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A NEW Z-TYPE NONCONFORMING ELEMENT IN ANY DIMENSIONS *

Wang Ming[1], Shi Zhong-ci[2] and Jinchao Xu[3]

Abstract. In this paper, some Z-type finite elements for the discretization of fourth order elliptic partial differential operator are viewed. A new Z-type nonconforming finite element, in n spatial dimensions with $n \geq 2$, is proposed. The new finite element is proved to be convergent for the biharmonic equation.

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Introduction

In this paper, we will propose a new Z-type nonconforming simplex finite element for n-dimensional fourth order partial differential equations with $n \geq 2$. This element uses the values of function and first order derivatives at vertices as degrees of freedom, that is, it uses the same degrees of freedom with the Zienkiewicz element [2 or 6]. But its shape function space is different from the one of the Zienkiewicz element.

As a nonconforming element for fourth order partial differential equations, the Zienkiewicz element is attractive. The first thing is its convergent property. In two dimensional case, the Zienkiewicz element is only convergent under the parallel line condition, and is divergent in general grids. The numerical experiments were given in [7] and the mathematical proof can be found in [10]. In three dimensional case, the Zienkiewicz element was proved to be divergent on a special grid [19]. Another attractive thing is the degrees of freedom of the Zienkiewicz element. It is convenient for numerical computations to take values of function and derivatives at vertices as degrees of freedom. Although on each single element the number of degrees of freedom of the Zienkiewicz element is not the least, the global number is.

Keywords and phrases: Nonconforming finite element, forth order elliptic equation, biharmonic

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[1] LMAM, School of Mathematical Sciences, Peking University, mwang@math.pku.edu.cn

[2] Institute of Computational Mathematics, CAS, shi@lsec.cc.ac.cn

[3] The School of Mathematical Sciences, Peking University, and Department of Mathematics, Pennsylvania State University, xu@math.psu.edu, http://www.math.psu.edu/xu/

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There are some modified triangular elements, with the same degrees of freedom of the Zienkiewicz element, proposed by different ways, such as, the quasi-conforming method [18,5], the generalized conforming method [9], the free formula method [1,4] and the energy orthogonal method [3]. We call these elements are of Z-type. The convergence analysis of these elements were given in [20,13,11,14] respectively. It is a little surprise that there were no convergent Z-type element proposed directly from the nonconforming element method in two dimensions.

In three dimensional case, two convergent Z-type elements were proposed in [19]. One is a nonconforming element and is not C^0 element, another is a quasi-conforming element.

The new Z-type element proposed in this paper is a nonconforming C^0 element, and it is constructed in a canonical fashion for two and higher dimensions.

The rest of the paper is organized as follows. Section 1 recalls the nonconforming element method. Section 2 reviews some Z-type finite elements. Section 3 gives a detailed descriptions of a new Z-type nonconforming element. Section 4 shows the convergence of the new element.

1. Preliminaries

Let Ω be a bounded polyhedroid domain in R^n $(n \geq 2)$ with boundary $\partial\Omega$. For a nonnegative integer s, Let $H^s(\Omega)$, $H_0^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev spaces, norm and semi-norm respectively. Let $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$.

For $f \in L^2(\Omega)$, we consider the following fourth order boundary value problem:

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^\top$ is the unit outer normal to $\partial\Omega$ and Δ is the standard Laplacian operator.

Set

$$\nabla = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\Big)^\top.$$

Define

$$a(v,w) = \int_\Omega \sum_{i,j=1}^n \frac{\partial^2 v}{\partial x_i \partial x_j}\frac{\partial^2 w}{\partial x_i \partial x_j}, \quad \forall v, w \in H^2(\Omega). \tag{2}$$

The weak form of problem (1) is: find $u \in H_0^2(\Omega)$ such that

$$a(u,v) = (f,v), \qquad \forall v \in H_0^2(\Omega). \tag{3}$$

For a subset $B \subset R^n$ and a nonnegative integer r, let $P_r(B)$ be the space of all polynomials with degree not greater than r.

Let (T, P_T, Φ_T) be a finite element where T is the geometric shape, P_T the shape function space and Φ_T the vector of degrees of freedom, and let Φ_T be P_T-unisolvent (see [6]). Let $\mathcal{T}_h$ be a triangulation of Ω with mesh size h. For each element $T \in \mathcal{T}_h$, let h_T be the diameter of the smallest ball containing T and ρ_T be the diameter of the largest ball contained in T.

Let $\{\mathcal{T}_h\}$ be a family of triangulations with $h \to 0$. Throughout the paper, we assume that $\{\mathcal{T}_h\}$ is quasi-uniform, namely it satisfied that $h_T \leq h \leq \eta\rho_T$, $\forall T \in \mathcal{T}_h$ for a positive constant η independent of h.

For each $\mathcal{T}_h$, let V_h and V_{h0} be the corresponding finite element spaces associated with (T, P_T, Φ_T) for the discretization of $H^2(\Omega)$ and $H_0^2(\Omega)$ respectively. In the case of nonconforming element, $V_h \not\subset H^2(\Omega)$ and $V_{h0} \not\subset H_0^2(\Omega)$.

For $v, w \in L^2(\Omega)$ that $v|_T, w|_T \in H^2(T)$, $\forall T \in \mathcal{T}_h$, we define

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T \sum_{i,j=1}^{n} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}. \tag{4}$$

The finite element method for problem (3) corresponding to the element (T, P_T, Φ_T) is: find $u_h \in V_{h0}$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}. \tag{5}$$

For any $v \in L^2(\Omega)$ that $v|_T \in H^m(\Omega)$, $\forall T \in \mathcal{T}_h$, we define the following mesh-dependent norm $\|\cdot\|_{m,h}$ and semi-norm $|\cdot|_{m,h}$:

$$\|v\|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} \|v\|_{m,T}^2\Big)^{1/2}, \quad |v|_{m,h} = \Big(\sum_{T\in\mathcal{T}_h} |v|_{m,T}^2\Big)^{1/2}.$$

For nonconforming elements, the basic mathematical theory has been established (see [6,8,15-17]). We will use them to discuss the convergence of the new element.

2. Some Z-Type Finite Elements

In this section, we review some Z-type finite elements. We begin with the general n-dimensional Zienkiewicz element.

Given an n-simplex T with vertices $a_i = (x_{1i}, x_{2i}, \cdots, x_{ni})^\top$, $1 \le i \le n+1$, denote by F_i $(1 \le i \le n+1)$ the $(n-1)$-dimensional subsimplex of T without a_i as its vertex, and by $\lambda_1, \lambda_2, \cdots, \lambda_{n+1}$ the barycentric coordinates of T. Denote by $|T|$ and $|F_i|$ the measures of T and F_i respectively.

For $1 \le i < j \le n+1$, let $a_{ij} = (a_i + a_j)/2$. For $1 \le i < j < k \le n+1$, let $a_{ijk} = (a_i + a_j + a_k)/3$, and define

$$\psi_{ijk}(v) = 6v(a_{ijk}) - 2\sum_{l=i,j,k} v(a_l) + \sum_{l=i,j,k} (a_l - a_{ijk})^\top \nabla v(a_l), \ \forall v \in C^1(T).$$

By ψ_{ijk} we may define a polynomial space

$$P_3'(T) = \{\, p \in P_3(T) \,|\, \psi_{ijk}(p) = 0, \ 1 \le i < j < k \le n+1 \,\}.$$

It can be verified that $P_3'(T) = P_2(T) + \operatorname{span}\{\lambda_i^2\lambda_j - \lambda_i\lambda_j^2 \mid 1 \le i < j \le n+1\}$.

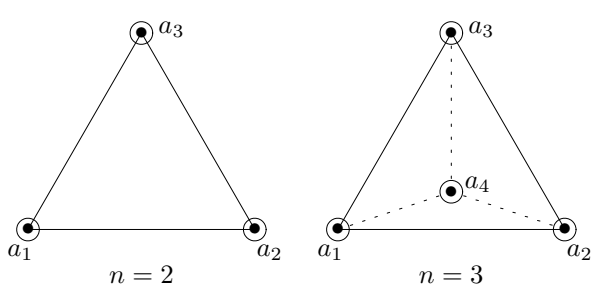

Fig. 1

The Zienkiewicz element

For the n-dimensional Zienkiewicz element, (T, P_T, Φ_T) is defined by (see Fig. 1)

1) T is an n-simplex.
2) $P_T = P_3'(T)$.
3) The components of Φ_T are:

$$v(a_j),\ 1 \le j \le n+1, \quad (a_j - a_i)^\top \nabla v(a_i),\ 1 \le i \ne j \le n+1, \quad \forall v \in C^1(T).$$

For the Zienkiewicz element, we can define the corresponding finite element spaces V_h^z and V_{h0}^z as follows: $V_h^z = \{v \in L^2(\Omega) \mid v|_T \in P_3'(T), \forall T \in T_h,\ v$ and ∇v are continuous at all vertices of elements in $\mathcal{T}_h\ \}$. $V_{h0}^z = \{v \in V_h^z \mid v$ and ∇v vanish at all vertices belonging to $\partial\Omega\}$.

From now to the end of this section, let $n = 2$. We will review some convergent Z-type finite elements in two dimensions.

TQC9 element

Given a triangle T, let Π_T^1 denote the linear interpolation operator with values of function at three vertices. For $p \in P_3'(T)$, define $\partial_T^{ij} p \in P_1(T)$, $i, j \in \{1, 2\}$, such that $\partial_T^{12} p = \partial_T^{21} p$ and for any $q \in P_1(T)$

$$\begin{cases} \int\limits_T q\partial_T^{11} p = \int\limits_{\partial T} q\Pi_T^1 \dfrac{\partial p}{\partial x_1}\nu_1 - \int\limits_T \dfrac{\partial q}{\partial x_1}\dfrac{\partial p}{\partial x_1}, \\ \int\limits_T q\partial_T^{22} p = \int\limits_{\partial T} q\Pi_T^1 \dfrac{\partial p}{\partial x_2}\nu_2 - \int\limits_T \dfrac{\partial q}{\partial x_2}\dfrac{\partial p}{\partial x_2}, \\ 2\int\limits_T q\partial_T^{12} p = \int\limits_{\partial T} q\big(\Pi_T^1 \dfrac{\partial p}{\partial x_2}\nu_1 + \Pi_T^1 \dfrac{\partial p}{\partial x_1}\nu_2\big) - \int\limits_T \big(\dfrac{\partial q}{\partial x_2}\dfrac{\partial p}{\partial x_1} + \dfrac{\partial q}{\partial x_1}\dfrac{\partial p}{\partial x_2}\big). \end{cases} \tag{6}$$

For $v_h \in V_h^z$ and $i, j \in \{1, 2\}$, define $\partial_h^{ij} v_h$ by, $\partial_h^{ij} v_h|_T = \partial_T^{ij}(v_h|_T)$, $\forall T \in T_h$. Define

$$\bar{a}_h(v, w) = \sum_{i,j=1}^{2} \int_\Omega \partial_h^{ij} v\, \partial_h^{ij} w, \quad \forall v, w \in V_h^z. \tag{7}$$

The finite element method for problem (1) with TQC9 element is: find $u_h \in V_{h0}^z$ such that

$$\bar{a}_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}^z. \tag{8}$$

A three dimensional analogue of TQC9 element can be found in [19].

The generalized conforming element

Define

$$\mathbf{P}_T = (\lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, \lambda_1^2\lambda_2 - \lambda_1\lambda_2^2, \lambda_2^2\lambda_3 - \lambda_2\lambda_3^2, \lambda_3^2\lambda_1 - \lambda_3\lambda_1^2)^\top.$$

For any $p \in P_3'(T)$, it can be written by

$$p = \Psi^\top \mathbf{P}_T \tag{9}$$

with $\Psi = (\psi_1, \psi_2, \cdots, \psi_9)^\top$.

For $p \in P_3'(T)$, define $\mathbf{d}(p) = (d_1(p), d_2(p), \cdots, d_9(p))^\top$ with

$$\begin{cases} d_1(p) = \int_{F_3} \frac{\partial p}{\partial \tau}, \qquad d_2(p) = \int_{F_1} \frac{\partial p}{\partial \tau}, \\ d_3(p) = 60\Big(\frac{1}{|F_3|}\int_{F_3}(\lambda_2 - \frac{1}{2})p + \frac{1}{|F_1|}\int_{F_1}(\lambda_3 - \frac{1}{2})p + \frac{1}{|F_2|}\int_{F_2}(\lambda_1 - \frac{1}{2})p\Big), \\ d_4(p) = \frac{1}{|F_3|}\int_{F_3} p, \quad d_5(p) = \frac{1}{|F_1|}\int_{F_1} p, \quad d_6(p) = \frac{1}{|F_2|}\int_{F_2} p, \\ d_7(p) = \int_{F_3} \frac{\partial p}{\partial \nu}, \quad d_8(p) = \int_{F_1} \frac{\partial p}{\partial \nu}, \quad d_9(p) = \int_{F_2} \frac{\partial p}{\partial \nu}. \end{cases} \tag{10}$$

We have that

$$\mathbf{d}(p) = C_T \Psi, \tag{11}$$

with C_T a nonsingular matrix.

On the other hand, we want to describe $\mathbf{d}(p)$ by the nodal parameter vector of the Zienkiewicz element, that is, by

$$\Phi_T(p) = \Big(p(a_1), \nabla p(a_1)^\top, p(a_2), \nabla p(a_2)^\top, p(a_3), \nabla p(a_3)^\top\Big)^\top.$$

Integrating $d_1(p)$ and $d_2(p)$ directly, $d_3(p)$ to $d_6(p)$ by the Hermitian integral formula and $d_7(p)$ to $d_9(p)$ by the numerical trapezoidal integral formula, we obtain that

$$\mathbf{d}(p) = G_T \Phi_T(p) + \epsilon(p)\mathbf{E}, \tag{12}$$

with

$$E = (0,0,0,0,0,0,1,1,1)^\top, \quad \epsilon(p) = O(h_T^3 |p|_{3,\infty,T})$$

and G_T a nonsingular matrix.

Combining (9), (11) and (12), we have

$$p = (\Phi_T(p))^\top (C_T^{-1} G_T)^\top \mathbf{P}_T + \epsilon(p)(C_T^{-1}\mathbf{E})^\top \mathbf{P}_T, \quad \forall p \in P_3'(T). \tag{13}$$

Omitting the error terms caused by numerical integral, we obtain an approximate polynomial $\Pi_T^g p$ for p, and

$$\Pi_T^g p = (\Phi_T(p))^\top (C_T^{-1} G_T)^\top \mathbf{P}_T, \quad \forall p \in P_3'(T). \tag{14}$$

Define an approximate function $\Pi_h^g v_h$ for $v_h \in V_h^z$ by $\Pi_h^g v_h|_T = \Pi_T^g(v_h|_T)$, $\forall T \in \mathcal{T}_h$. The finite element method for problem (1), by the 9-parameter generalized conforming element, is to find $u_h \in V_{h0}^z$ such that

$$a_h(\Pi_h^g u_h, \Pi_h^g v_h) = (f, v_h), \quad \forall v_h \in V_{h0}^z. \tag{15}$$

Define $V_h^g = \Pi_h^g V_{h0}^z$. One can also solve the problem: to find $\tilde{u}_h \in V_h^g$ such that

$$a_h(\tilde{u}_h, v_h) = (f, v_h), \quad \forall v_h \in V_h^g. \tag{16}$$

TRUNC element

Given a triangle T, let Π_T^z be the interpolation operator corresponding to the Zienkiewicz element. For $v \in C^1(T)$, $\Pi_T^z v$ can be written as

$$\Pi_T^z v = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1 + a_7(\lambda_1^2\lambda_2 - \lambda_1\lambda_2^2) + a_8(\lambda_2^2\lambda_3 - \lambda_2\lambda_3^2) + a_9(\lambda_3^2\lambda_1 - \lambda_3\lambda_1^2).$$

Now we split $\Pi_T^z v$ into two parts: $\Pi_T^c v$ and $\Pi_T^z v - \Pi_T^c v$ by

$$\Pi_T^c v = a_1\lambda_1 + a_2\lambda_2 + a_3\lambda_3 + a_4\lambda_1\lambda_2 + a_5\lambda_2\lambda_3 + a_6\lambda_3\lambda_1 \tag{17}$$

that is, $\Pi_T^c v$ represents a full quadratic polynomial on T.

For $v_h \in V_h^z$, it can be split into two parts,

$$v_h = \bar{v}_h + v_h', \tag{18}$$

where for any $T \in \mathcal{T}_h$,

$$\bar{v}_h|_T = \Pi_T^c(v_h|_T), \quad v_h'|_T = v_h|_T - \Pi_T^c(v_h|_T). \tag{19}$$

Define a new discrete bilinear form,

$$a_h^t(v_h, w_h) = a_h(\bar{v}_h, \bar{w}_h) + a_h(v_h', w_h'), \quad \forall v_h, w_h \in V_h^z. \tag{20}$$

The TRUNC element for problem (1) is to find $u_h \in V_{h0}^z$ such that

$$a_h^t(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}^z. \tag{21}$$

The Bergan element

Now we define an element (T, P_T, Φ_T) by

1) T is a triangle.
2) $P_T = \text{span}\ \{\ \lambda_1, \lambda_2, \lambda_3, \lambda_1\lambda_2, \lambda_2\lambda_3, \lambda_3\lambda_1, (\lambda_1 - \lambda_2)^3, (\lambda_2 - \lambda_3)^3, (\lambda_3 - \lambda_1)^3\}$.
3) The components of Φ_T are:

$$v(a_j),\ 1 \le j \le n+1, \quad (a_j - a_i)^\top \nabla v(a_i),\ 1 \le i \ne j \le n+1, \quad \forall v \in C^1(T).$$

We can define the corresponding finite element spaces V_h^b and V_{h0}^b as follows: $V_h^b = \{v \in L^2(\Omega) \mid v|_T \in P_T, \forall T \in T_h,\ v$ and ∇v are continuous at all vertices of elements in $\mathcal{T}_h\ \}$. $V_{h0}^b = \{v \in V_h^b \mid v$ and ∇v vanish at all vertices belonging to $\partial\Omega\}$.

For any $v_h \in V_h^b$, it can be written as

$$v_h = \bar{v}_h + v_h' \tag{22}$$

where for any $T \in \mathcal{T}_h$,

$$\bar{v}_h|_T = \ b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3 + b_4\lambda_1\lambda_2 + b_5\lambda_2\lambda_3 + b_6\lambda_3\lambda_1 \tag{23}$$

representing a full quadratic polynomial on T, and

$$v_h'|_T = b_7(\lambda_1 - \lambda_2)^3 + b_8(\lambda_2 - \lambda_3)^3 + b_9(\lambda_3 - \lambda_1)^3 \tag{24}$$

being a cubic polynomial. Define $\tilde{v}_h$ by

$$\tilde{v}_h|_T = \Pi_T^c(v_h|_T), \quad \forall T \in T_h, \tag{25}$$

where Π_T^c given by (17). Define a discrete bilinear form,

$$a_h^b(v_h, w_h) = a_(\tilde{v}_h, \tilde{w}_h) + a_h(v_h', w_h'), \quad \forall v_h, w_h \in V_h^b. \tag{26}$$

The Bergan element for problem (1) is to find $u_h \in V_{h0}^b$ such that

$$a_h^b(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_{h0}^b. \tag{27}$$

3. A New Z-Type Nonconforming Element

In this section, we give a detailed description of our new Z-type nonconforming element in n-dimensions ($n \geq 2$).

Given an n-simplex T, let q_0 be the bubble function defined by

$$q_0 = \lambda_1 \lambda_2 \cdots \lambda_{n+1}.$$

For $1 \leq i < j \leq n+1$, we define

$$q_{ij} = \lambda_i^2 \lambda_j - \lambda_i \lambda_j^2 + \frac{(2n-1)!}{n!}\Big(\frac{(n-1)n}{n+1}(\lambda_i - \lambda_j) + \sum_{\substack{1 \leq k \leq n+1 \\ k \neq i, k \neq j}} \frac{(\nabla\lambda_i - \nabla\lambda_j)^\top \nabla\lambda_k}{\|\nabla\lambda_k\|^2}(n\lambda_k - 1)\Big) q_0. \tag{28}$$

A new Z-type nonconforming element, NZT element for short, is defined by (T, P_T, Φ_T) with

1) T is an n-simplex.
2) $P_T = P_2(T) + \text{span}\{\ q_{ij} \mid 1 \leq i < j \leq n+1\}$.
3) The components of Φ_T are:

$$v(a_j),\ 1 \leq j \leq n+1, \quad (a_j - a_i)^\top \nabla v(a_i),\ 1 \leq i \neq j \leq n+1, \quad \forall v \in C^1(T).$$

Let $\nu^{(i)}$ denote the unit out normal of $(n-1)$-subsimplex F_i of T ($1 \leq i \leq n+1$). By certain computation, we can obtain that

$$\frac{1}{|F_i|}\int_{F_i} \frac{\partial p}{\partial \nu^{(i)}} = \frac{1}{n}\sum_{\substack{1 \leq k \leq n+1 \\ k \neq i}} \frac{\partial p}{\partial \nu^{(i)}}(a_k), \quad 1 \leq i \leq n+1,\ \forall p \in P_T. \tag{29}$$

Lemma 3.1. *For NZT element, Φ_T is P_T-unisolvent.*

Proof. Let $p \in P_T$ and

$$p(a_j) = 0,\ 1 \leq j \leq n+1; \quad (a_j - a_i)^\top \nabla p(a_i) = 0,\ 1 \leq i \neq j \leq n+1.$$

We only need to show that $p \equiv 0$. Let $q_1, \cdots, q_l$ be a basis of $P_2(T)$. Then p can be wriiten as

$$p = \sum_{i=1}^{l} c_i q_i + \sum_{1 \leq i < j \leq n+1} c_{ij} q_{ij}$$

where c_i and c_{ij} are constants. Set $\tilde{q}_{ij} = \lambda_i^2\lambda_j - \lambda_i\lambda_j^2$ and

$$\tilde{p} = \sum_{i=1}^{l} c_i q_i + \sum_{1\le i<j\le n+1} c_{ij}\tilde{q}_{ij}.$$

Then

$$p = \tilde{p} + \sum_{1\le i<j\le n+1} c_{ij}(q_{ij} - \tilde{q}_{ij}).$$

It can be verified that

$$(q_{ij} - \tilde{q}_{ij})(a_k) = 0,\ 1 \le k \le n+1; \quad (a_m - a_k)^\top \nabla(q_{ij} - \tilde{q}_{ij})(a_k) = 0,\ 1 \le k \ne m \le n+1.$$

Thus, $\tilde{p}$ satisfies

$$\tilde{p}(a_j) = 0,\ 1 \le j \le n+1; \quad (a_j - a_i)^\top \nabla\tilde{p}(a_i) = 0,\ 1 \le i \ne j \le n+1.$$

On the other hand, $\tilde{p} \in P_3'(T)$. Thus $\tilde{p} \equiv 0$, that is,

$$c_i = 0,\ 1 \le i \le l; \quad c_{ij} = 0,\ 1 \le i < j \le n+1.$$

It follows that $p \equiv 0$. □

For $1 \le i \ne j \le n+1$, we define

$$\begin{cases} p_{ij} = \dfrac{1}{2}\lambda_i\lambda_j(1 + \lambda_i - \lambda_j), \\ \qquad + \dfrac{(2n-1)!}{2n!}\Big(\dfrac{(n-1)n}{n+1}(\lambda_i - \lambda_j) + \displaystyle\sum_{\substack{1\le k\le n+1\\ k\ne i, k\ne j}} \dfrac{(\nabla\lambda_i - \nabla\lambda_j)^\top\nabla\lambda_k}{\|\nabla\lambda_k\|^2}(n\lambda_k - 1)\Big)q_0 \\ p_i = \lambda_i^2 + 2\displaystyle\sum_{\substack{1\le j\le n+1\\ j\ne i}} p_{ij}. \end{cases} \tag{30}$$

Let δ_{ij} be the Kronecker delta. It can be verified that for $1 \le i \ne j \le n+1$ and $1 \le k \ne l \le n+1$,

$$\begin{cases} p_i(a_k) = \delta_{ik}, \quad (a_l - a_k)^\top\nabla p_i(a_k) = 0, \\ p_{ij}(a_k) = 0, \quad (a_l - a_k)^\top\nabla p_{ij}(a_k) = \delta_{ik}\delta_{jl}. \end{cases} \tag{31}$$

That is, q_i and p_{ij} are the nodal basis functions respect to the degrees of freedom. The corresponding interpolation operator Π_T can be written by,

$$\Pi_T v = \sum_{1\le i\le n+1} p_i v(a_i) + \sum_{1\le i\ne j\le n+1} p_{ij}(a_j - a_i)^\top\nabla v(a_i), \quad \forall v \in C^1(T). \tag{32}$$

For NZT element, we can define the corresponding finite element spaces V_h and V_{h0} as follows: $V_h = \{v \in L^2(\Omega) \mid v|_T \in P_T, \forall T \in T_h,\ v$ and ∇v are continuous at all vertices of elements in $\mathcal{T}_h\}$. $V_{h0} = \{v \in V_h \mid v$ and ∇v vanish at all vertices belonging to $\partial\Omega\}$.

We claim that $V_h \subset H^1(\Omega)$ and $V_{h0} \subset H_0^1(\Omega)$. Let $v_h \in V_h$, and let F be a common $(n-1)$-dimensional subsimplex of $T, T' \in \mathcal{T}_h$. By definition, the restrictions of $v_h|_T$ and $v_h|_{T'}$ on F are all in $P_3'(F)$, and they and their first order derivatives are equal at all vertices of F respectively. Thus $v_h|_T = v_h|_{T'}$ on F, that is, $v_h \in C^0(\bar{\Omega})$, and this leads to that $v_h \in H^1(\Omega)$. Using similar argument, we can show $v_h \in H_0^1(\Omega)$ when $v_h \in V_{h0}$.

Given any $(n-1)$-dimensional subsimplex F and $v_h \in V_h$, let us define the jump of ∇v_h across F as follows:

$$[\nabla v_h] = \nabla v_h|_T - \nabla v_h|_{T'}$$

if $F = T \cap T'$ for some $T, T' \in \mathcal{T}_h$ and

$$[\nabla v_h] = \nabla v_h|_T$$

if $F = T \cap \partial\Omega$.

The following lemma is a direct consequence of equality (29) and the definitions of V_h and V_{h0}.

Lemma 3.2. *Let V_h and V_{h0} be the finite element spaces corresponding to NZT element. If F is a common $(n-1)$-dimensional subsimplex of $T, T' \in \mathcal{T}_h$, then*

$$\int_F [\nabla v_h] = 0, \quad \forall v_h \in V_h. \tag{33}$$

If an $(n-1)$-dimensional subsimplex F of $T \in \mathcal{T}_h$ is on $\partial\Omega$ then

$$\int_F [\nabla v_h] = 0, \quad \forall v_h \in V_{h0}. \tag{34}$$

4. Convergence Analysis

In this section, we discuss the convergence property of NZT element. Let V_h and V_{h0} be the finite element spaces corresponding to NZT element. First, we consider the error estimates for finite element spaces.

Theorem 4.1. *Let V_{h0} be the finite element space corresponding to NZT element. Then there exists a constant C independent of h such that*

$$\inf_{v_h \in V_{h0}} \sum_{m=0}^{3} h^m |v - v_h|_{m,h} \le Ch^3 |v|_{3,\Omega}, \ \forall v \in H^3(\Omega) \cap H_0^2(\Omega), \tag{35}$$

$$\inf_{v_h \in V_h} \sum_{m=0}^{3} h^m |v - v_h|_{m,h} \le Ch^3 |v|_{3,\Omega}, \ \forall v \in H^3(\Omega). \tag{36}$$

Proof. For $v \in H^3(\Omega) \cap H_0^2(\Omega)$, let $w_h \in L^2(\Omega)$ such that $\forall T \in \mathcal{T}_h$, $w_h|_T \in P_2(T)$ and

$$\int_T q w_h dx = \int_T q v dx, \quad \forall q \in P_2(T).$$

By the interpolation theory, we have

$$|v - w_h|_{m,h} \le Ch^{3-m} |v|_{3,\Omega}, \quad 0 \le m \le 3. \tag{37}$$

Given a set $B \subset R^n$, let $\mathcal{T}_h(B) = \{ T \in \mathcal{T}_h \mid B \cap T \neq \emptyset \}$ and $N_h(B)$ be the number of the elements in $\mathcal{T}_h(B)$.

Now we define $v_h \in V_{h0}$ as follows: for any $T \in \mathcal{T}_h$, if vertex a_i of T is in Ω then

$$v_h(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} (w_h|_{T'})(a_i),\ \nabla v_h(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \nabla (w_h|_{T'})(a_i).$$

Obviously, v_h is well-defined. We will show that

$$|v - v_h|_{m,h} \le Ch^{3-m}|v|_{3,\Omega}, \quad 0 \le m \le 3. \tag{38}$$

Let $T \in \mathcal{T}_h$, by a standard scaling argument, we have

$$|p|^2_{m,T} \le Ch^{n-2m} \sum_{i=1}^{n+1} \Big(|p(a_i)|^2 + h^2 \|\nabla p(a_i)\|^2 \Big), \quad 0 \le m \le 3,\ \forall p \in P_T. \tag{39}$$

Set $\phi_h = w_h - v_h$. Obviously, $\phi_h|_T \in P_T$.

If vertex a_i of T is in Ω, the definition of v_h leads to that

$$(\phi_h|_T)(a_i) = \frac{1}{N_h(a_i)} \sum_{T' \in \mathcal{T}_h(a_i)} \Big((w_h|_T)(a_i) - (w_h|_{T'})(a_i) \Big).$$

For $T' \in \mathcal{T}_h(a_i)$ there exist $T_1, \cdots, T_J \in \mathcal{T}_h(a_i)$ such that $T_1 = T$, $T_J = T'$ and $\tilde{F}_j = T_j \cap T_{j+1}$ is a common $(n-1)$-dimensional subsimplex of T_j and T_{j+1} and $a_i \in \tilde{F}_j$, $1 \le j < J$. By the inverse inequality, we have

$$\begin{aligned}
\Big| (w_h|_T)(a_i) - (w_h|_{T'})(a_i) \Big|^2 &= \Big| \sum_{j=1}^{J-1} \big((w_h|_{T_j})(a_i) - (w_h|_{T_{j+1}})(a_i) \big) \Big|^2 \\
&\le C \sum_{j=1}^{J-1} \Big| (w_h|_{T_j})(a_i) - (w_h|_{T_{j+1}})(a_i) \Big|^2 \\
&\le Ch^{1-n} \sum_{j=1}^{J-1} \Big| w_h|_{T_j} - w_h|_{T_{j+1}} \Big|^2_{0,\tilde{F}_j} \\
&\le Ch^{1-n} \sum_{j=1}^{J-1} \Big(\Big| v - w_h|_{T_j} \Big|^2_{0,\tilde{F}_j} + \Big| v - w_h|_{T_{j+1}} \Big|^2_{0,\tilde{F}_j} \Big)
\end{aligned}$$

By the interpolation theory, we obtain

$$\Big| (w_h|_T)(a_i) - (w_h|_{T'})(a_i) \Big|^2 \le Ch^{6-n} \sum_{j=1}^{J} |v|^2_{3,T_j}$$

Since $N_h(T)$ is bounded, we have

$$|(\phi_h|_T)(a_i)|^2 \le Ch^{6-n} \sum_{T' \in \mathcal{T}_h(T)} |v|^2_{3,T'} \tag{40}$$

If vertex a_i of T is on $\partial\Omega$ then there exists $T' \in \mathcal{T}_h(a_i)$ with an $(n-1)$-dimensional subsimplex F of T' on $\partial\Omega$ and $a_i \in F$. By the definitions of w_h and v_h, we have

$$|(\phi_h|_T)(a_i)| = |(w_h|_T)(a_i) - (w_h|_{T'})(a_i) + (w_h|_{T'})(a_i)| \le |(w_h|_T)(a_i) - (w_h|_{T'})(a_i)| + |(w_h|_{T'})(a_i)|.$$

By the inverse inequality and the interpolation theory

$$|(w_h|_{T'})(a_i)|^2 \le Ch^{1-n}|w_h|_{T'}|_{0,F}^2 = Ch^{1-n}|v - w_h|_{T'}|_{0,F}^2 \le Ch^{6-n}|v|_{3,T'}^2.$$

By a similar analysis for $|(w_h|_T)(a_i) - (w_h|_{T'})(a_i)|$, we conclude that (40) is also true in this case.

Similarly, we can show that

$$\sum_{i=1}^{n+1} \|\nabla(\phi_h|_T)(a_i)\|^2 \le Ch^{4-n} \sum_{T' \in \mathcal{T}_h(T)} |v|_{3,T'}^2. \tag{41}$$

Combining (39), (40) and (41), we have

$$h^{2m}|\phi_h|_{m,T}^2 \le Ch^6 \sum_{T' \in \mathcal{T}_h(T)} |v|_{3,T'}^2.$$

Summing the above inequality over all $T \in \mathcal{T}_h$, we obtain that

$$h^{2m}|\phi_h|_{m,h}^2 \le Ch^6 \sum_{T \in \mathcal{T}_h} \sum_{T' \in \mathcal{T}_h(T)} |v|_{3,T'}^2.$$

Consequently

$$h^{2m}|\phi_h|_{m,h}^2 \le Ch^6 |v|_{3,\Omega}^2. \tag{42}$$

Inequality (38) follows from (42) and (37).

Using similar argument, we can show (36). □

Lemma 4.2. *Let V_{h0} be the finite element space corresponding to NZT element. Then there exists a constant C independent of h such that for $v \in H^3(\Omega)$*

$$|a_h(v, v_h) - (\Delta^2 v, v_h)| \le Ch|v|_{3,\Omega}|v_h|_{2,h}, \quad \forall v_h \in V_{h0}. \tag{43}$$

Proof. Let $v_h \in V_{h0}$ and $\phi \in H^1(\Omega)$. Given $T \in \mathcal{T}_h$ and an $(n-1)$-dimensional subsimplex F of T, let $P_F^0 : L^2(F) \to P_0(F)$ be the L^2-orthogonal projection.

Let $i, j \in \{1, 2, \cdots, n\}$. By Lemma 3.2 and Green's formula we have

$$\begin{aligned}
\sum_{T \in \mathcal{T}_h} \int_T \Big(\phi \frac{\partial^2 v_h}{\partial x_i \partial x_j} + \frac{\partial \phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big) &= \sum_{T \in \mathcal{T}_h} \int_{\partial T} \phi \frac{\partial v_h}{\partial x_j}\nu_i = \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F \phi \frac{\partial v_h}{\partial x_j}\nu_i \\
&= \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F \phi \Big(\frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j}\Big)\nu_i \\
&= \sum_{T \in \mathcal{T}_h} \sum_{F \subset \partial T} \int_F (\phi - P_F^0 \phi)\Big(\frac{\partial v_h}{\partial x_j} - P_F^0 \frac{\partial v_h}{\partial x_j}\Big)\nu_i
\end{aligned}$$

Using the Schwarz inequality and the interpolation theory we obtain that

$$\begin{aligned}\Big|\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F(\phi-P_F^0\phi)\Big(\frac{\partial v_h}{\partial x_j}-P_F^0\frac{\partial v_h}{\partial x_j}\Big)\nu_i\Big| \\ \le \sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\|\phi-P_F^0\phi\|_{0,F}\Big\|\frac{\partial v_h}{\partial x_j}-P_F^0\frac{\partial v_h}{\partial x_j}\Big\|_{0,F} \\ \le C\sum_{T\in\mathcal{T}_h}h|\phi|_{1,T}|v_h|_{2,T}\le Ch|\phi|_{1,\Omega}|v_h|_{2,h}.\end{aligned}$$

Consequently, for $i,j\in\{1,2,\cdots,n\}$,

$$\Big|\sum_{T\in\mathcal{T}_h}\int_T\Big(\phi\frac{\partial^2 v_h}{\partial x_i\partial x_j}+\frac{\partial\phi}{\partial x_i}\frac{\partial v_h}{\partial x_j}\Big)\Big|\le Ch|\phi|_{1,\Omega}|v_h|_{2,h},\quad\forall\phi\in H^1(\Omega),\ \forall v_h\in V_{h0}. \tag{44}$$

Using (44) and the following equality,

$$\begin{aligned}a_h(v,v_h)-(\Delta^2 v,v_h)=&\sum_{i=1}^n\sum_{T\in\mathcal{T}_h}\int_T\Big(\Delta v\frac{\partial^2 v_h}{\partial x_i^2}+\frac{\partial\Delta v}{\partial x_i}\frac{\partial v_h}{\partial x_i}\Big)\\ &+\sum_{1\le i\ne j\le n}\sum_{T\in\mathcal{T}_h}\int_T\Big(\frac{\partial^2 v}{\partial x_i\partial x_j}\frac{\partial^2 v_h}{\partial x_i\partial x_j}+\frac{\partial^3 v}{\partial x_i^2\partial x_j}\frac{\partial v_h}{\partial x_j}\Big)\\ &-\sum_{1\le i\ne j\le n}\sum_{T\in\mathcal{T}_h}\int_T\Big(\frac{\partial^2 v}{\partial x_i^2}\frac{\partial^2 v_h}{\partial x_j^2}+\frac{\partial^3 v}{\partial x_i^2\partial x_j}\frac{\partial v_h}{\partial x_j}\Big),\end{aligned} \tag{45}$$

we obtain the conclusion of the lemma. □

Theorem 4.3. *Let V_{h0} be the finite element space corresponding to NZT element, and let u and u_h be the solutions of problems (3) and (5) respectively. Then*

$$\lim_{h\to 0}\|u-u_h\|_{2,h}=0, \tag{46}$$

and there exists a constant C independent of h such that

$$\|u-u_h\|_{2,h}\le Ch|u|_{3,\Omega} \tag{47}$$

when $u\in H^3(\Omega)$.

Proof. From Lemma 3.2 we see that NZT element passes the F-E-M-Test in [12]. Hence NZT element passes the generalized patch test. By Lemma 3.2 and the fact that $H_0^2(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in norm $\|\cdot\|_{2,\Omega}$, we obtain

$$\lim_{h\to 0}\inf_{v_h\in V_{h0}}\|v-v_h\|_{2,h}=0,\quad\forall v\in H_0^2(\Omega).$$

Thus (46) is true by the result in [16].

By the generalized Poincare-Friedrichs inequality [17] and the Strang Lemma (see [6] or [15]), we have

$$\|u-u_h\|_{2,h} \le C\left(\inf_{w_h\in V_{h0}}\|u-w_h\|_{2,h}+\sup_{w_h\in V_{h0}\, w_h\neq 0}\frac{|\,a_h(u,w_h)-(f,w_h)|}{\|w_h\|_{2,h}}\right).$$

Then (47) follows from (35) and (43). □

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