EXPERIMENT 1

DC Breakdown of Gases: Paschen Curves

I. General

To study plasma formation, one must either make them, model them with equations, or rely on nature to create them. In this experiment we choose the first option. This gives us control of the system and is not subject to the vagaries of incomplete models or cloudy nights. That does not mean our situation is simple. On the contrary, it is rather complex and rather ambitious, in part because of our desire to unify the descriptions of many systems and situations into a simple picture, perhaps even a single equation. Among the choices we have in the experiment are: what gases and pressures are chosen; what is the size and geometry of the apparatus; what power is available (thermal or electrical, e.g., DC, RF or microwave, high or low voltage); what time scales are important (transient or quasi-steady state); and what temperatures are attainable (both plasma and material).

In this experiment we shall learn how to form relatively cool (2 eV) plasmas by applying a DC voltage between two flat, parallel electrodes. This is similar to the method used to make plasmas in fluorescent lights and neon signs. What we shall learn is that there is a general law (Paschen's Law) that tells us the easiest way to make plasmas, meaning, under what conditions a plasma can be formed at a relatively modest voltage, 100 to 500 volts. This modest voltage is dangerous, though the power supply can only provide 10 ma, about a factor of 10 below the **lethal** value. Do not operate this equipment without an experienced colleague or teacher present.

A Paschen-like law can be developed for plasma formation by RF or microwave fields. The student is encouraged to explore this through readings.

Primary goal: of this experiments: To examine the dependence of DC breakdown voltage on gas type, gas pressure, and electrode separation

II. The physical picture of DC breakdown

On the earth, most gases are almost completely electrically neutral, to the extent that the vast majority of the electrons are bound to their atoms and each atom carries no net charge. (Counter examples are when lightning, sparks, intense flames or the aurora occur.) For a gas to behave as a plasma requires that many of its atoms be ionized. The term *many* is rather casual and qualitative; the requirement for a gas to behave as a plasma may be as weak as only one free electron for each million neutral gas atoms.

There are different names given to types of plasmas, depending on the degree of ionization. When more than ~90% of the atoms present in a gas are ionized, that is, imprecisely, called a *fully ionized* plasma. When fewer than about one free electron per 10,000 gas atoms exists, a plasma is called *weakly ionized*. In between those limits, a plasma is termed *partially ionized*. Below one in a million, the ionized gas rarely behaves like a plasma in that it does not sustain plasma oscillations or shield out electrical fields.

Ordinary air actually does have some free electrons and ions, formed by flames, ultraviolet light, the residue of sparks from static electricity or friction, cosmic ray impacts, energetic debris from radioactive materials commonly present in building materials and certain types of rocks, etc. In these ordinary conditions, there are relatively few free electrons, about one for each 10^{12} gas atoms. Let us consider what happens to the few electrons and ions in this situation when two nearby metal plates are raised to a high voltage. Perhaps you know from experience that, at a high enough voltage, a spark occurs. The voltage at which the spark first occurs is called the *breakdown voltage*, V_B, because the air is said to have broken down into its constituent electrons and ions. What we want to do is understand what determines that exact voltage. To find the answer, we simplify the picture by first considering a single free electron in the gas between the two metal plates, the electrodes.

Connect a DC power supply to the two plates, giving an electrical bias, a potential difference, between them. If we assume that the plates are very large and parallel, then the potential will rise linearly from the cathode (assumed here to be at $V = 0$, i.e., ground potential) to the anode, which has the positive bias voltage from the DC power supply, see figure 1. A linearly varying potential gives a constant electric field.

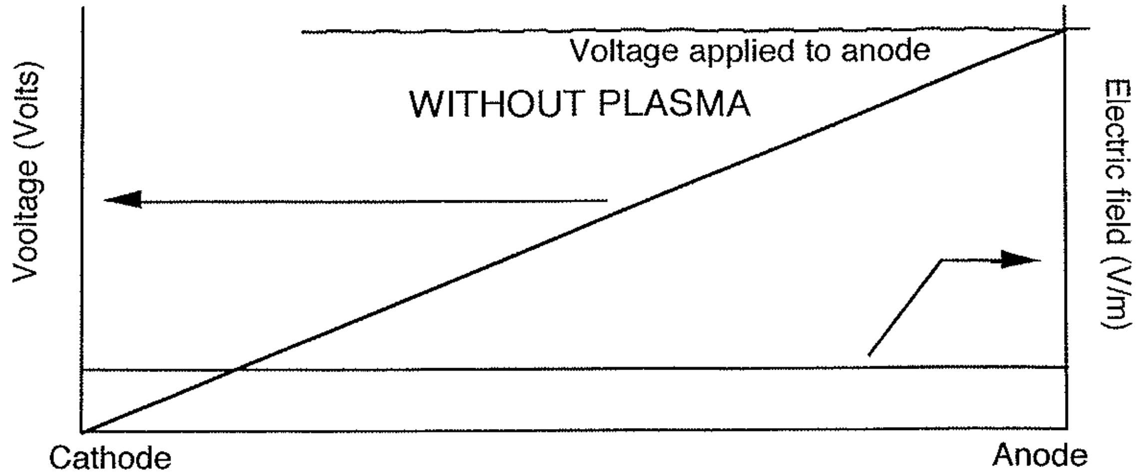

Figure 1. Voltage and electric field between two parallel metal plates in the absence of a plasma.

An electron between the plates feels the force due to the electric field and starts to accelerate towards the anode, see Figure 2. It moves in a nearly straight line. If the density of the gas is low enough or the electron close enough to the anode to begin with, the electron may accelerate all the way to the anode without colliding with an air molecule. If not, the electron will collide with gas atoms and molecules along its path. If the electron has little energy, it will simply bounce off (elastic collision) the atoms. (For this discussion, we use the words *atoms* and *molecules* interchangeably.) With more energy, the electron can either excite the atom, possibly cause light to be emitted subsequently, or ionize it. That is what is shown schematically in Figure 2. At each ionization event, another free electron is generated and added to the electron flow towards the plate. The amount of energy required to ionize an atom is called its ionization potential, V_i. Values of V_i for some atoms are given in Table 1. The amount of energy gained between each collision, ΔW, is the distance (parallel to E) traveled between collisions, λ_i, times the electric field, E, and the electron's charge, -e.

$$\Delta W = e\,\lambda_i\,E \tag{1}$$

If the gas is very dense, then the electron does not gain enough energy between collisions and cannot ionize the gas. From this simple picture we can see that there are two cases where ionization (the seed of plasma formation) will be poor: 1) when the gas is too dense - so that collisions are too frequent and the electron has too little energy to ionize gas atoms; and 2) when the gas is too tenuous - then the electron can reach the anode without striking any gas atoms along its path. Thus, we expect that a moderate pressure provides the optimal situation for ionization. Our next goal is to quantify this picture.

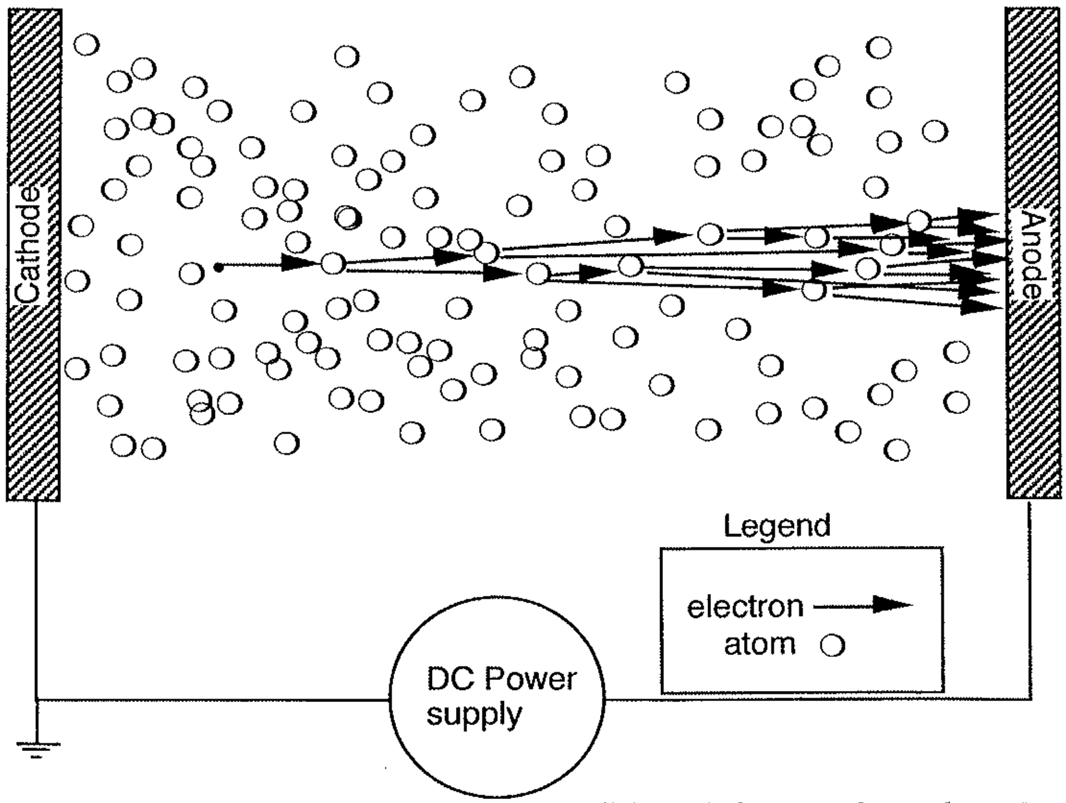

Figure 2. A single electron, between two biased parallel metal plates may be accelerated to high enough energy that it ionizes gas atoms, generating a cascade of electrons flowing towards the anode. All the ions thus generated flow in the opposite direction and impact on the cathode. Some of these ion impacts cause secondary electrons to be ejected.

The electron motion is only half the story. Essentially all the electrons in this burst are absorbed by the anode. If the electrons were the only particles to consider, there would be no plasma because the electrons would be rapidly swept out of gas by the electric field. The other particles in this picture are the ions. An ion is generated for every electron created by ionization. Because of their positive charge, the ions stream in the opposite direction, towards the cathode. They do so more slowly, because of their greater mass. When the ions hit the metal cathode, they may cause the ejection of electrons. This is called secondary electron emission. For each ion impact, the probability of ejecting an electron, γ, is small, about 1-10% at the energies of concern. The ejected electrons, called secondaries, are themselves accelerated by the electric field and flow towards the anode. In doing so, the process of ionization has begun anew.

(The terminology applied to the electrons emitted from the cathode is a continuing cause of confusion. To the surface physicist the electrons are clearly *secondary* electrons. To the plasma physicist, they are *primary* energetic electrons which ionize the gas. The electrons produced by ionization are sometimes called the *ultimate* electrons.)

The important question is whether more than 1 electron was liberated from the cathode by the impact of all the ions generated by that first electron. With $\gamma \sim 0.05$, more than 20 ions need to have been created within the gas to produce, on impacting the cathode, statistically, more than one secondary electron. When this occurs, the ions created in the gas by the secondary electrons will exceed those created by that first electron. The whole process creates more and more electrons with each succeeding generation. This is the breakdown process.

Table 1: Ionization potentials and Paschen curve coefficients A,B of some atoms and molecules. A. von Engels, *Ionized Gases*

Atom or molecule	Ionization potential, V_i (eV)	A ($m^{-1}\,mT^{-1}$)	B ($V\,m^{-1}\,mT^{-1}$)
H_2	15.4	0.54	14
He	24.6	0.28	3.4
N_2	15.8	1.2	34
Air	~13	1.5	36
Ne	21.6		
Ar	15.8	1.2	18

III. Mathematical derivation of the breakdown voltage: Paschen's Law

The mathematical description of breakdown begins with writing down the equations describing the currents, both electron and ion, as functions of position.

The electron density and current grow exponentially, as a function of distance towards the anode, in a characteristic distance given by the $1/\alpha$, where α is the average number of collisions *per* unit length times the probability that a collision will be of the ionizing variety. Stated mathematically

$$dn_e/dx = \alpha\, n_e \qquad (2a)$$

or $$n_e = n_{e0}\, e^{\alpha x}. \tag{2b}$$

α, for historical purposes, is known as the first Townsend coefficient.

The total current density in the gas is equal to the sum of the ion (J_{ic}) and electron (J_{eo} + J_{es}) current densities, where the subscripts mean *e* = electron, *i* = ion, *o* = initial, *c* = cathode, *a* = anode, and *s* = secondary. At the cathode

$$J_c = J_{eo} + J_{es} + J_{ic}\,. \tag{3}$$

Note that $$J_{es} = \gamma\, J_{ic}\,, \qquad \text{thus} \tag{4}$$

$$J_c = J_{eo} + (1 + 1/\gamma) J_{es}\,. \tag{5}$$

Now $$J_a = J_c \sim J_{ea} \tag{6}$$

because of conservation of current and because there are no secondary ions emitted from the anode. Finally, from equation 2, the electron current at the anode equals $e^{\alpha d}$ *times* the electron current at the cathode (which are separated from each other by a distance *d*). Combining these expressions, eliminating J_{es}, gives an expression for the electron current at the anode.

$$J_{ea} = \left[\frac{1}{1-\gamma(e^{\alpha d}-1)}\right] e^{\alpha d} J_{eo} \tag{7}$$

There are three parameters which control the amount of current: γ, α and *d*. Clearly, when the denominator goes to zero, the current goes to infinity. This corresponds to breakdown. For $\gamma << 1$ it occurs when

$$\gamma\, e^{\alpha d} = 1 \tag{8}$$

Now we must find the connection between this condition and the gas pressure and the electrode spacing. The first Townsend coefficient is related to the mean-free-path, λ, by

$$\alpha = \frac{1}{\lambda} e^{-V_i/E\lambda}. \tag{9}$$

The exponential term represents the probability that an electron will gain more energy than $\Delta W = V_i$, by traversing a distance greater than the mean -free-path λ. Recall that $\lambda = 1/n_n \sigma_n$, (where σ_n is the collision cross section and n_n the neutral gas density) and that the gas density can be obtained from the perfect gas law, $p = n_n k T_n$. Then equation (9) can be rearranged as

$$\alpha = Ape^{(-Bp/E)} \tag{10}$$

where $A = \sigma_n/kT_n$ and $B = \sigma_n V_i/kT_n$. Note for fixed E, α has a maximum at a certain pressure, p. This occurs when $E/p = B$. Values of A and B are given in Table 1.

Paschen's law is obtained from substituting equation 10 into equation 8 and using $E = V_B/d$. The result is

$$V_B = \frac{Bpd}{\ln[Apd/\ln(1/\gamma)]} \tag{11}$$

The optimum (lowest voltage) condition for breakdown occurs at $dV_B/d(pd) = 0$. There we find

$$(pd)_{min} = e(1)\frac{\ln(1/\gamma)}{A} \tag{12}$$

$$(V_B)_{min} = e(1)B\frac{\ln(1/\gamma)}{A} \tag{13}$$

The goal of the experiment is to find values for V_B as a function of pd, hence to get values for A, B, and γ. Since there are two equations, but three unknowns, we need to eliminate one. For the purposes here, we assume that someone else has measured γ. In Table 2 are listed values of γ for ions impacting Fe, a reasonable approximation for stainless steel in this context.

Table 2. Secondary electron emission coefficient, γ, for the impact of various low-energy (< 1000 eV) ions on iron.

Gas	He	Ne	Ar	Air
γ	0.015	0.022	0.058	0.020

J.D. Cobine, *Gaseous Conductors*, Dover Publications, New York, 1958.

DC Breakdown of Gases - Paschen Curves

I. Safety

In this experiment we shall learn how to make plasma by applying a DC voltage between two electrodes. This is similar to the method used to make plasmas in fluorescent lights and neon signs. **Dangerous voltages** may be present when making many types of plasma, including those used in this experiment. Do not operate any equipment until you are confident that you can do so safely.

A particularly helpful guideline when using equipment is to work with *one hand only* so you can focus all your attention on the single switch you are turning, making sure it is the right switch and that you are moving it the proper amount in the proper direction. Keeping your other hand in your pocket will prevent potential electrical currents across your chest.

I.1. Electrical

Potentially lethal power supplies are used in this experiment. They can provide ten of milliamperes of current at thousands of volts. The electrical leads to these supplies are shielded by thick insulation. In *dry* air, good insulation can shield 400V/ 0.001". In addition to the power supplies themselves, the wall outlets can provide equally lethal currents at voltages of 110, 208 and 480 VAC.

I.2. Mechanical

Perhaps the simplest way to become injured in any environment is to ignore the law of gravity. How might this happen? Slipping on an oily spot on the floor and grabbing onto a rack of equipment is a possible way. Using a chair for a ladder is another classic way to lose your balance.

I.3. Gas Cylinders

There are no gas cylinders used in this experiment. We are using only the air in the room as our working gas. All spent gases are vented to the outside world via 3" PVC pipes. Along with these gases exit small amounts of oil vapor produced by the vacuum pumps.

I.4. Implosions and explosions

The vacuum systems we use are made of glass. The more fragile parts can be broken by a sharp impact; glass fragments, moving at near-sonic speeds could result. Care is required.

I.5. Radiation

We do not use radioactive sources in this lab. The power supplies used are limited to below 2 kV, thus reducing the possibility for generating hard (penetrating) xrays by Bremsstrahlung. The plasmas we generate can produce UV radiation. This, though potentially dangerous, does not cause problems because all of our plasmas are in borosilicate glass that absorb UV effectively.

II. Goals of the experiment

To examine how plasma formation depends on applied voltage, gas pressure, and electrode separation.

To become familiar with vacuum apparatus, high-voltage DC power supplies, and basic plasma techniques.

III. Apparatus

The plasma will be formed in a glass-walled chamber. An electrical system is necessary to form and characterize the plasma, and a vacuum system is needed to evacuate the chamber to the desired pressure for plasma formation.

Two stainless steel plates inside the glass chamber are used as the electrodes between which the plasma will be formed. A DC voltage is applied across these electrodes to cause breakdown and form the plasma. The separation between the anode and cathode may be changed by adjusting the grounded (anode) electrode. The electrodes are connected to a high voltage power supply through a 'ballast' resistor (whose purpose is to limit the flow of current through the plasma).

The chamber is evacuated by a mechanical pump. Pressure is measured with a convectron gauge. Pressures as low as 20×10^{-3} Torr (20 mT) can be obtained in the chamber (where 1 atm = 10^5 Pa = 760 Torr). Air is introduced into the chamber through the needle valve.

IV. Exercises

A. <u>Vacuum equipment</u>

1. Familiarize yourselves with the vacuum system. Identify the vacuum pump, pressure gauge, and gas flow control valves. *Use the needle valve* ***carefully!*** Do not close it fully.

2. Practice controlling the pressure using the valves.

3. Determine how the seals for the electrode work within the vacuum chamber.

4. **Never** turn the vacuum pump on or off without first <u>closing</u> the valve on the pump. Failure to do so will cause oil from the pump to be pulled into the glass tube.

B. Electrical setup

1. Identify the high-voltage power supply.

2. Starting from the back of the power supply, trace the high-voltage output through the wire attached to the cathode of the plasma chamber. Make sure the wire is *securely* connected to the cathode and that the resistor (see Figure 4) is in series with the chamber and the output wire. Trace where the ground line goes.

3. Note: the power supply will produce a high *negative* voltage (relative to ground) on the cathode in the chamber.

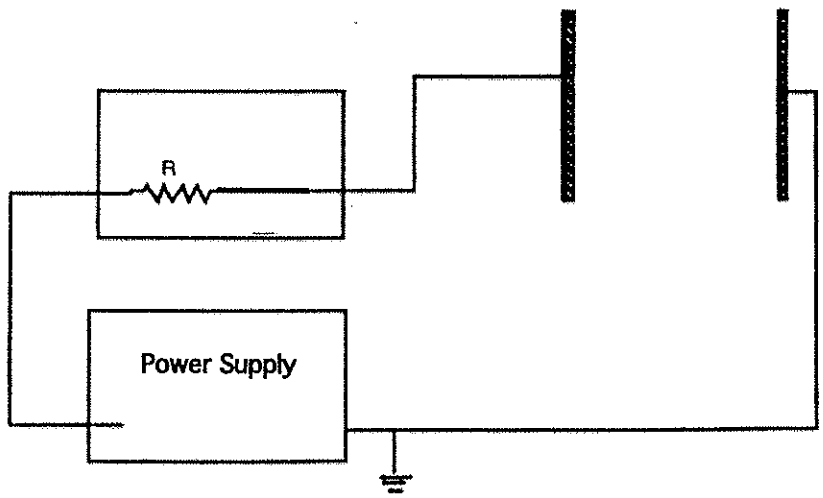

Figure 4. Schematic of the electrical circuit.

C. Lighting a plasma

1. Adjust the plate spacing in the vacuum chamber so that the two plates are 3" apart.

2. SLOWLY increase the voltage. At some point plasma will form in the vacuum chamber. (DO NOT exceed a setting of 1500V on the power supply! If the plasma has not formed by the time you have dialed up to 1500V, stop and let one of us know. How do you know that plasma has formed?

3. Once you have plasma, the voltage where it appeared is the "BREAKDOWN VOLTAGE."

V. Experiment

Breakdown Voltage

1. Measure the breakdown voltage, V_B, as a function of pressure for a variety of large and small electrode spacings. Describe the shape and color of the glowing and dark regions of the plasma as functions of pressure and voltage.

   ⇨ Always dial the power supply down to zero volts before changing anything else!! ⇦

2. Plot V_B vs. pd. Is there a minimum in your data? If not, take additional data for the smallest possible values of pd. Compare your experimental values to those calculated from Paschen's Law (see below).

Derivation of the Paschen curve law
ALPhA Laboratory Immersion

Arturo Dominguez

July 3, 2014

1 Objective

If a voltage differential is supplied to a gas as shown in the setup (Figure 1), an electric field is formed. If the electric field applied is strong enough, an avalanche process (the Townsend avalanche) is started which leads to the breakdown of the gas and the formation of plasma. This document describes the physics of this process and derives the law (Paschen's law) that predicts the voltage differential that needs to be supplied in order to create the plasma.

2 Experimental Setup

In Figure 1 the experimental setup for the Paschen curve experiment is shown. A pair of parallel plate electrodes are placed inside a vessel that contains a gas which can be air but can also be a more pure gas, like He, Ar, Ne, etc. While the geometry of the setup is irrelevant to the qualitative behavior of the breakdown voltage, the simple 1D geometry results in a simpler comparison with theory. The variables that can be controlled are: the pressure of the gas, p, the distance between the electrodes, d, and the voltage between the electrodes, V, as well as the gas contained in the vessel.

3 Qualitative description

As the voltage difference is applied, the electric field will accelerate any free charges that exist. Free electrons exist in the system due to random events from a variety of mechanisms including the triboelectric effect or through astronomical particles traversing the vessel and ionizing neutral particles. If an electron can gain more than the ionizing energy of the gas, U_I (approximately $14eV$ for Nitrogen), the electron can ionize the neutral particle and create a new free electron and a free ion. The new free electron can repeat the process and create a chain reaction called the Townsend Avalanche. The ion is accelerated towards the cathode and as it hits it there is a chance that it will

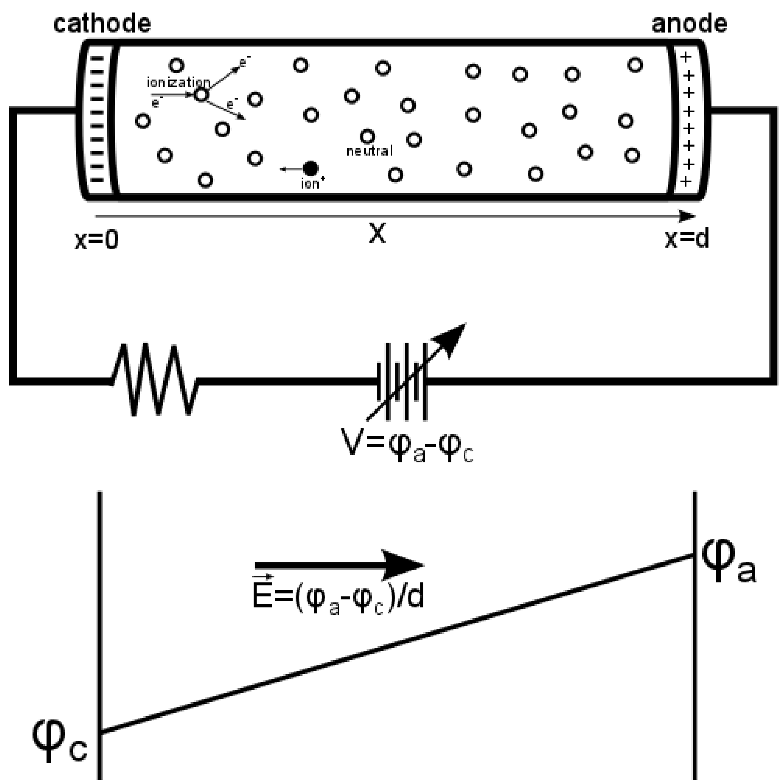

Figure 1: The standard DC discharge setup shows an electron ionizing a neutral particle. When the process becomes self sustaining, the Townsend Avalanche is initiated and a plasma is formed.

free an electron from the cathode. This process is called *secondary emission* and is responsible for the production of electrons that can sustain the plasma. Once a threshold condition is met, the secondary electrons suffice to begin the Townsend Avalanche.

4 Derivation of Paschen's Law

Quantitatively, the electron avalanche can be described using the *rate of ionization per unit length*, α. For example, if $\alpha = 2cm^{-1}$ then within a cm of electron travel along the x-axis, there will, on average, be 2 collisions with neutral particles that result in ionizing events. This results in the following equation describing the increase of electron current density, $\Gamma_e(x)$:

$$d\Gamma_e(x) = \Gamma_e(x)\alpha dx, \tag{1}$$

which integrates to:

$$\Gamma_e(x) = \Gamma_e(0)e^{\alpha x}. \tag{2}$$

Since there is no current leaving or entering the vessel except through the electrodes, and there is no charge accumulation, the continuity equation results in:

$$\Gamma(0) = \Gamma(d), \tag{3}$$

that is, the current density at the cathode ($x = 0$) is the same as that at the anode ($x = d$). Assuming a single ion species, Equation 3 can be rewritten as:

$$\Gamma_e(0) + \Gamma_i(0) = \Gamma_e(d) + \Gamma_i(d) \tag{4}$$

$$\Gamma_i(0) = \Gamma_e(d) - \Gamma_e(0) + \Gamma_i(d). \tag{5}$$

Using the fact that there is no input of ions from the anode, $\Gamma_i(d) = 0$. Using this constraint, and substituting Equation 2 into Equation 5 results in:

$$\Gamma_i(0) = \Gamma_e(0)(e^{\alpha d} - 1). \tag{6}$$

As an ion reaches the cathode, the probability of it releasing a secondary electron is given by the coefficient γ, therefore:

$$\Gamma_e(0) = \gamma\Gamma_i(0). \tag{7}$$

At the threshold point where the secondary emission sustains the plasma and the Townsend Avalanche is formed, Equations 6 and 7 are balanced and $\Gamma_i(0)$ can be substituted, leading to:

$$\frac{\Gamma_e(0)}{\gamma} = \Gamma_e(0)(e^{\alpha d} - 1) \tag{8}$$

$$\frac{1}{\gamma} = e^{\alpha d} - 1 \tag{9}$$

$$\alpha d = \ln(1 + 1/\gamma). \tag{10}$$

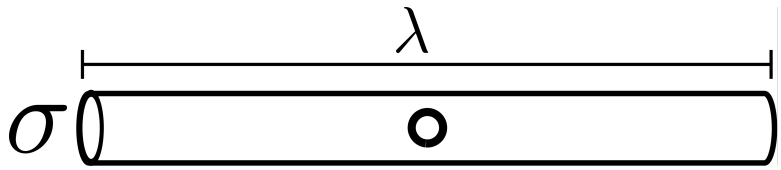

Figure 2: The volume associated with an individual neutral gas particle is given by $vol = 1/n = \sigma\lambda$.

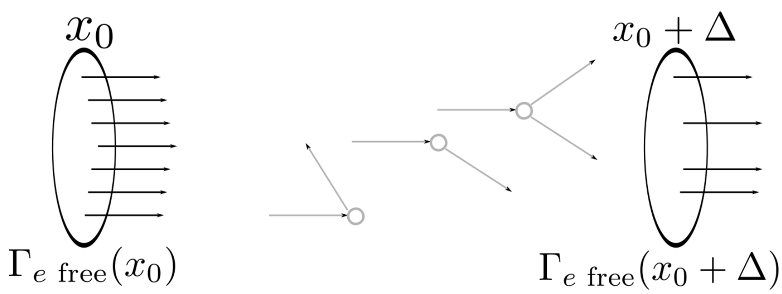

Figure 3: The volume associated with an individual neutral gas particle is given by $vol = 1/n$.

Equation 10 gives one constraint on the ionization coefficient.
While we've defined the role of α, we haven's discussed a physical derivation of what its value should be. In the following section we will tackle that.

As an electron is accelerated through the gas, the distance it travels, on average, before it collides with a neutral particle is given by the *mean free path* (mfp), λ. The volume occupied by a single neutral particle is given by $vol = 1/n$, where n is the neutral density, as shown in Figure 2. The face of the cylinder is the cross section of the collision between the neutral particle and the electron, or $\sigma \approx \pi(r_e + r_n)^2 \approx \pi r_n^2$ where r_e and r_n are the electron and neutral particle's radii respectively. Therefore, the volume occupied per neutral is: $vol = 1/n = \sigma\lambda$. Since the neutral gas follows the ideal gas law, $p = nk_BT$, where p is the neutral pressure, T is the temperature of the neutrals and k_B is the *Boltzmann constant*, the mean free path can be rewritten as:

$$\lambda = \frac{1}{\sigma n} = \frac{k_BT}{\sigma p} \tag{11}$$

λ is the rate of collisions per length in the x-axis of electrons hitting neutral particles. Note that if the electron is energetic enough, the collision will ionize the particle, but this is not generally true.
If you follow the flow of electrons at a position x_0, one can define the current density of *free* electrons, $\Gamma_{e \text{ free}}(x_0 + \Delta)$ which is the current density of free electrons from x_0 that have reached $x_0 + \Delta$ without having collided with a neutral.

In Figure 3 it is observed that as the electrons travel across the length, Δ, some will collide with neutrals and are lost from the $\Gamma_{e\ \mathrm{free}}$. The rate of collisions per unit length is, as was discussed earlier, λ. Therefore, the differential equation that determines $\Gamma_{e\ \mathrm{free}}(x)$ is given as:

$$d\Gamma_{e\ \mathrm{free}}(x) = -\Gamma_{e\ \mathrm{free}}(x)\frac{dx}{\lambda} \tag{12}$$

$$\Gamma_{e\ \mathrm{free}}(x_0+\Delta) = \Gamma_{e\ \mathrm{free}}(x_0)e^{-\Delta/\lambda} \tag{13}$$

$$\frac{\Gamma_{e\ \mathrm{free}}(x_0+\Delta)}{\Gamma_{e\ \mathrm{free}}(x_0)} = e^{-\Delta/\lambda} \tag{14}$$

Note that the RHS of Equation 14 is independent of x_0, hence, at any point in x, the probability that a free electron has traversed a distance of *at least* Δ is given by:

$$P(\text{distance traveled} \geq \Delta) = e^{-\Delta/\lambda} \tag{15}$$

Now, let's get to α. If every collision between an electron and a neutral resulted in an ionization, then $\alpha = 1/\lambda$, but only the proportion of those electrons that have enough energy to ionize the neutral particle, U_I, will cause the ionization, therefore:

$$\alpha = \frac{P(\text{electrons with energy } \geq U_I)}{\lambda} \tag{16}$$

Since the electrons gain energy by being accelerated down the electric field, E, then we can define λ_I using:

$$U_I = eE\lambda_I \tag{17}$$

$$\lambda_I = \frac{U_I}{eE} \tag{18}$$

$$\lambda_I = \frac{U_I d}{eV} \tag{19}$$

where $E = V/d$ has been used. λ_I is, therefore, the distance that an electron must travel in the electric field in order to gain the necessary energy, U_I, to ionize the neutral upon collision. Therefore, Equation 20 can be rewritten as:

$$\alpha = \frac{P(\text{distance traveled} \geq \lambda_I)}{\lambda} = \frac{e^{-\lambda_I/\lambda}}{\lambda}, \tag{20}$$

where Equation 15 has been used. Multiplying d on both sides of Equation 20, we obtain:

$$\alpha d = d\frac{e^{-\lambda_I/\lambda}}{\lambda} \tag{21}$$

$$\ln(1+1/\gamma) = \frac{d}{k_B T/\sigma p}e^{(-U_I d/eV)/(k_B T/\sigma p)} \tag{22}$$

$$\ln(1+1/\gamma) = \left(\frac{\sigma}{k_B T}\right)(pd)e^{(-U_I\sigma/ek_B T)(pd/V))} \tag{23}$$

$$\ln(\ln(1+1/\gamma)) - \ln(Apd) = \frac{-Bpd}{V} \tag{24}$$

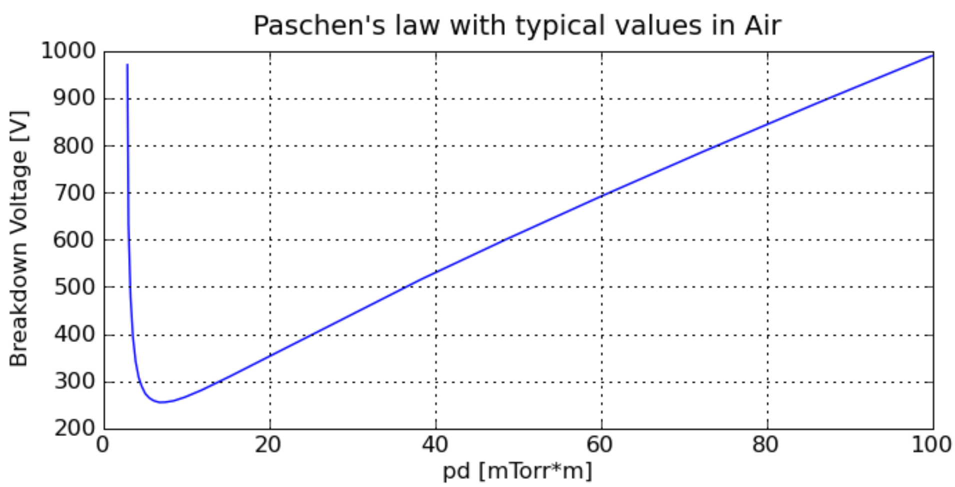

Figure 4: Equation 25 using $A = 1.5mTorr^{-1}m^{-1}$, $B = 36V/(mTorr \times m)$ and $\gamma = 0.02$

where $A \equiv \sigma/k_BT$ and $B \equiv U_I\sigma/ek_BT$ and we have used Equations 10, 11 and 19. Finally, since V is really the voltage right at the point of breakdown, we can change its notation to V_{BD} (breakdown voltage) and put it on the LHS:

$$V_{BD} = \frac{Bpd}{\ln(Apd) - \ln(\ln(1 + 1/\gamma))} \tag{25}$$

This is the final form of Paschen's Law. If we take pd as the abscissa $(x - axis)$ and V_{BD} as the ordinate $(y - axis)$ then the range (as we will see) is $pd = (\ln(1 + 1/\gamma)/A, \infty)$.

For $A = 1.5mTorr^{-1}m^{-1}$, $B = 36V/(mTorr \times m)$ and $\gamma = 0.02$ which are typical values in air at room temperature using stainless steel electrodes, Equation 25 leads to the plot shown in Figure 4.

Note the existence of a minimum in the curve. The pd and V_{BD} at the minimum can be found using the fact that at the critical value, $\frac{dV_{BD}}{d(pd)} = 0$:

$$\begin{aligned} \frac{dV_{BD}}{d(pd|_{\min})} &= \frac{BD - \frac{1}{pd|_{\min}}(Bpd|_{\min})}{D^2} = 0 && (26) \\ B(D-1) &= 0 && (27) \\ D &= \ln(\ln(1+1/\gamma)) - \ln(Apd|_{\min}) = 1 && (28) \\ pd|_{\min} &= \frac{\ln(1+1/\gamma)}{A}e && (29) \\ V_{BD\ \min} &= \frac{B\ln(1+1/\gamma)}{A}e && (30) \end{aligned}$$

where $D \equiv \ln(\ln(1 + 1/\gamma)) - \ln(Apd|_{\min})$ is the denominator of Equation 25 and e in Equations 29 and 30 is *Euler's number*.

Using Equations 10 and 11, Equation 29 can be rewritten as:

$$\alpha|_{\min} = \frac{e^{-1}}{\lambda} \tag{31}$$

Or, comparing it to Equation 20, at the minimum: $\lambda_I = \lambda$. This makes sense intuitively, it's taking the electrons the mfp to accelerate just enough to ionize the neutrals. If you were to increase λ_I, a lot of the collisions would not lead to ionization, if you increase λ then the rate of collisions would drop. (from the denominator in Equation 20).
We also see that when $D = 0$, $V_{BD} \to \infty$, this occurs when:

$$\ln(\ln(1 + 1/\gamma)) - \ln(Apd|_{\text{inf}}) = 0 \tag{32}$$

$$\ln(1 + 1/\gamma) = Apd|_{\text{inf}} \tag{33}$$

$$\alpha|_{\text{inf}} = \frac{1}{\lambda} \tag{34}$$

This result also makes sense, since the rate of ionizing collisions can't be greater than the rate of collisions ($1/\lambda$). This sets the lower limit on pd as $pd > \ln(1 + 1/\gamma)/A$ as discussed before.

5 Discussion of the minimum

The conditions for the minimum are shown in Equations 29 and 30. Its existence can be understood by looking at the extremes: if pd is too big, since $\lambda \sim 1/p$, λ is very small, therefore the electrons will collide too much and won't acquire the ionizing energy U_I necessary. On the other hand, if pd is too small, that is λ is big compared to the distance between the electrodes d, the electrons will collide with the anode before they have a chance of ionizing the gas.
There are many implications of Paschen's law, e.g., when constructing a fluorescent light bulb, where the distance between the electrodes d, is set by the application, the pressure is set so as to be close to the minimum in order to minimize the necessary voltage needed, leading to lower power consumption. The typical pressure inside a fluorescent bulb is $\approx 2Torr$..
Another consequence of Paschen's law is that if we're dealing with atmospheric plasmas ($p \approx 760Torr$) the distance between electrodes for the minimum condition is calculated (using the same conditions as in Figure 4) to be $\sim 10\mu m$. This leads to the label of *microplasmas* for these type of discharges and their consequential difficulty in attaining experimentally.