Plasma waves

Troy Carter
Dept. of Physics and Astronomy, UCLA

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UCLA

Importance of plasma waves

- Along with single particle motion, understanding of linear waves are foundation for physical intuition for behavior of plasmas
- Waves play direct role in important physical processes: RF heating in fusion plasmas, particle acceleration by waves in space, plasma turbulence in astrophysical objects

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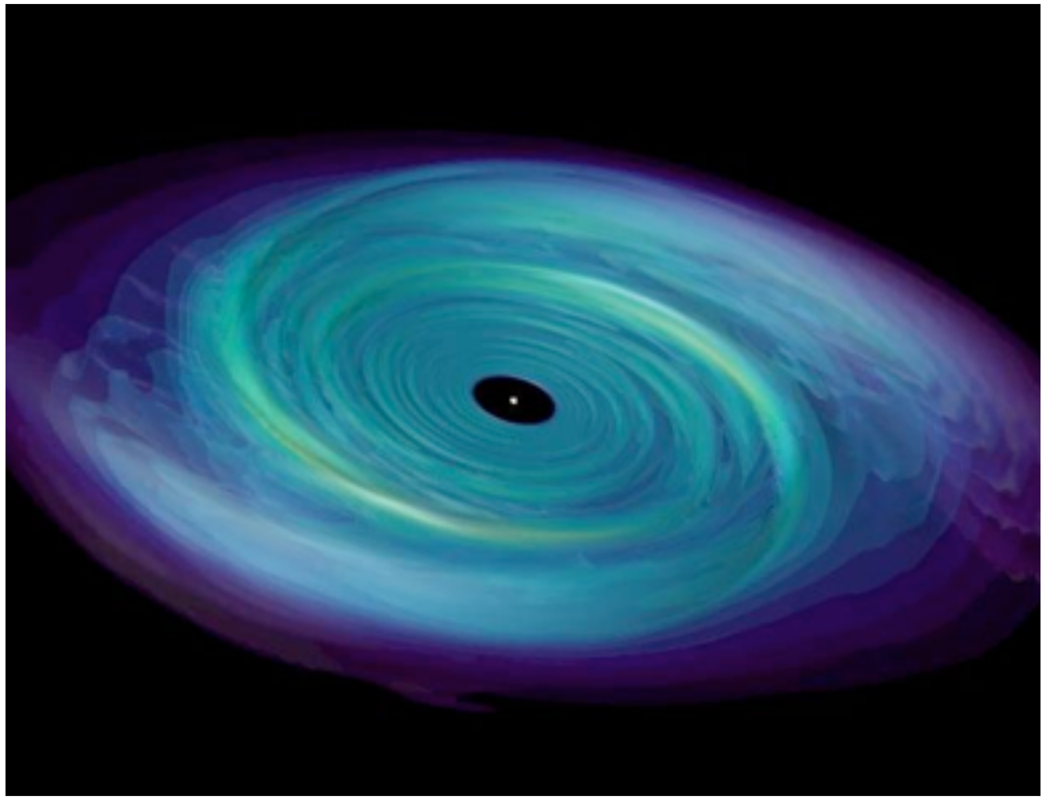


Importance of plasma waves

- Along with single particle motion, understanding of linear waves are foundation for physical intuition for behavior of plasmas
- Waves play direct role in important physical processes: RF heating in fusion plasmas, particle acceleration by waves in space, plasma turbulence in astrophysical objects
- Wave is collective response of plasma to perturbation, however, intuition for waves starts with considering single particle response to electric/magnetic fields that make up the wave
- Focus on magnetized plasmas: particle response is anisotropic, orientation of wave E-field wrt background magnetic field is essential in determining response

Wave equation, plasma dielectric model for linear waves

- Treat plasma as conducting medium; will lead to dielectric description (but start by treating plasma charge and currents as free)

$$\nabla \times \mathbf{B} = \mu_o \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_o \frac{\partial \mathbf{j}}{\partial t} = 0$$

- Plasma effects buried in current, need model to relate current to E: choose linear, tensor conductivity

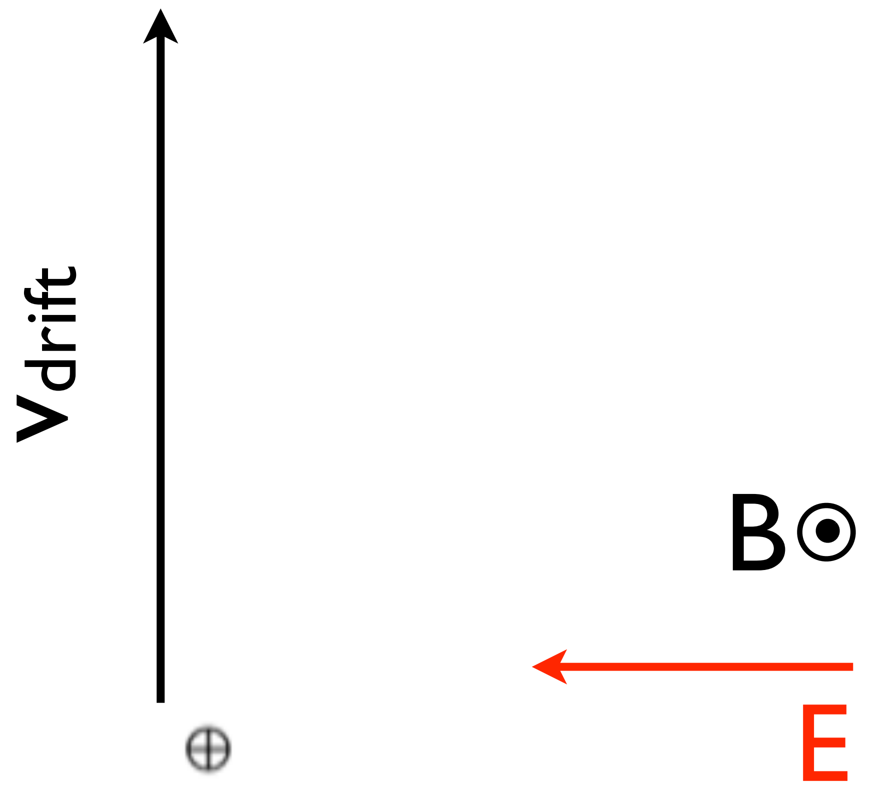
$$\mathbf{j} = \sigma \cdot \mathbf{E}$$

- Need plasma response model to determine the conductivity tensor

Single particle response to wave fields

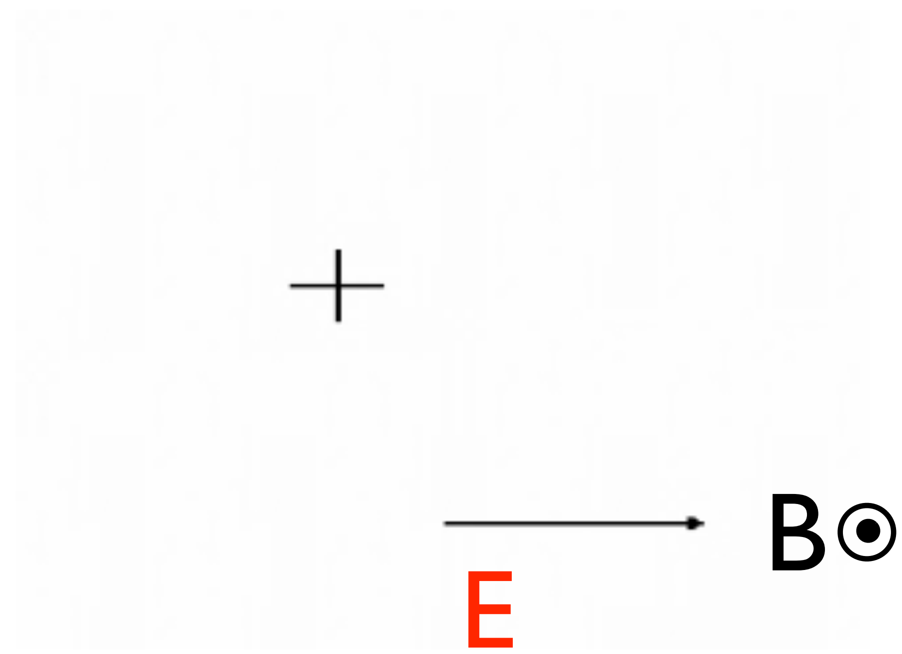
- Conductivity tensor tells us plasma response to applied electric field; useful to think about single particle orbits
- In particular for magnetized plasmas and wave electric fields that are perpendicular to B
- Two drifts matter (in uniform plasma): ExB drift and polarization drift
  - ExB drift is the dominant particle response for low frequency wave fields $\omega < \Omega_c$
  - Polarization drift is dominant at higher frequencies

ExB and Polarization Drifts



ExB drift, DC E Field

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$



Polarization drift, ExB drift removed

$$\mathbf{v}_p = \frac{1}{\Omega}\frac{\partial}{\partial t}\frac{\mathbf{E}_\perp}{B}$$

- No currents from ExB at low freq (ions and electrons drift the same); above ion cyclotron freq, ions primarily polarize, no ExB, can get ExB current from electrons

Model for plasma conductivity

- Use cold, two-fluid model; formally cold means:

$v_\phi \gg v_{\mathrm{th,e}}, v_{\mathrm{th,i}}$

$$\begin{aligned} n_s m_s \frac{d\mathbf{v}_s}{dt} &= n_s q_s \left(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}\right) \\ \mathbf{j} &= \sum_s n_s q_s \mathbf{v}_s \equiv \sigma \cdot \mathbf{E} \end{aligned}$$

- Assume plane wave solution (uniform plasma), linearize the equations:

$$f(\mathbf{r}, t) = f \exp\left(i\mathbf{k} \cdot \mathbf{r} - i\omega t\right)$$

$$f = f_0 + f_1 + \ldots \quad ; \quad f_1 \ll f_o$$

- Ignore terms higher than first order in perturbation

Plasma model, cont.

Choose $\mathbf{B} = B_0\hat{z}$, $\mathbf{E} = \mathbf{E}_1 = E_x\hat{x} + E_z\hat{z}$

Ion momentum equation becomes:

$$\begin{aligned} -i\omega v_x - \Omega_i v_y &= \frac{eE_x}{m_i} \\ \Omega_i v_x - i\omega v_y &= 0 \end{aligned}$$

$$\Omega_i = \frac{eB}{m_i}$$

Solve for v_x, v_y:

$$v_x = \frac{-i\omega}{\Omega_i^2 - \omega^2}\frac{e}{m_i}E_x \quad \text{(polarization)}$$

$$v_y = \frac{-\Omega_i}{\Omega_i^2 - \omega^2}\frac{e}{m_i}E_x \quad \text{(E}\times\text{B)}$$

For the parallel response: $v_z = \dfrac{ie}{\omega m_i}E_z$ (inertia-limited response)

Plasma model, cont.

Back to the wave equation, rewrite with plane wave assumption:

$$-\mathbf{k}\times\mathbf{k}\times\mathbf{E}-\frac{\omega^2}{c^2}\mathbf{E}-i\omega\mu_o\sigma\cdot\mathbf{E}=0$$

Can rewrite in the following way:

$$\mathbb{M}\cdot\mathbf{E}=0$$

$$\mathbb{M}=(\hat{k}\hat{k}-\mathbb{I})n^2+\epsilon \qquad n^2=\frac{c^2k^2}{\omega^2}\ \text{index of refraction}$$

$$\epsilon=\mathbb{I}+\frac{i\sigma}{\epsilon_o\omega}\ \text{dielectric tensor}$$

unit tensor

Cold plasma dispersion relation

Using the cold two-fluid model for σ, the dielectric tensor becomes:

$$\epsilon = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$$S = 1 - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2} - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} \quad \text{(polarization)}$$

$$D = \frac{\Omega_i \omega_{pi}^2}{\omega(\omega^2 - \Omega_i^2)} - \frac{\Omega_e \omega_{pe}^2}{\omega(\omega^2 - \Omega_e^2)} \quad \text{(E}\times\text{B response)}$$

$$P = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2} \quad \text{(inertial response)}$$

Defining θ to be the angle between $\mathbf{k}$ and $\mathbf{B_o}$, the wave equation becomes:

$$\begin{pmatrix} S - n^2\cos^2\theta & -iD & n^2\sin\theta\cos\theta \\ iD & S - n^2 & 0 \\ n^2\sin\theta\cos\theta & 0 & P - n^2\sin^2\theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$\det \mathbb{M} = 0$ provides dispersion relation for waves – allowable combinations of ω and $\mathbf{k}$

Alternate dielectric formulation: circularly polarized waves

- Previous dielectric formulation based on linearly polarized basis, some physical insight can be gained by considering a circularly polarized basis (wrt B)

$$\hat{r}, \hat{l} = \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}} \quad \epsilon = \begin{pmatrix} R & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & P \end{pmatrix}$$

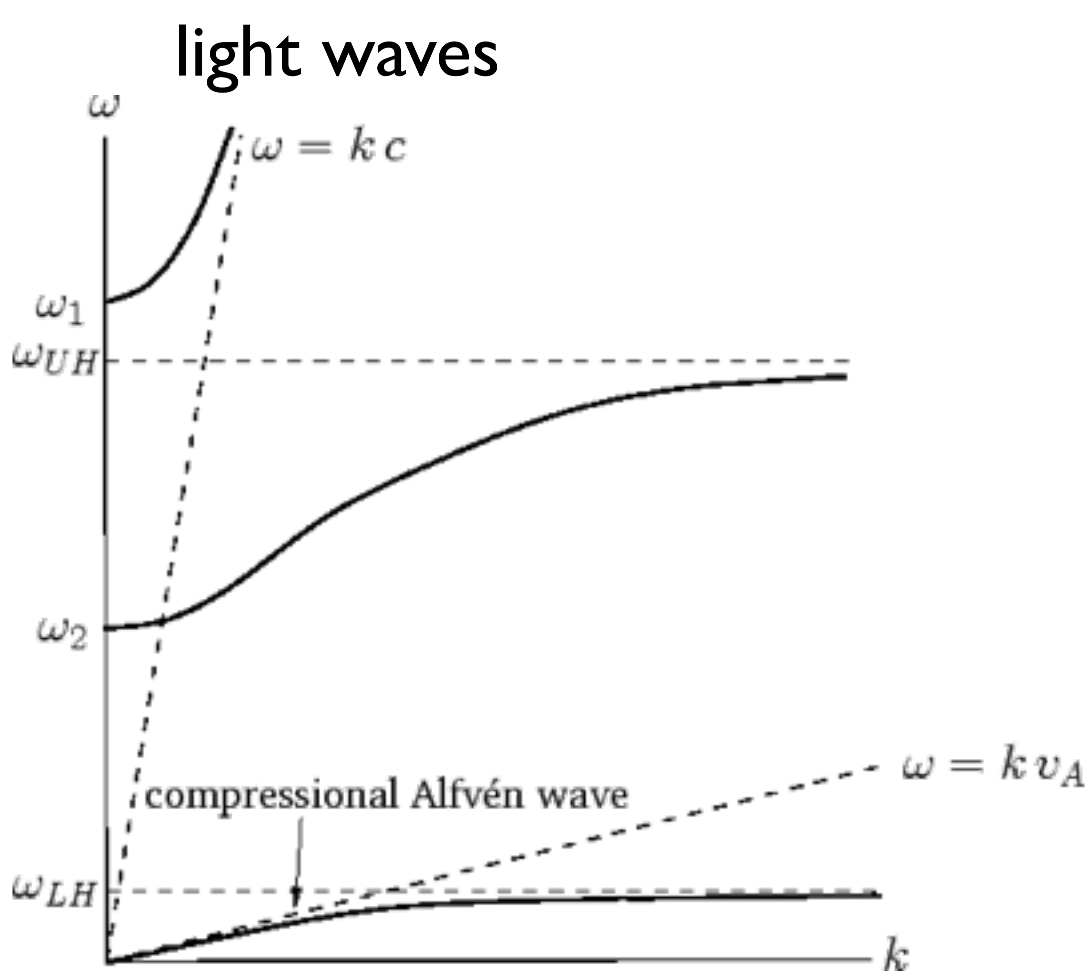
$$R = 1 - \frac{\omega_{pi}^2}{\omega^2}\frac{\omega}{\omega - \Omega_i} - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{\omega - \Omega_e}$$

$$L = 1 - \frac{\omega_{pi}^2}{\omega^2}\frac{\omega}{\omega + \Omega_i} - \frac{\omega_{pe}^2}{\omega^2}\frac{\omega}{\omega + \Omega_e}$$

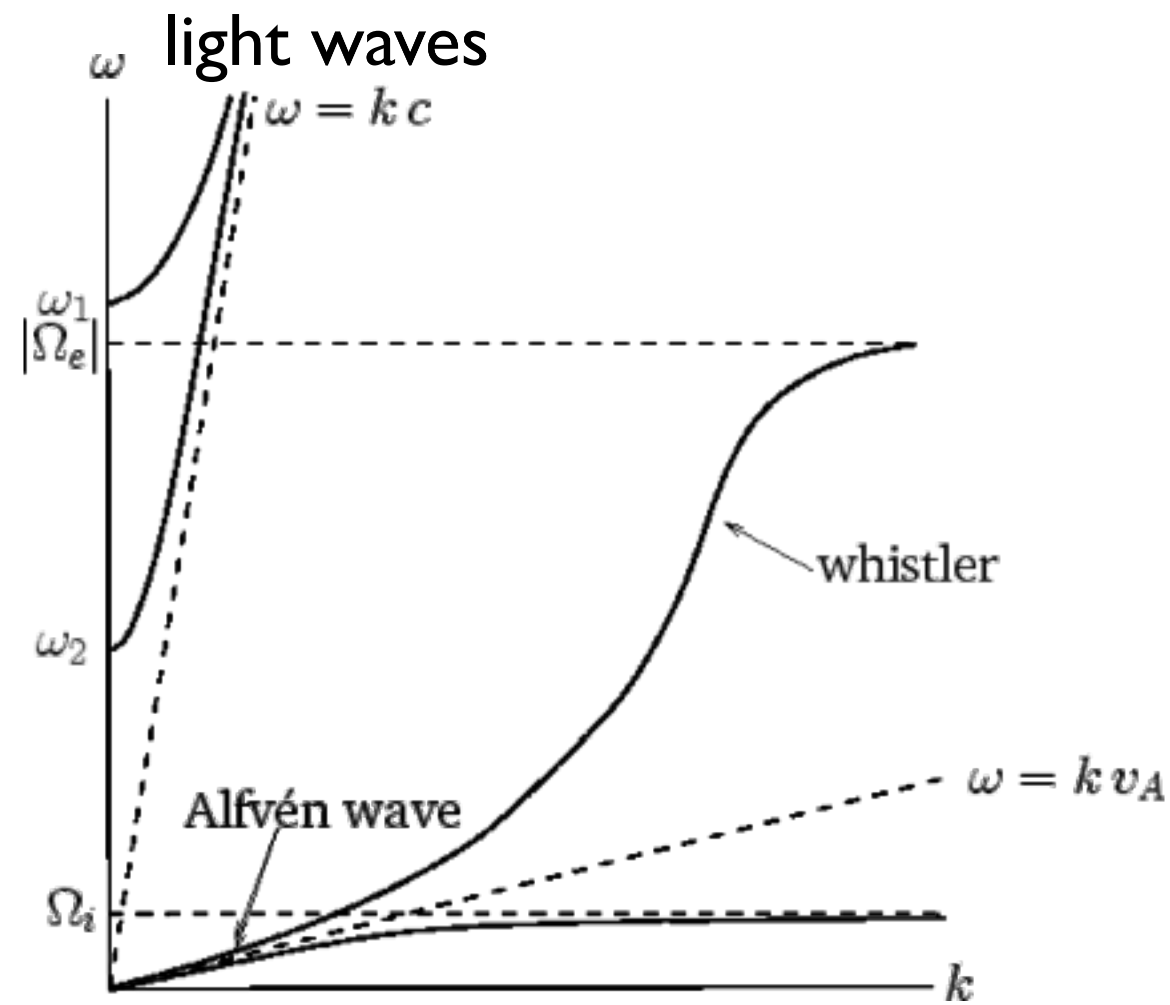
- Physical insight: ions gyrate LH, electrons RH. LHP waves have stronger ion interaction (and cyclotron resonance), RHP waves interact with electrons

- Same set of waves contained in either basis, some are easier to extract from CP

Cold plasma wave zoology

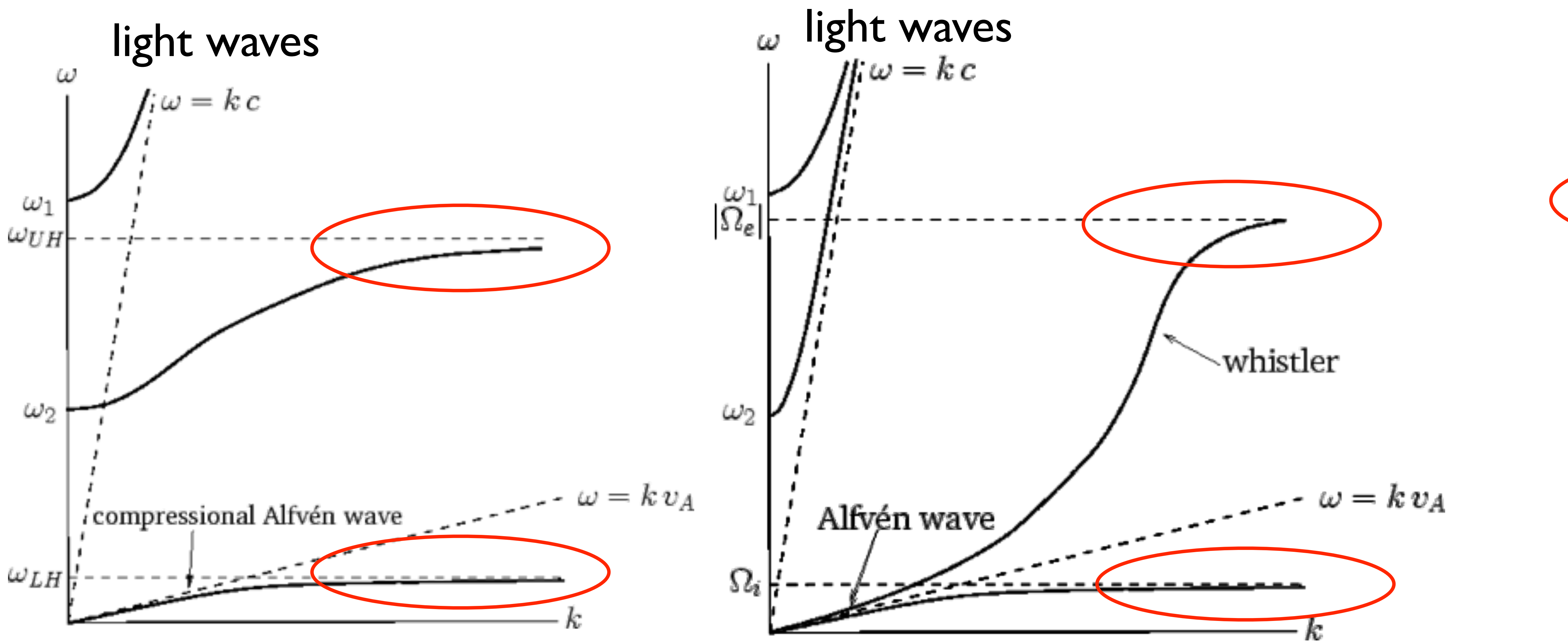


Perp. propagating ($\mathbf{k} = k_x \hat{x}$)



Parallel propagating ($\mathbf{k} = k_z \hat{z}$)

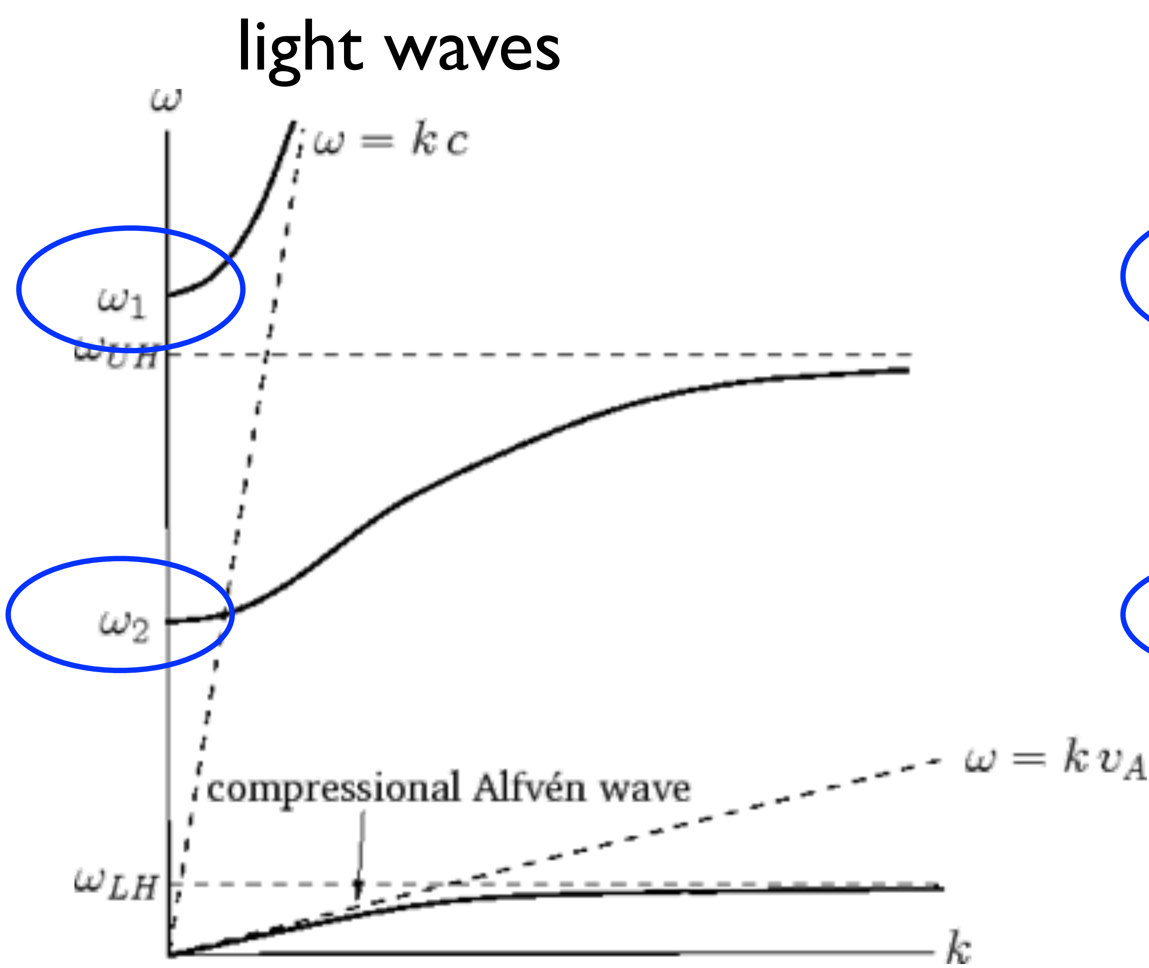
Cold plasma wave zoology



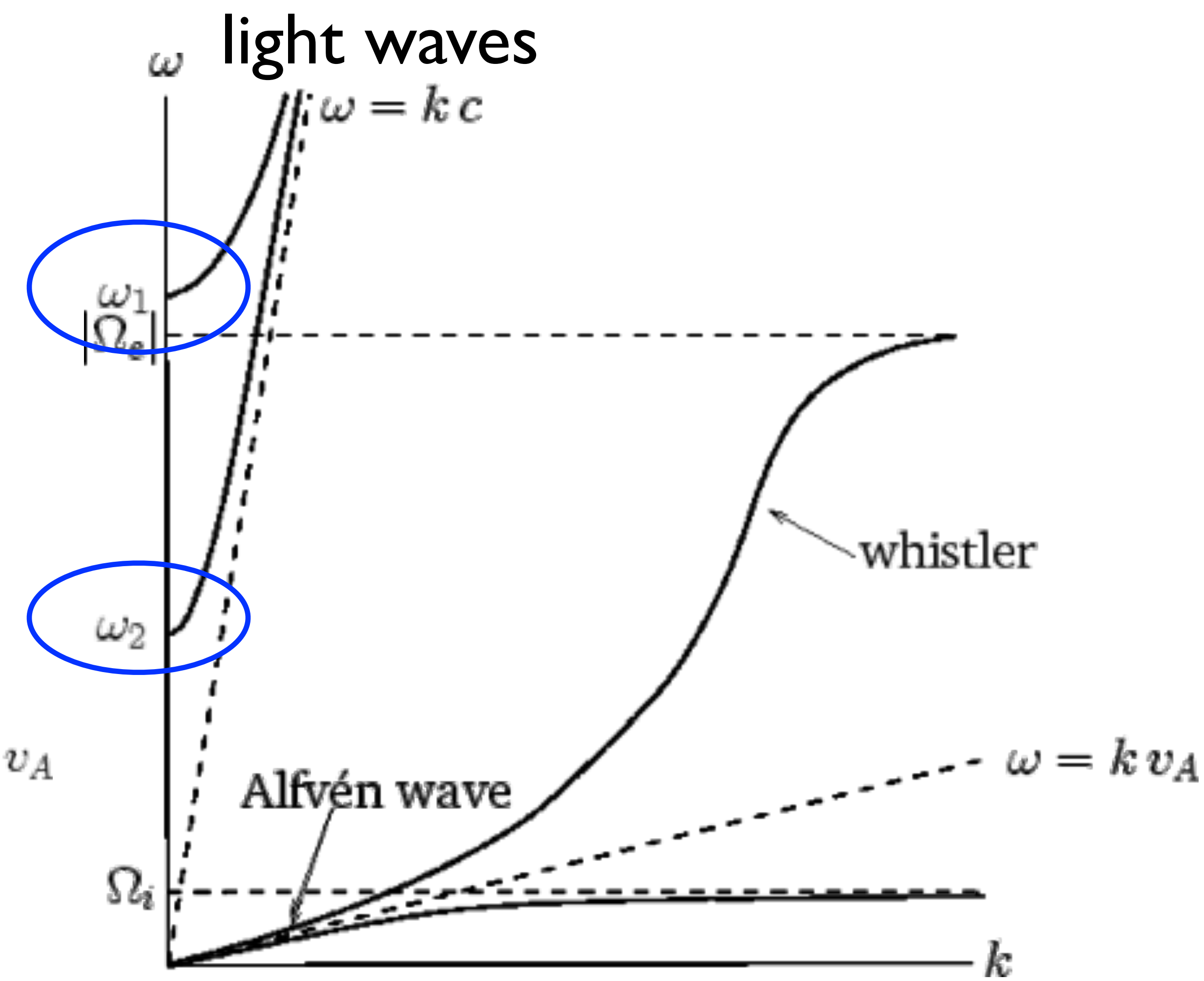
Perp. propagating $(\mathbf{k} = k_x \hat{x})$ Parallel propagating $(\mathbf{k} = k_z \hat{z})$

- Resonance, $n^2 \rightarrow \infty$, when waves resonate with particle motion

Cold plasma wave zoology



Perp. propagating $(\mathbf{k} = k_x \hat{x})$

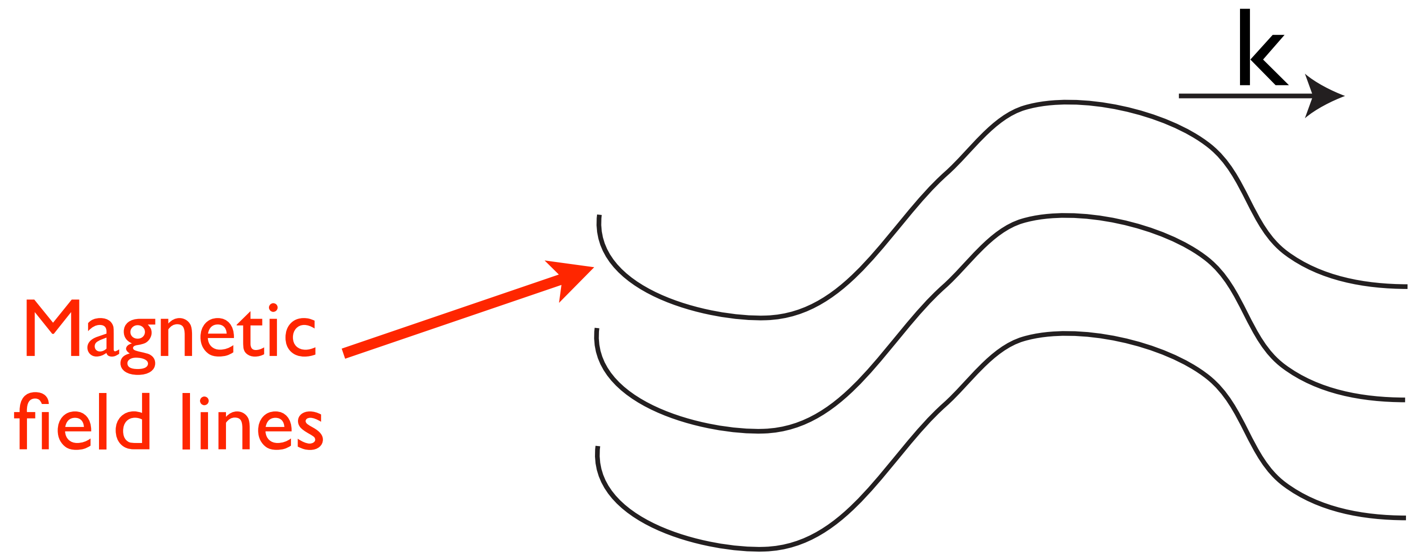


Parallel propagating $(\mathbf{k} = k_z \hat{z})$

- Resonance, $n^2 \rightarrow \infty$, when waves resonate with particle motion
- Cutoffs for $n^2=0$, transverse EM waves will not propagate below these frequencies (evanescent)

Low frequency waves: Alfvén waves

- For freq. much less than ion cyclotron frequency, primary waves are Alfvén waves



Shear Alfvén wave

- Primary motion: ExB motion of electrons and ions together (D→0)
- To pull this out of our cold plasma model:

$$\mathbf{k} = k_z \hat{z} \ (\theta = 0)$$

Shear wave in cold plasma model

$$\begin{pmatrix} S - n^2 & 0 & 0 \\ 0 & S - n^2 & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$n^2 = S = \cancel{1} - \frac{\omega_{pi}^2}{\cancel{\omega^2} - \Omega_i^2} - \cancel{\frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2}} \approx \frac{\omega_{pi}^2}{\Omega_i^2} = \frac{c^2}{v_A^2}$$

$$\omega^2 = k_\parallel^2 v_A^2 \quad ; \quad v_A^2 = \frac{B^2}{\mu_o m_i n_i}$$

- Like wave on string: magnetic field plays role of tension, plasma mass → string mass

Alfvén waves from MHD

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$ Continuity

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \vec{j} \times \vec{B}$$ Momentum

$$\vec{E} + \vec{v} \times \vec{B} = 0$$ Ohm's Law (electron momentum)

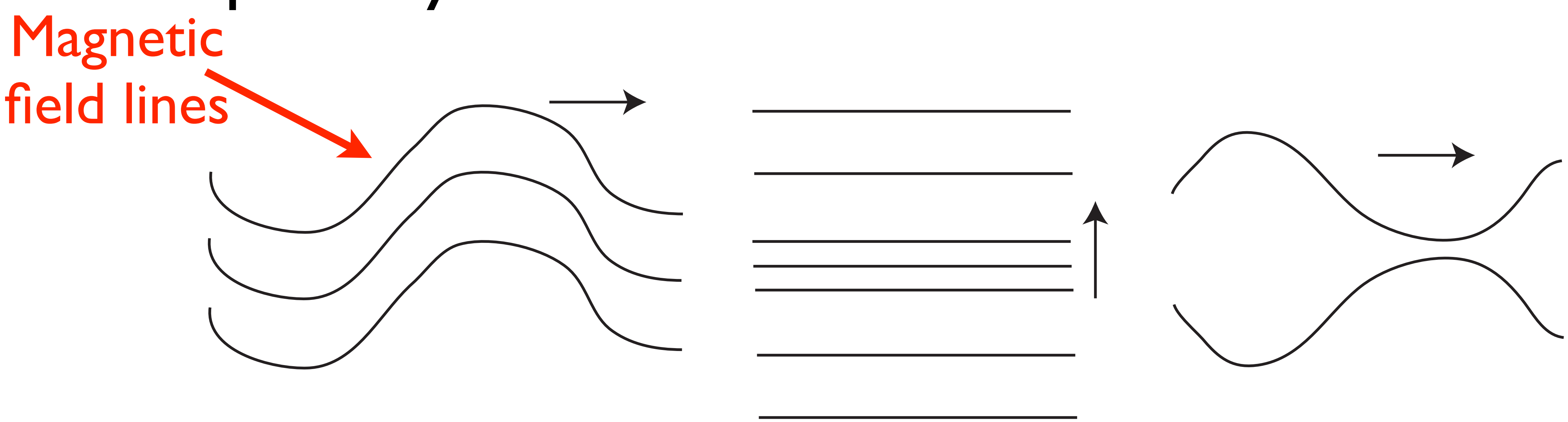
$$\frac{d}{dt} \left(\frac{p}{\rho^\gamma} \right) = 0$$ Pressure closure (adiabatic)

+ Maxwell's Equations

- Linearizing this system reveals four waves: fast and slow magnetosonic waves, the shear Alfvén wave, and the entropy wave

MHD Waves

- For freq. much less than ion cyclotron frequency, primary waves are Alfvén waves



Shear Alfvén wave

$$\omega^2 = k_\parallel^2 v_A^2$$

Compressional
Alfvén wave
(fast magnetosonic)

Slow magnetosonic

$$\omega^2 = \frac{k^2}{2}\left(c_s^2 + v_A^2 \pm \sqrt{c_s^4 + v_A^4 - 2c_s^2 v_A^2 \cos 2\theta}\right)$$

sound wave response (in fast/slow modes)
not in our cold model

Shear wave dispersion derivation

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}\right)=-\nabla p+\vec{j}\times\vec{B}=-\nabla\left(p+\frac{B^2}{2\mu_o}\right)+\frac{\vec{B}\cdot\nabla\vec{B}}{\mu_o}$$

magnetic pressure

magnetic tension

$$\frac{\partial\vec{B}}{\partial t}=-\nabla\times\vec{E}=\nabla\times\vec{v}\times\vec{B}$$

- We are looking for the shear wave, so we'll make appropriate assumptions:

$\vec{k}\cdot\delta\vec{v}=0$ incompressible motion

$\vec{B}=B_o\hat{z}+\delta B\hat{x}$ no field line compression, linearly polarized

$\delta B,\delta v\propto\exp(i\vec{k}\cdot\vec{r}-i\omega t)$ plane waves

$\delta p=0$ follows from the first assumption, adiabatic assumption

Shear wave dispersion derivation, cont

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\cancel{\vec{v}\cdot\nabla\vec{v}}\right)=-\nabla\left(\cancel{p}+\cancel{\frac{B^2}{2\mu_o}}\right)+\frac{\vec{B}\cdot\nabla\vec{B}}{\mu_o}$$

$$-i\omega\rho\delta\vec{v}=\frac{ik_{\|}B_o\delta B}{\mu_o}\hat{x}$$

$$\frac{\partial\vec{B}}{\partial t}=\nabla\times\vec{v}\times\vec{B}$$

$$-i\omega\delta B=ik_{\|}\delta v B_o$$

- Combine these two to get:

$$\boxed{\omega^2=k_{\|}^2\frac{B^2}{\mu_o\rho}=k_{\|}^2 v_A^2}$$

Currents in MHD AW

$$\vec{j} = \frac{1}{\mu_o}\nabla \times \vec{B} = \frac{1}{\mu_o} i\vec{k} \times (\delta B\hat{x}) \qquad \delta\vec{E} = -\delta\vec{v} \times B_o\hat{z} = -\delta v B_o\hat{y}$$

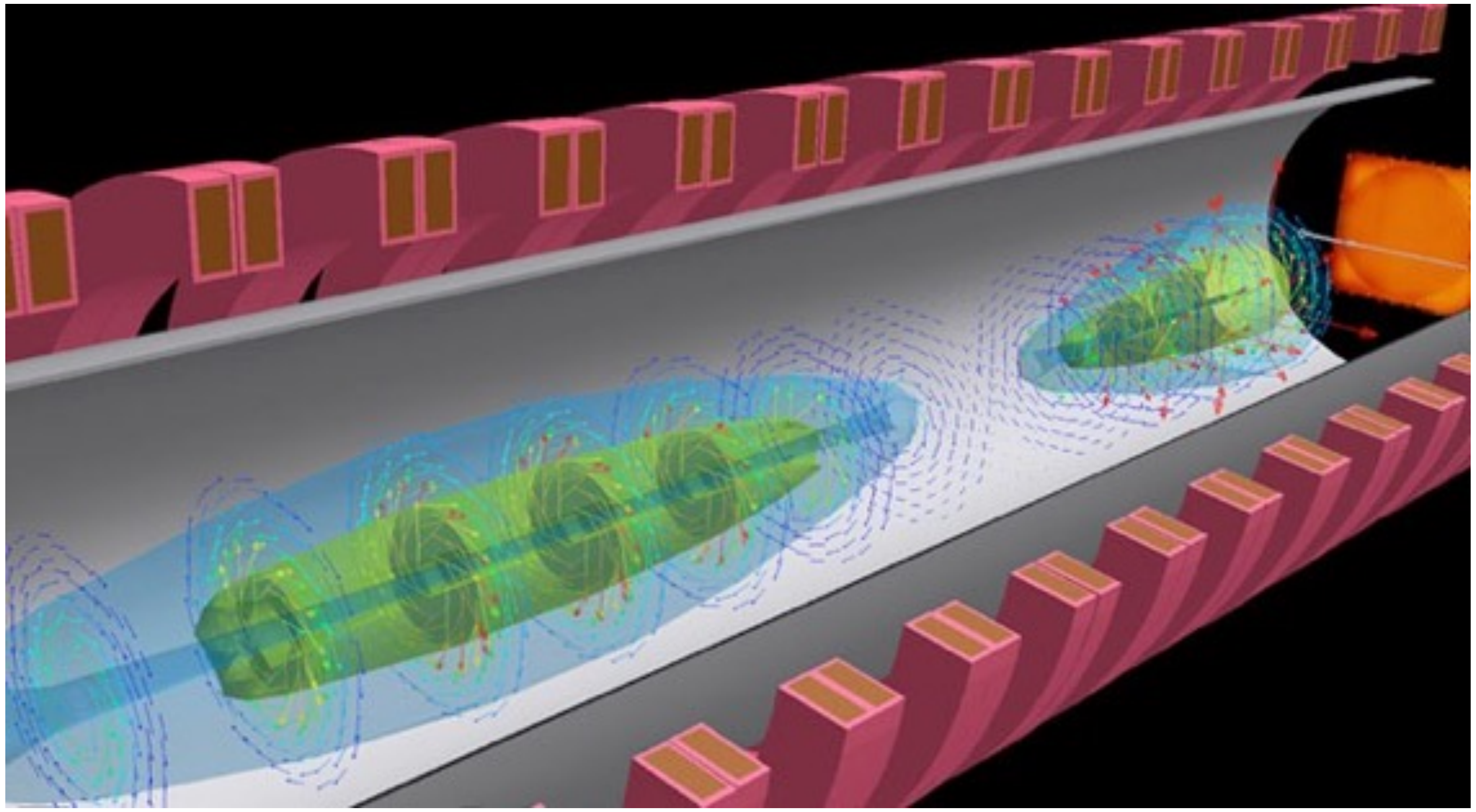
$$\delta\vec{E} = \frac{k_\parallel B_o^2}{\omega\rho\mu_o}\delta B\hat{y}$$

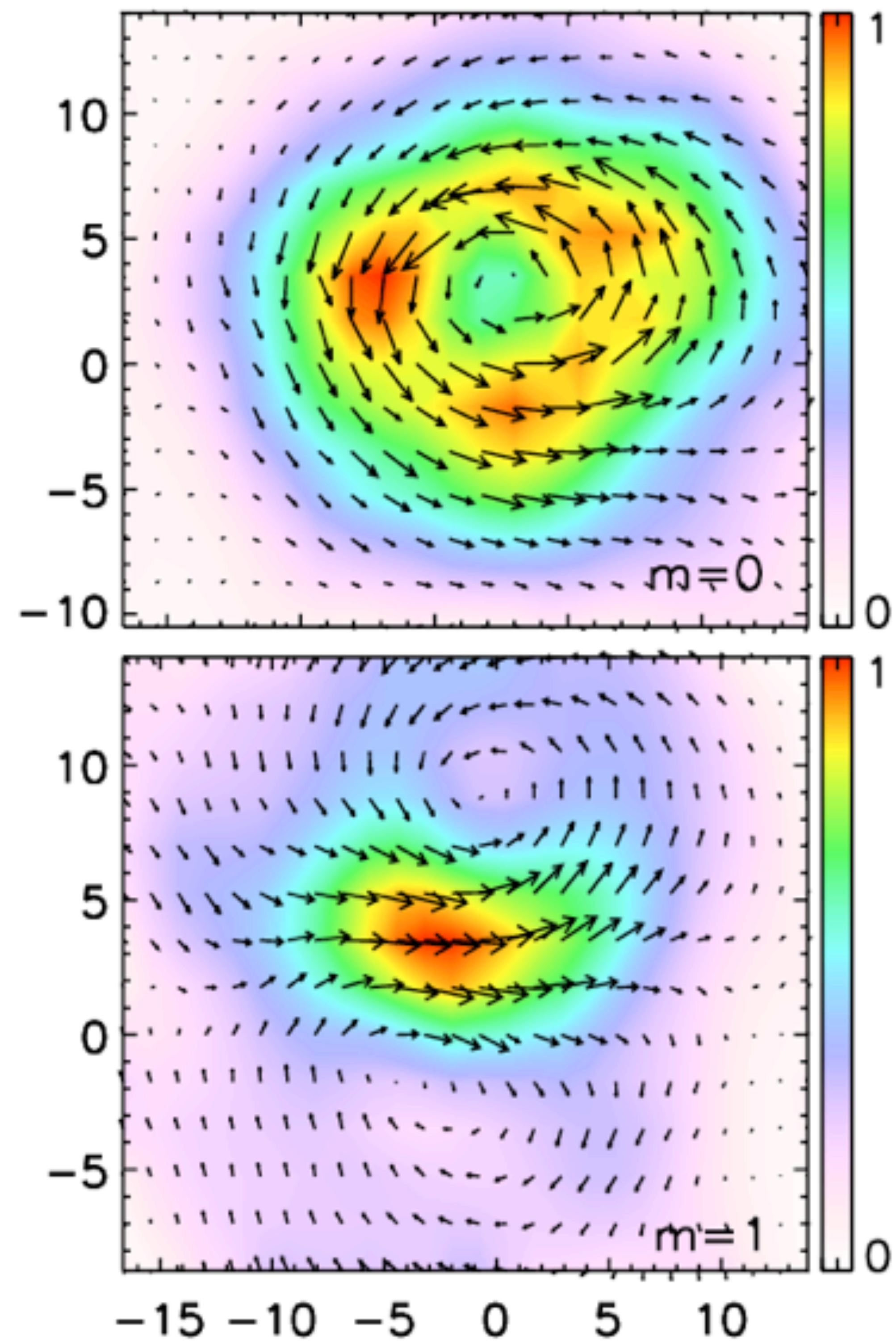
$$\boxed{\vec{j} = -\frac{i\omega ne}{\Omega_i}\frac{\delta\vec{E}}{B_o} - \frac{ik_y\delta B}{\mu_o}\hat{z}}$$

Polarization current

- Current in $k_\perp=0$ AW is entirely due to ion polarization current: no field aligned current
- As $k_\perp$ is introduced, current closes along the field (inductively driven)

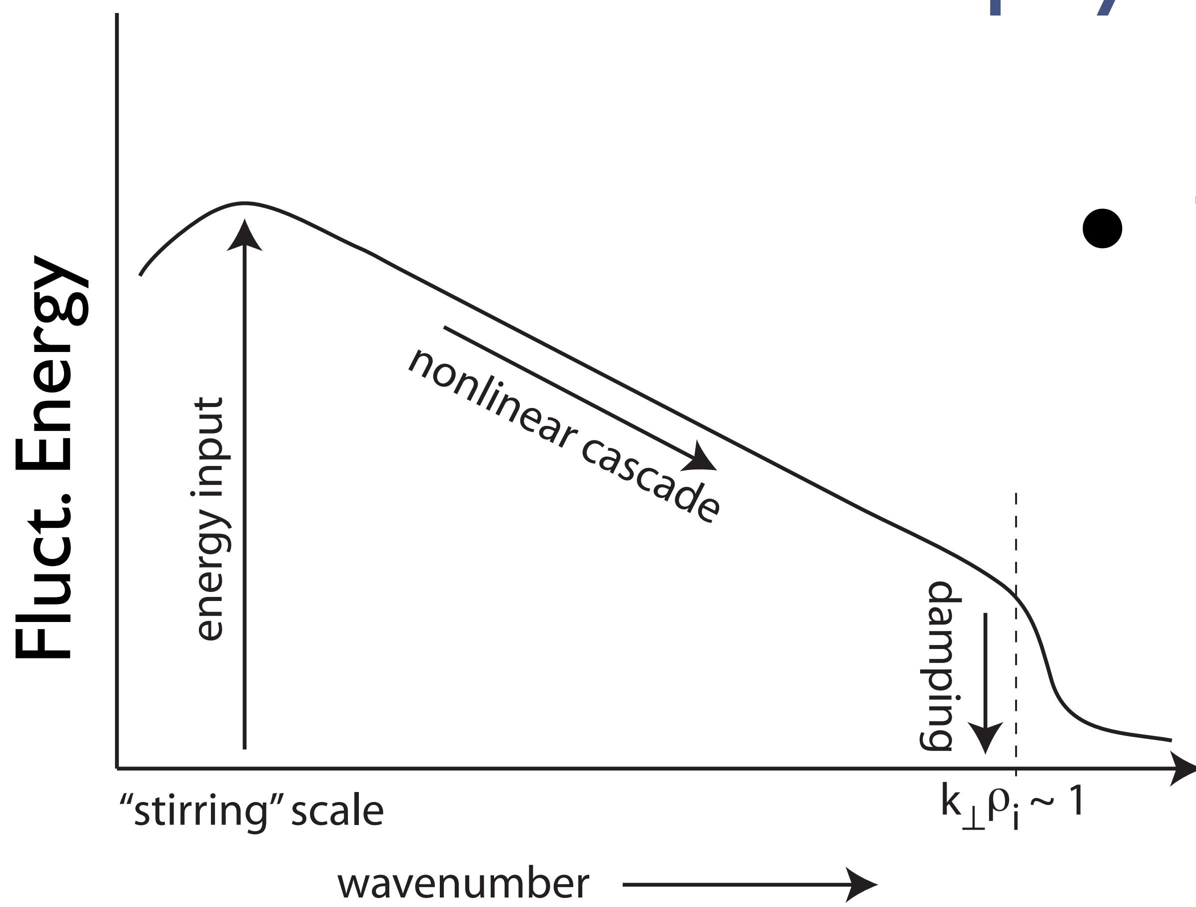
Alfvén waves in the lab





- Above: 3D measurements of shear AW fields and currents in LAPD at UCLA
- Right: Movie of B of SAW eigenmodes in cross-field plane in LAPD
- AWs important in fusion devices: AEs excited by fast particles (e.g. alphas)

Alfvén waves in astrophysics: MHD turbulence



- Turbulence in magnetized plasma can be thought of as a collection of weakly interacting linear waves

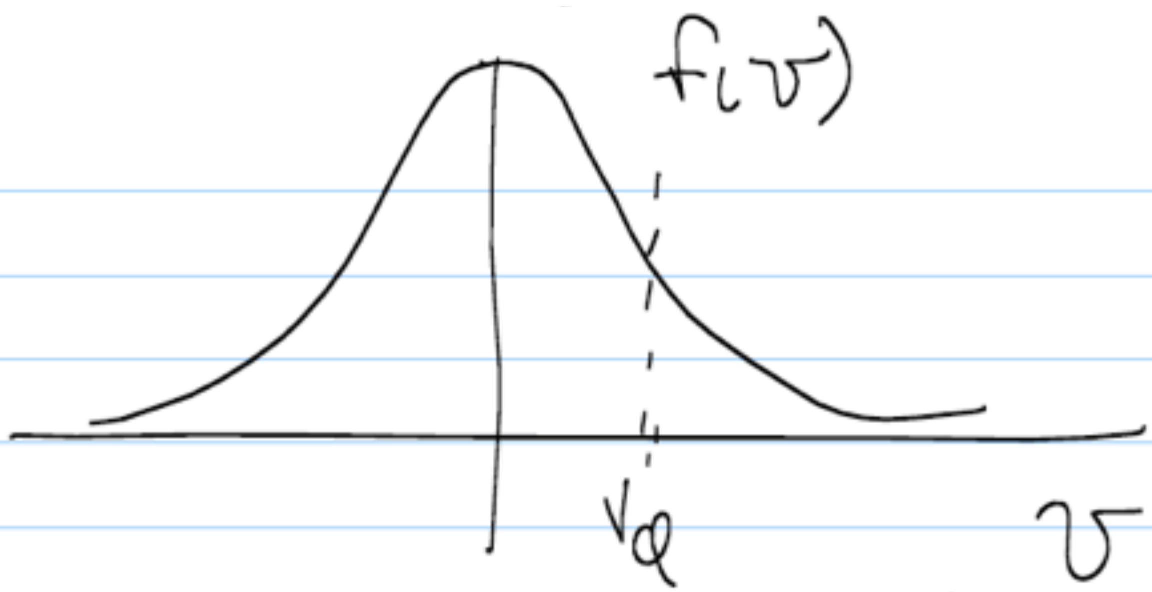
- Accretion disk: large scale instability (MRI) injects Alfvén waves at small k

- AWs nonlinearly interact (shred each other apart) to generate smaller and smaller scale waves, until dissipation scale is reached, waves damp, energy goes into particles

- Theory needed to explain luminosity; MHD turbulence important in other astro contexts (e.g. solar wind, ISM)

Collisionless damping: Landau Damping

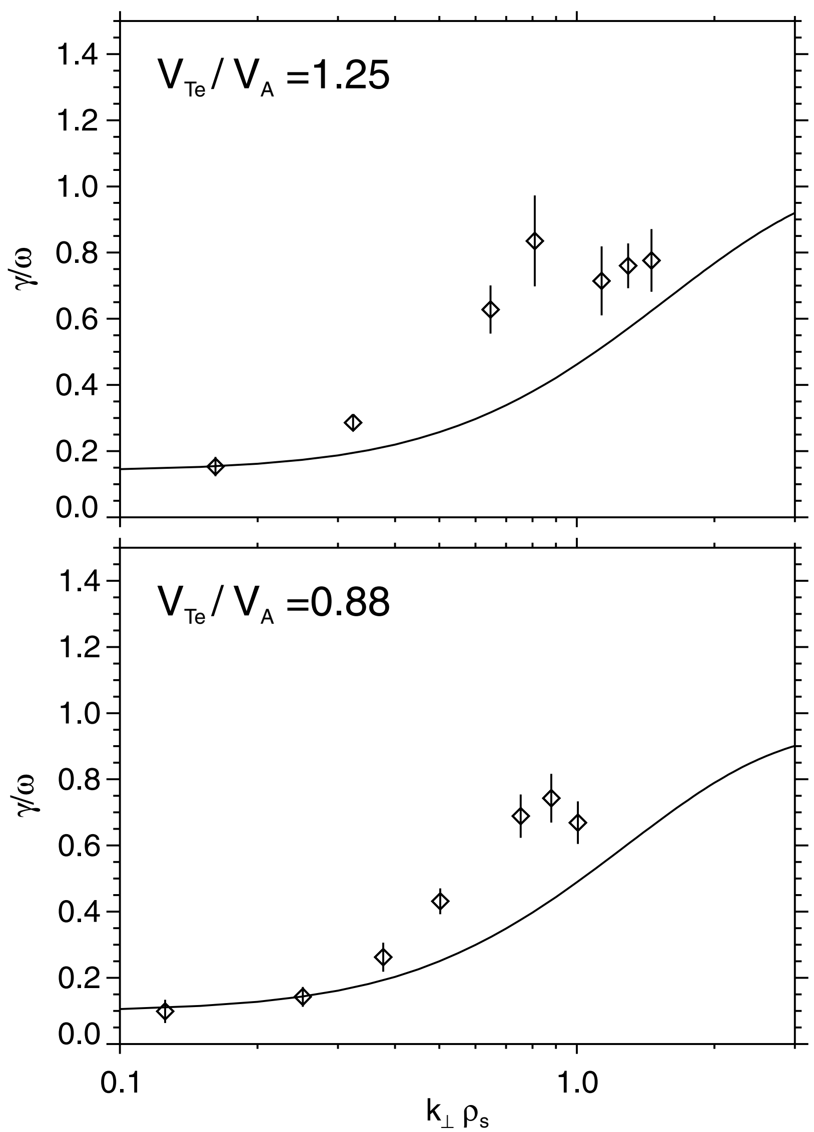
- Warm plasma: particles have finite thermal speed, some particles might have speed comparable to wave phase speed, can resonate



Resonant particles ride with the wave, see "DC" E-field, can exchange energy

- Particles slightly slower than wave get accelerated, take energy from wave; particles moving slightly faster are decelerated, give energy back to wave
- Can have wave damping or growth: $\gamma \propto \left.\frac{\partial f}{\partial v}\right|_{v=v_\phi}$

Collisionless damping of Alfvén waves



LAPD data

- Ideal MHD AW does not have $E_{||}$, can not Landau damp
- AW with large $k_\perp$ (violating MHD assump.) develops $E_{||}$, can model with generalized Ohm's law:

$$\vec{E} + \left(V - \frac{\vec{j}}{ne} \right) \times \vec{B} + \frac{1}{ne} \nabla p_e = \frac{m_e}{e^2 n} \frac{\partial \vec{j}}{\partial t}$$

ρ_s

$\frac{c}{\omega_{pe}}$

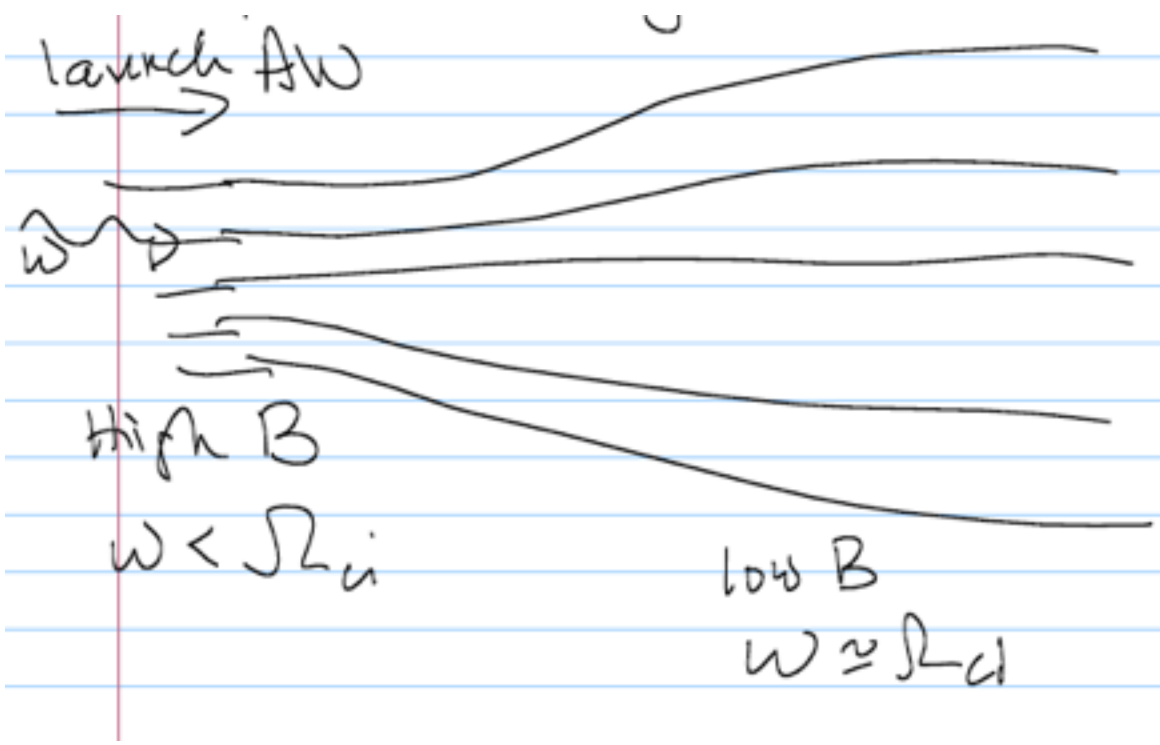
- Get dispersive kinetic Alfvén wave, which can Landau damp

- Damping of AW cascade only happens at small enough scale, requiring NL cascade to damp energy in MRI

Cyclotron resonance and heating

- Another collisionless damping mechanism: waves with freq. near the cyclotron frequency can resonate with particle gyration; with finite T get absorption

- Used as heating mechanism for fusion plasmas. One of earliest ideas: heating by SAW in magnetic "beach"



- Launch SAW into decreasing field (e.g. from end of mirror machine)

- Waves absorbed where

$$\omega \approx \Omega_i$$

- Modern schemes use fast wave to heat ions (no parallel access to tokamak core)

Higher frequency: whistler waves

Primarily parallel propagating mode with $\Omega_i < \omega < \Omega_e$

Electrons $E \times B$, ions polarize, get finite S and D

Can show in this limit:

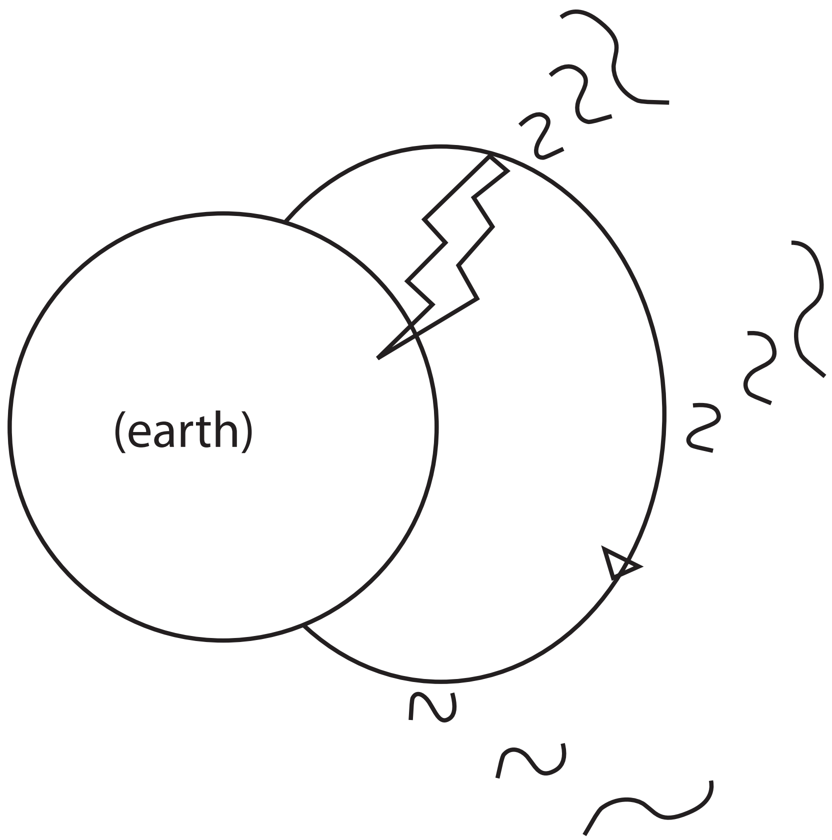
$$\omega \approx \frac{k^2 v_A^2}{\Omega_i}$$

the group velocity of this mode:

$$v_g = \frac{\partial \omega}{\partial k} \approx \frac{2c\sqrt{\omega \Omega_e}}{\omega_{pe}}$$

- Dispersive waves: phase/group velocity depends on frequency of wave

Whistlers in the magnetosphere



- Lightning strikes excite broad range of RF waves in magnetosphere
- Some whistler waves are born at strike site, propagate along earth's dipole field

- Because of dispersion, higher frequency waves go faster than lower frequency: higher freq at front of wave packet
- Whistler in magnetosphere are in audible range of frequencies: picked up by radio/telephone operators in WWI/II; chirp downward in frequency (hence "whistler")

Even higher freq: modified light waves

For $\omega > \omega_{\rm pe}$, have modified light waves, no plasma normal modes (too fast for electrons to keep up!)

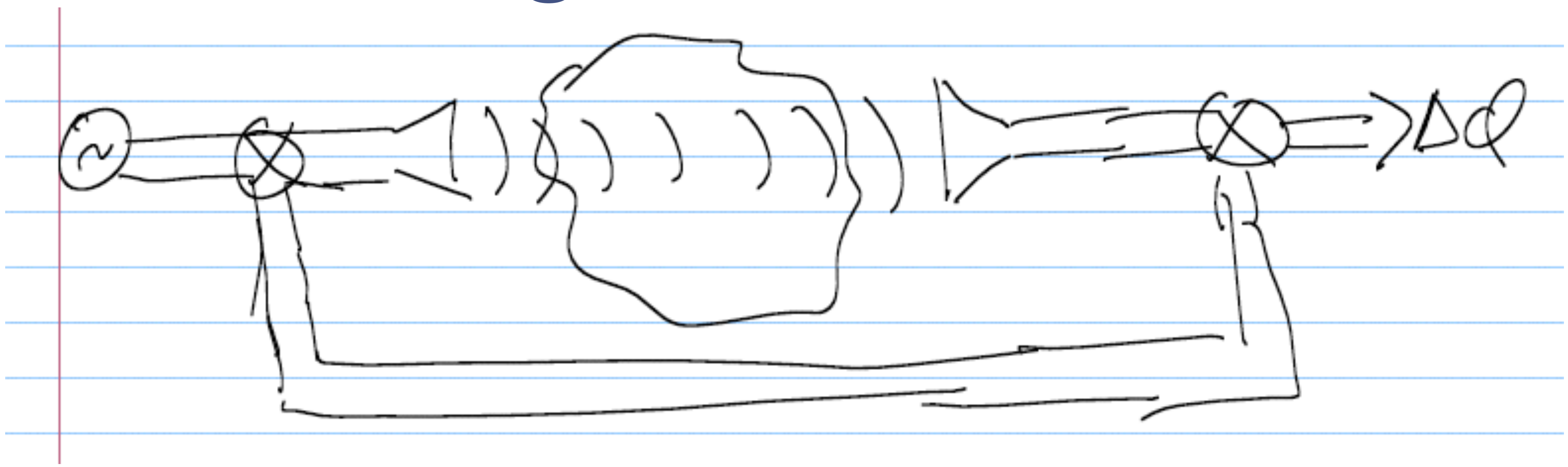
Example: "O-mode" transverse waves, $\mathbf{k} = k\hat{x}$, $\mathbf{E} = E_z\hat{z}$

$$n^2 = P = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\omega^2} \approx 1 - \frac{\omega_{pe}^2}{\omega^2}$$

$$\omega = \frac{kc}{\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}}$$

- Wave cutoff below plasma frequency (evanescent)
- Phase velocity faster than c (wavelength longer in plasma)

Waves as diagnostics: interferometer



- Phase speed of EM wave in plasma depends on density, compare wave sent through plasma with wave in vacuum: plasma wave will advance phase faster, develop phase shift w/ respect to vacuum wave

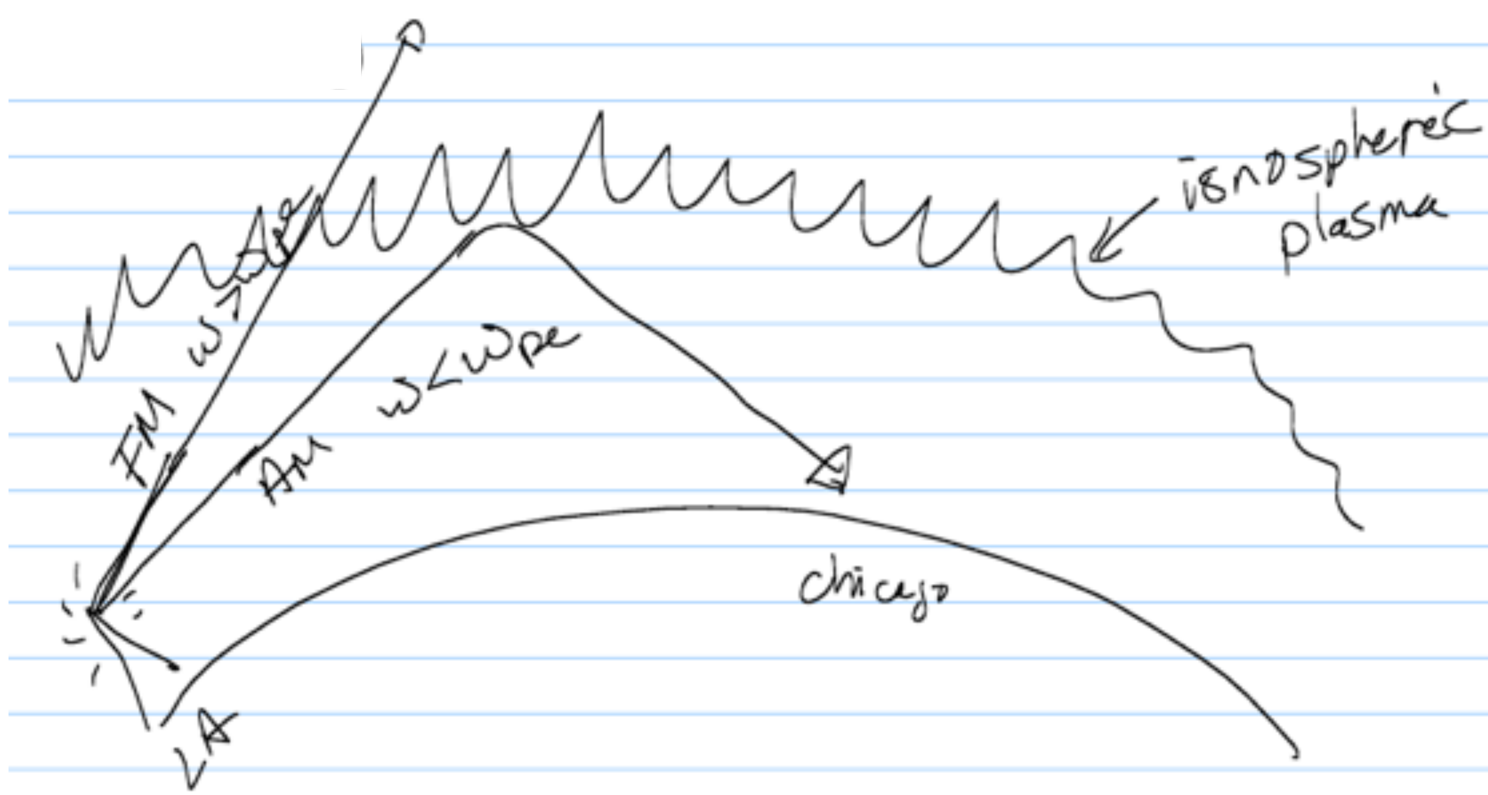
$$\Delta\phi \propto \int n_e dl$$

- Standard lab diagnostic, also used in astronomy

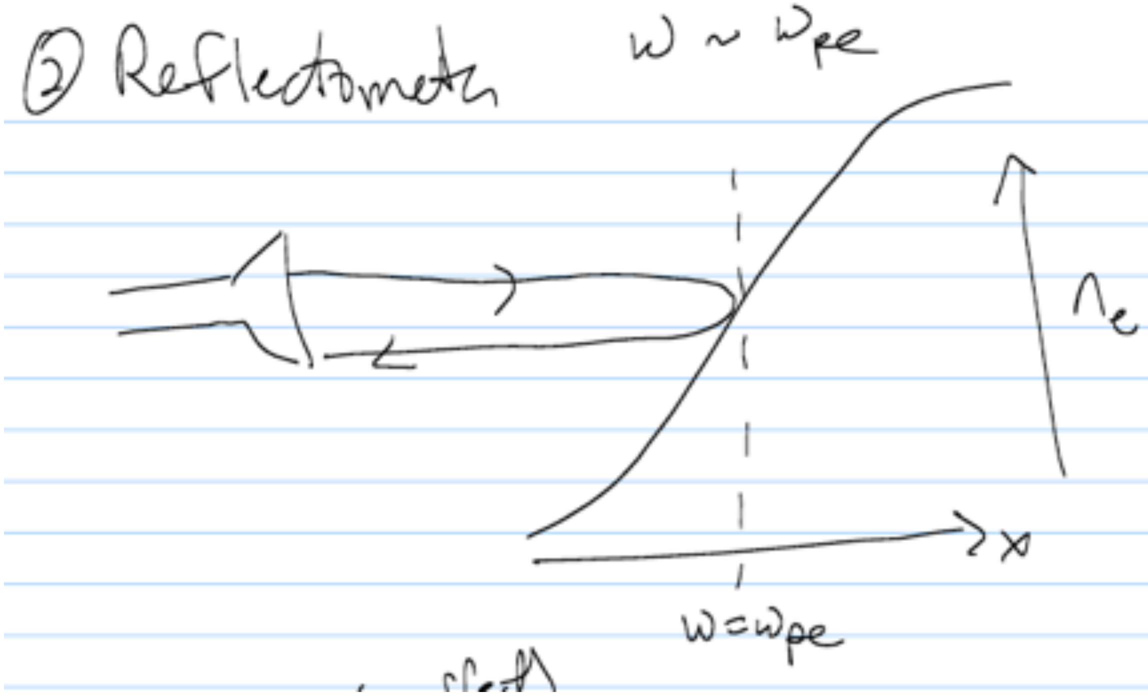
Reflection of light waves by plasma

- Transverse EM waves are cutoff for frequencies below the plasma freq. - incident waves are reflected (similar concept can be applied to reflection from conductors/ metals)

- Natural example: AM radio wave reflection from ionosphere



Reflection as plasma diagnostic

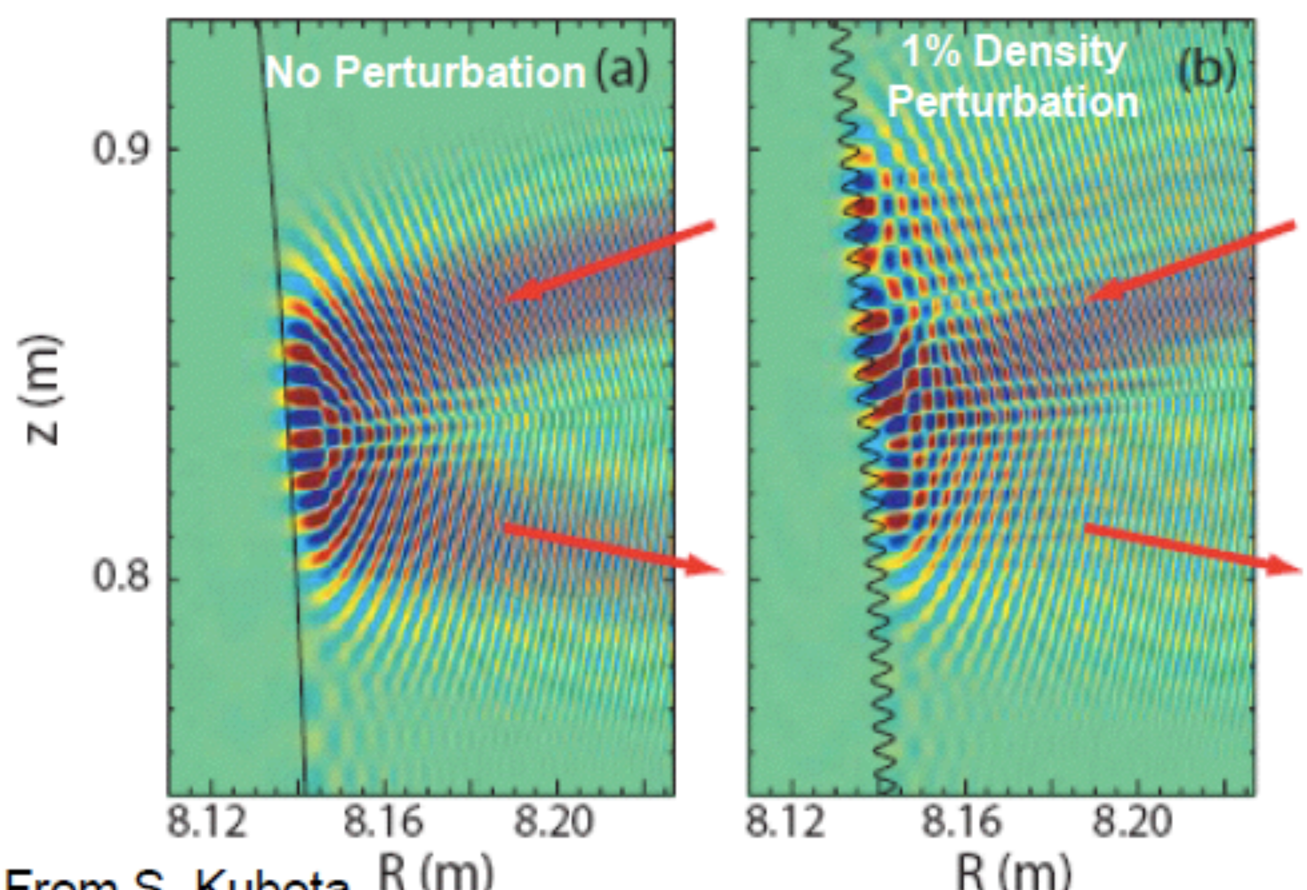


Propagate EM wave (microwave range) into plasma, up dens. grad.

Reflects at cutoff, radar ranging of cutoff surface location (scan freq for profile)

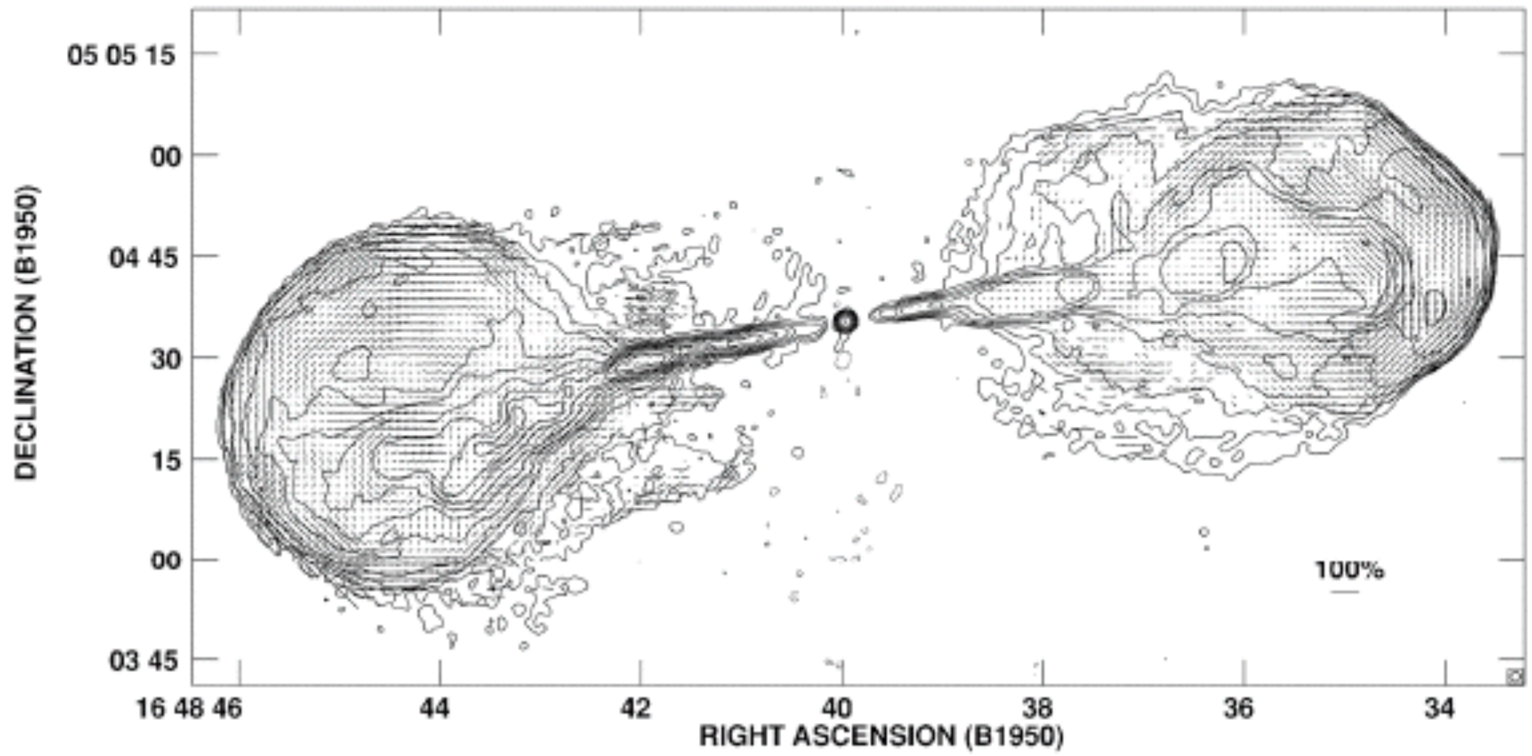
Right: simulation of microwave propagation to cutoff in tokamak

Can also measure turbulence: both through scattering & modulation of reflection



From S. Kubota

Faraday Rotation



- Recall the alternate, circularly polarized formulation of the dielectric
- RHP and LHP components of an EM wave acquire different phase velocity in plasma, polarization rotates while propagating along B
- Measure line-of-sight B in lab plasmas, astrophysical plasmas (above: astro jet)