}_1$, $\beta = (m-1) \in A_1$. Let $v_h \in M_h^m$ be the function such that for $(n-1)$-dimensional subsimplex F_σ of T, $d_{T,F_\sigma,\beta}(v_h) = 1$, and the other degrees of freedom of v_h on all elements in $\mathcal{T}_h$ are zero. Then $v_h^{T'}|_F \equiv 0$ and by (4.8) and (4.6),

$$v_h^T|_F = \frac{(-1)^{m-1}}{(m-1)!\,\|\nabla\lambda_1\|^{m-1}}\lambda_1^{m-1}\Big(1 - \frac{n}{m}\lambda_1\Big) \not\equiv 0.$$

This completes the proof. $\square$

5. Concluding remarks

The construction of the consistent approximation of Sobolev spaces with minimal degree piecewise polynomials is motivated by the theoretical consideration and the interest in application to practical problems. In this paper, a new consistent approximation to m-th order Sobolev of n-dimensions with $n \ge m \ge 1$ is proposed in a canonical fashion, and the convergence and the error estimate for application of $2m$-th order elliptic problems in R^n are shown by a general convergent theory. The new class of nonconforming elements has several attractive properties, such as

- consistent approximation with minimal degree piecewise polynomials;
- the degrees of freedom fit perfectly well;
- recovers the well-known nonconforming linear elements for $m = 1$ and Morley element for $m = 2$ in a canonical fashion;
- the inclusion property.

The work in this paper is hoped to shed some new insight to the finite element theory. In addition to its theoretical interest, the new type of finite element is potentially useful in practice.

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Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University

E-mail address: mwang@math.pku.edu.cn

Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, and Department of Mathematics, Pennsylvania State University

E-mail address: xu@math.psu.edu

URL: http://www.math.psu.edu/xu/