Geodesic Acoustic Mode Induced by Energetic Particles in Tokamak

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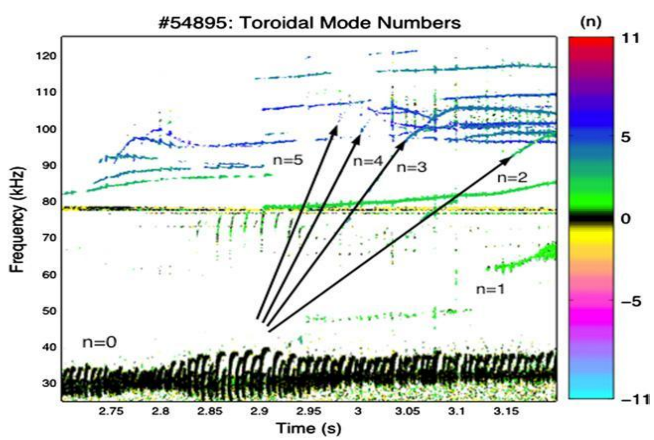
Outline

1. $n = 0$ GAM observed in JET and interprated as energetic particle driven GAM(EGAM) by Berk et al. (Nucl. Fusion **46**, 2006)
2. GAM-like mode driven by suprathermal ion observed in DIII-D and simulation done by Nazikian et al. (PRL **101**, 2008)
3. A fluid-kinetic model constructed and numerical simulation done for $n = 0$ EGAM by Fu (PRL **101**, 2008)



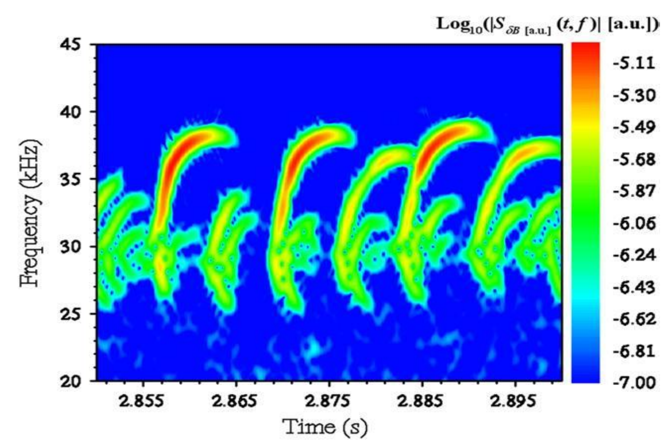


Figure: Berk et al. Nucl. Fusion, 2006

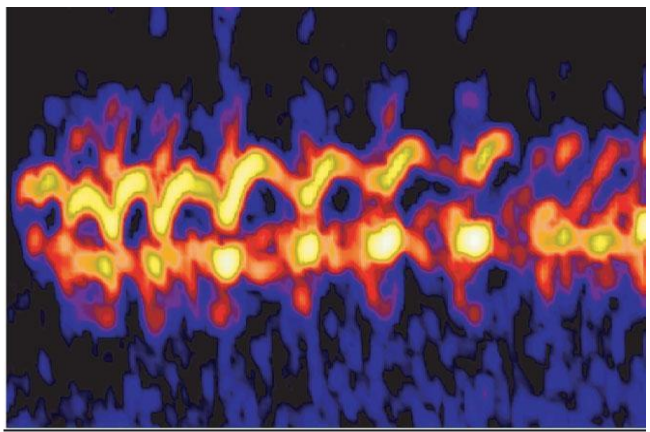


Figure: Nazikian et al. PRL **101**, 2008

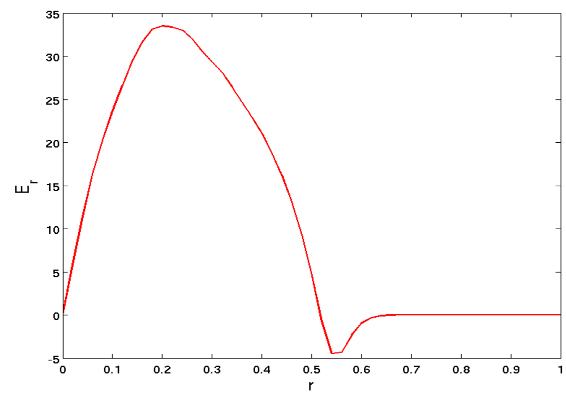


Figure: Global Mode (Fu PRL **101**, 2008)

Electrostatic perturbation

- Kinetic Equation

$$\frac{df_j}{dt} = \frac{e_j}{m_j}\frac{\partial F_j}{\partial E} \vec{v} \cdot \nabla\phi, j = \text{thermal e, ion and energetic ion}$$

- Solution by integrating over equilibrium orbits

$$f_j = \frac{e_j}{m_j}\frac{\partial F_j}{\partial E}[\phi(r,\theta)e^{-i\omega t} + i\omega\int_{-\infty}^{t} dt' e^{-i\omega t'}\phi(r(t'),\theta(t'))].$$

- Quadratic form

$$\mathcal{L} = \int d\Gamma\phi^{+}(r,\theta,t)\sum_j e_j f_j(r,\theta,t,\vec{v}) = \mathcal{L}_e + \mathcal{L}_i + \mathcal{L}_h$$

Treat 3 species differently!

- Fourier expansion into orbit harmonics

$$
\begin{aligned}
\mathcal{L}_j &= \int d\Gamma \phi^+(r,\theta,t) e_j f_j(r,\theta,t,E,\mu,P_\phi) \\
&= \int d\Gamma \frac{e_j^2}{m_j} \frac{\partial F_j(r,E,\mu,P_\phi)}{\partial E} [\phi^+\phi - \sum_{-\infty}^{\infty} \frac{\omega}{\omega - p\,\omega_{bj}} \phi^+_{-p}\phi_p] \\
&= \int d\Gamma \frac{e_j^2}{m_j} \frac{\partial F_j(r,E,\mu,P_\phi)}{\partial E} \sum_{-\infty}^{\infty} [(1 - \frac{\omega}{\omega - p\,\omega_{bj}}) \phi^+_{-p}\phi_p]
\end{aligned}
$$

- Each species moves in different bounce frequency ω_b W.R.T. mode frequency ω, treat the 2nd term differently.

Thermal Electrons

- Bounce frequency $\omega_{be} \gg \omega$, keep the 0th harmonic, treat higher ones perturbatively
$$\frac{\omega}{\omega - p\,\omega_{be}} = \begin{cases} 1 & p = 0 \\ -[\frac{\omega}{p\,\omega_{be}} + (\frac{\omega}{p\,\omega_{be}})^2 + ...] & p \neq 0. \end{cases}$$
- F_e Maxwellian

$$\mathcal{L}_e = \int d\Gamma \frac{e_e^2}{m_e} \frac{\partial F_e(r,E)}{\partial E} [\phi^+(r,\theta)\phi(r,\theta) - \phi_0^+\phi_0]$$
$$\approx \int dr^3 \frac{n_e(r) e_e^2}{m_e T_e} \sum_{m\neq 0} \phi_{-m}^+ \phi_m$$

Notice: $\phi_{m=0}$(the 0^{th} Fourier component in θ) disappears in $\mathcal{L}_e$. To bring T_e in, we have to keep $\phi_{m=\pm 1}$ in $\mathcal{L}_e$.

Thermal Ions

- Bounce frequency $\omega_{bi} \ll \omega$,

$$
\begin{aligned}
1 - \frac{\omega}{\omega - p\,\omega_{bi}} &= -\frac{p\,\omega_{bi}}{\omega} - (\frac{p\,\omega_{bi}}{\omega})^2 - \ldots. \\
&\sim \frac{1}{\omega}\frac{d}{dt} - \frac{1}{\omega^2}\frac{d}{dt}\frac{d}{dt} - \ldots \\
&\sim \frac{i}{\omega}\vec{v}\cdot\nabla - \frac{1}{\omega^2}\vec{v}\cdot\nabla\,\vec{v}\cdot\nabla - \ldots
\end{aligned}
$$

  the $1^{st} \sim$ understood inside quadratic form $\int d\tau \phi^+ \ldots \phi$

- F_i Maxwellian

$$
\mathcal{L}_i = \int d\Gamma \frac{e_i^2}{m_i}\frac{\partial F_i(r,E)}{\partial E}\Big[\frac{i}{2\omega}(\phi^+\,\vec{v}\cdot\nabla\phi - \phi\,\vec{v}\cdot\nabla\phi^+) \\
-\frac{1}{\omega^2}\vec{v}\cdot\nabla\phi^+\,\vec{v}\cdot\nabla\phi\Big]
$$

Polarization Response for Thermal Ion

- Polarization term emerging from FLR

$$\frac{1}{4\pi}\int d^3 r \phi^+(\vec{r})\nabla_\perp \frac{\omega_{pi}^2}{\omega_{ci}^2}\cdot\nabla_\perp\phi(\vec{r})$$

- In terms of $m = 0$ component ϕ, $\mathcal{L}_e$ and $\mathcal{L}_i$ contribute to $\mathcal{L}$

$$\frac{1}{4\pi}\int dr^3 \phi^+ \frac{1}{r}\frac{\partial}{\partial r}\frac{\omega_{pi}^2}{\omega_{ci}^2}(1-\frac{\Omega_{GAM}^2}{\omega^2})\frac{\partial\phi}{\partial r}$$

$$\Omega_{GAM}^2 \equiv \frac{1}{m_i R^2}(\frac{7T_i}{2}+2T_e)(1+\frac{1}{2q^2})$$

Energetic Ion Response (Trapped & Passing)

- Bounce frequency $\omega_{bh} \sim \omega$. No perturbation expansion parameter available, keep all terms in f_h.
- $D(\omega)$ is self-adjoint in $\mathcal{L}_j \equiv \int d\Gamma \phi^+ D(\omega)\phi$, $\phi^+ \to \phi$ in $\mathcal{L}_h$.
- Only $\phi(m = 0$ Fourier component of $\phi(r, \theta)$ in θ) is kept in $\mathcal{L}_h$ since we find $\phi_{\pm 1}/\phi_0 \sim k_\perp \rho_s \ll 1$, $\rho_s \equiv \sqrt{\frac{T_e}{m_i \omega_{ci}^2}}$

$$\mathcal{L}_h = \int d\Gamma \frac{e_h^2}{m_h} \frac{\partial F_h(r, E, \mu, P_\phi)}{\partial E} \left\{ \left[(\phi - \phi_0)^2 \right]_0 - 2\omega^2 \sum_{p=1}^{\infty} \frac{\phi_p^2}{\omega^2 - p^2 \omega_{bh}^2} \right\}$$

Integro-differential Equ. v.s. Differential Equ.

- Solve integro-differential Equation $\mathcal{L}[\phi, \phi'] = 0$ if the orbit width Δ_b is comparable with the mode width

$$k_\perp \Delta_b \sim 1$$

- Differential equation can be obtained in the thin orbit limit,

$$|k_\perp|\Delta_b \ll 1, \quad |\frac{1}{F_h}\frac{\partial F_h}{\partial r}|\Delta_b \ll 1, \quad |\frac{1}{\phi}\frac{\partial \phi}{\partial r}|\Delta_b \ll 1$$

In this limit one can expand $\phi(r)$ in $\mathcal{L}_h$ around r_0, the guiding center for trapped particles or the reference position for passing particles

$$\mathcal{L}_h[\phi] \Longrightarrow \mathcal{L}_h[\phi, \phi', \phi'', \phi''']$$

Variation of $\mathcal{L}$ w.r.t. ϕ leads to 4^{th} order differential equation for ϕ

Assumptions

- Orbit width is thin but comparable to the mode width, so deal with integro-differential equation directly.
- Effects from outside of energetic particles are neglected, so $\frac{d\phi}{dr} = 0$ at the boundary of energetic particles
- Energetic particles are equally distributed around the drift center r_0 corresponding to $P_{\phi 0}$
- Contribution is mainly from particles in the range where F_h is linearly increasing in E

$$F_h(E, \mu, P_\phi) \propto (E - E_{min})\delta(\mu - \mu_0)\delta(P_\phi - P_{\phi 0})$$

Pendulum Equation

$$\ddot{\theta} = -\omega_{b0}^2 \sin\theta\,; \quad \omega_{b0} \equiv \sqrt{\mu B_0 \epsilon}/(qR)$$
$$-\theta_T \le \theta \le \theta_T,\ \theta_T - -\text{turning angle}$$
$$0 \le \theta_T \le \theta_r,\ \theta_r - -\text{largest excursion poloidal angle}$$

- Exact Bounce Frequency

$$\omega_b(k^2(\theta_T)) = \frac{\pi/2}{K(k^2(\theta_T)}\omega_{b0};\ k \equiv \sin(\theta_T/2)$$
$$K(k^2(\theta_T)) = \int_0^{\pi/2} d\eta(1 - k^2(\theta_T)\sin^2\eta)^{-1/2}$$

- Angle

$$\theta(t \mid \theta_T) = 2\arcsin\{k\,\mathrm{sn}(\omega_{b0}t|k^2)\} \approx 2k\,\mathrm{sn}(\omega_{b0}t|k^2),$$
$$= \frac{4\pi}{K(k^2)}\sum_{n=0}^{\infty}\frac{p^{n+1/2}}{1-p^{2n+1}}\sin[(1+2n)\omega_b t] \approx \theta_T\sin(\omega_b t);$$
$$p \equiv \exp[-\pi K(1-k^2)/K(k^2)]$$

Varying of Minor Radius

- Canonical Angular Momentum Conservation

$$MR_0 v_{\|} - e\psi(r_0 + \delta r) = -e\psi(r_0) \Longrightarrow \delta r(t|\theta_T) = \frac{Mv_{\|}}{eB_p}$$

- Energy conservation

$$v_{\|} = \pm\sqrt{2(E - \mu B)} = \pm\sqrt{2\mu B_0 \epsilon(cos\theta - cos\theta_T)}$$

$$\delta r(t|\theta_T) = \Delta(\theta_T)\mathrm{cn}(\omega_{b0} t|k^2(\theta_T)),\ \Delta(\theta_T) \equiv \frac{q\sqrt{\mu B_0 \epsilon}}{\epsilon\,\omega_{c_i}} 2k(\theta_T)$$

$$= \quad \Delta(\theta_T)\frac{8k(\theta_T)\pi}{K[k^2(\theta_T)]}\sum_{n=0}^{\infty}\frac{p^{n+1/2}}{1+p^{(1+2n)}}\cos[(1+2n)\omega_b t]$$

$$\approx \quad \Delta(\theta_T)\ \cos(\omega_b t), \qquad k(\theta_T) \ll 1.$$

- *Symmetries of Orbits*

$$\begin{aligned}\theta(t) &= \theta(\frac{T}{2} - t);\\ \theta(-t) &= -\theta(t);\\ \delta r(t \pm \frac{T}{2}) &= \delta r(t);\\ \delta r(-t) &= \delta r(t)\end{aligned}$$

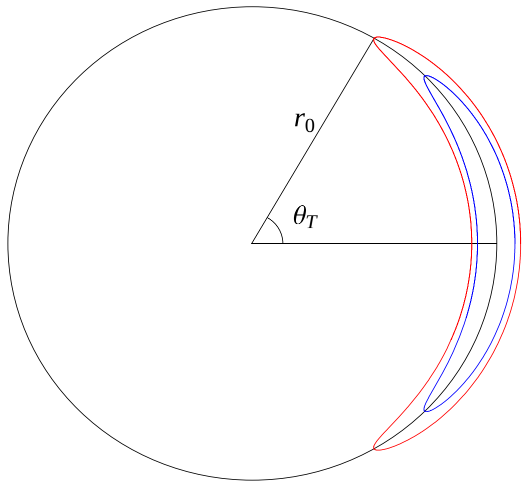


Figure: banana orbits

- Expand $\phi(r_0 + \delta r)$ in terms of basis

$$\phi(r_0 + \delta r) = \sum_0^{\infty} C_n \sin[\frac{(1+2n)\pi}{2} \frac{\delta r}{\Delta(\theta_r)}]$$

- Quadratic Form for ϕ

$$\frac{1}{4\pi}\int d^3r\frac{\phi}{r}\frac{\partial}{\partial r}r\frac{\omega_{pi}^2}{\omega_{c_i}^2}(1-\frac{\Omega_{GAM}^2}{\omega^2})\frac{\partial\phi}{\partial r}+\int dEd\tau\,\frac{e_h^2}{M_h}\frac{\partial F_h}{\partial E}\cdot$$
$$\left[\phi^2+\sum_{q,p}\frac{\omega}{p\omega_b-\omega}\phi_q\phi_p\exp(i(p+q)\omega_b t)\right]$$

- In terms of basis functions, a variational algebraic dispersion function is obtained

$$D(\omega,\alpha)\equiv(1-\frac{\Omega_{GAM}^2}{\omega^2})\sum_n C_n^2(1+2n)^2-\alpha\sum_{n,n'}C_nC_{n'}(A_{nn'}+B_{nn'}),$$

$$\alpha\equiv\frac{4}{\pi^3}\frac{q^2n_h}{\epsilon^2n_i}$$

- Mode equation

$$(1-\frac{\Omega_{GAM}^2}{\omega^2})C_n(1+2n)^2-\alpha\sum_{n'=0}^{N}C_{n'}(A_{n',n}+B_{n',n})=0$$

Matrix $\mathcal{A}$

$$
\begin{aligned}
& A_{nn'}[k^2(\theta_r)] \\
\equiv\ & \frac{8}{k^2(\theta_r)I(k^2(\theta_r))}\int_0^{k^2(\theta_r)} dzK(z)\int_0^1 dx \\
& \cdot\Big\{\cos\left[\pi(n'-n)\sqrt{z}\,\mathrm{cn}(K(z)x\mid z)/k(\theta_r)\right] \\
& -\cos\left[\pi(1+n+n')\sqrt{z}\,\mathrm{cn}(K(z)x\mid z)/k(\theta_r)\right]\Big\} \\
\approx\ & \frac{8}{k^2(\theta_r)I(k^2(\theta_r))}\int_0^{k^2(\theta_r)} dzK(z) \\
& \cdot\{J_0[(n'-n)\pi\frac{\sqrt{z}}{k(\theta_r)}]-J_0[(1+n+n')\pi\frac{\sqrt{z}}{k(\theta_r)}]\}.
\end{aligned}
$$

$$
I(k^2(\theta_r))\equiv\frac{4}{\pi k^2(\theta_r)}\int_0^{k^2(\theta_r)} dzzK(z)
$$

Matrix $\mathcal{B}$

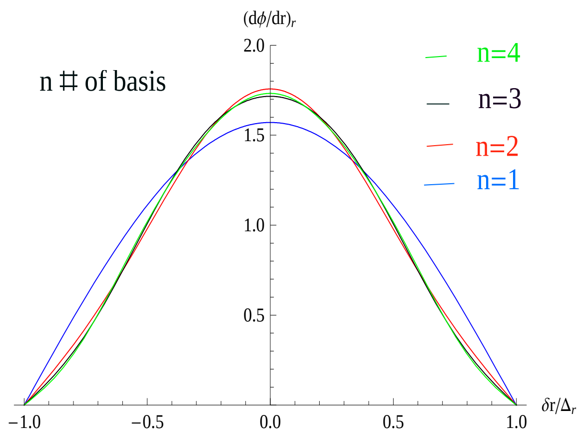
$$
\begin{aligned}
& B_{n,n'}[\omega, k^2(\theta_r)] \\
\equiv \quad & \frac{8}{k^2(\theta_r) I(k^2(\theta_r))} \sum_{p=1,3,\ldots} \int_0^{k^2(\theta_r)} dz \frac{K(z)(\omega/\omega_{b0})^2}{p^2\pi^2/4K^2(z) - (\omega/\omega_{b0})^2} \\
& \cdot \int_{-1}^{1} dx \exp(-ip\frac{\pi}{2}x) \sin\left[(1+2n)\frac{\pi}{2}\frac{\sqrt{z}}{k(\theta_r)} \mathrm{cn}(K(z)x \mid z)\right] \\
& \cdot \int_{-1}^{1} dy \exp(ip\frac{\pi}{2}y) \sin\left[(1+2n')\frac{\pi}{2}\frac{\sqrt{z}}{k(\theta_r)} \mathrm{cn}(K(z)y \mid z)\right] \\
\approx \quad & \frac{32}{k^2(\theta_r) I(k^2(\theta_r))} \sum_{p=1,3,\ldots} \int_0^{k^2(\theta_r)} dz \frac{K(z)(\omega/\omega_{b0})^2}{p^2\pi^2/4K^2(z) - (\omega/\omega_{b0})^2} \\
& \cdot J_p\left[(1+2n)\frac{\pi}{2}\sqrt{z}/k(\theta_r)\right] \cdot J_p\left[(1+2n')\frac{\pi}{2}\sqrt{z}/k(\theta_r)\right].
\end{aligned}
$$

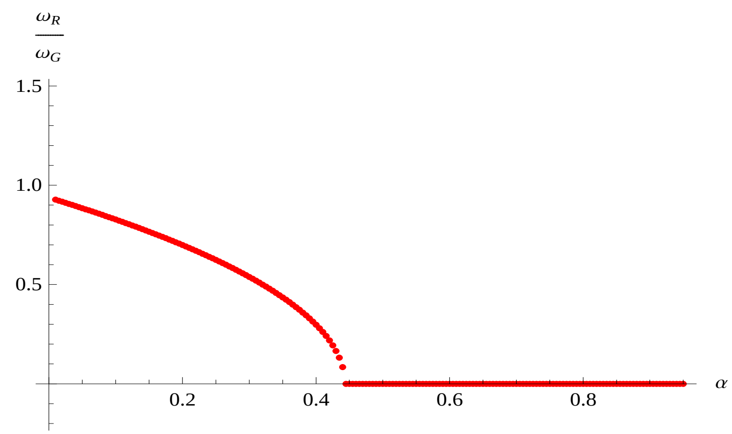
Numerical Results

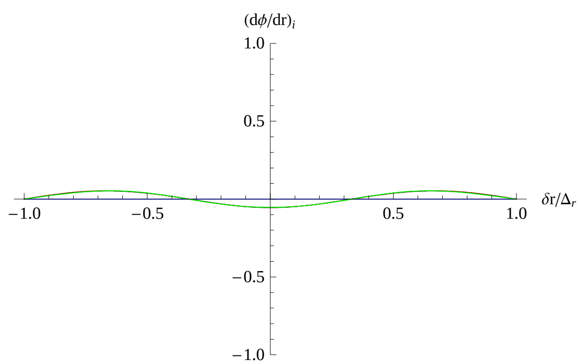
$$\omega_{GAM}(r_0)/\omega_{b0} = 1.1,\ \alpha = 0.2,\ \theta_r = 1$$

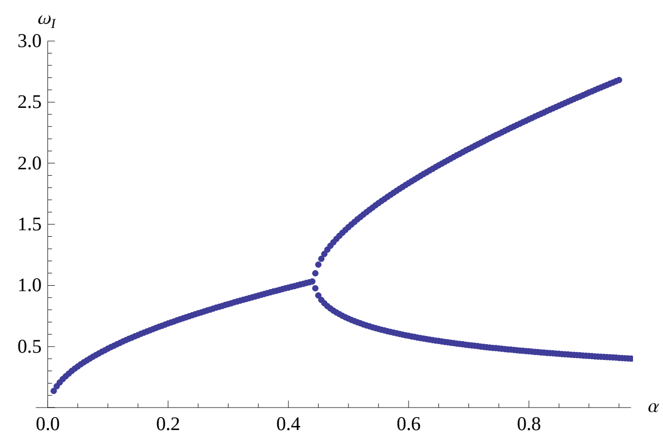
A few basis needed

ω v.s. α









$$\omega/\omega_{b0} = 0.767 + 0.695\ i$$

- Specifying the expansion point r_0 for F and ϕ
  Canonical angular momentum conservation

$$P_\phi = M_h R v_\parallel - e_h \psi(r) = M_h \overline{R v_\parallel} - e_h \psi(r_0),$$

r_0 is the position where the mechanical angular momentum is the bounce average of itself.

$$\delta r = \pm \frac{q}{\omega_{ch} r_0} (\overline{R v_\parallel} - R v_\parallel)$$

- For Strongly passing particle, $v_\perp \to 0$. projection of toroidal rotation leads to

$$\delta r = \pm \frac{q v_\parallel}{\omega_{ch}} \cos\theta = \pm \frac{q}{\omega_{ch}} \sqrt{2\mu B_0 \Lambda} \cos(\omega_b t), \Lambda \equiv \frac{E}{\mu B_0}$$

- Equivalently, noncanonical angular momentum specifies flux surface $\psi(r_0)$ around which expansion is taken.

$$\overline{P}_\phi = P_\phi - M_h \overline{R v_\parallel} = -e_h \psi(r_0),$$

Expansion

- Expanding F and ϕ in $\mathcal{L}_h$ around r_0
  F is a function of $E, \mu, \overline{P}_\phi$

$$\frac{\partial F(E,\mu,\overline{P}_\phi)}{\partial E}|_{\mu,P_\phi} = \frac{\partial F(E,\mu,\overline{p}_\phi)}{\partial E}|_{\mu,\overline{P}_\phi} + \frac{\partial F(E,\mu,\overline{p}_\phi)}{\partial \overline{p}_\phi}|_{E,\mu}\frac{\partial \overline{P}_\phi}{\partial E}|_{P_\phi}$$

$$= \frac{\partial F(E,\mu,\overline{p}_\phi)}{\partial E}|_{\mu,\overline{P}_\phi} + O(\frac{R_0\Delta_b}{rr_h}); \ r_h \text{ scale of hot particle distribution}$$

$$\approx \frac{\partial \tilde{F}(E,\mu,r_0)}{\partial E}|_{\mu,r_0}$$

$$= \frac{\partial \tilde{F}(E,\mu,r)}{\partial E} - \delta r\frac{\partial^2 \tilde{F}(E,\mu,r)}{\partial r\partial E} + \frac{1}{2}\delta r^2\frac{\partial^3 \tilde{F}(E,\mu,r)}{\partial r^2\partial E}$$

Substituting

$$\phi(r) = \phi(r_0) + \delta r\phi'(r_0) + \frac{1}{2}\delta r^2\phi''(r_0) + \frac{1}{6}\delta r^3\phi'''(r_0)$$

into $\mathcal{L}_h$ and then replacing $\phi(r_0)$ by

$$\phi(r_0) = \phi(r) - \delta r\phi'(r) + \frac{1}{2}\delta r^2\phi''(r) - \frac{1}{6}\delta r^2\phi'''(r)$$

All Fourier Transfers on $\phi(r)$ become Fourier Transfers on orbits $\delta r, \delta r^2$ and δr^3

Mode Equation

Taking variation of $\mathcal{L}$ w.r.t. ϕ leads to 4^{th} order ODE about ϕ, a total derivative about $\Psi(r) \equiv \phi'$, so we have a 2^{nd} *order ODE* about Ψ

$$
\begin{aligned}
0 &= \frac{M_h}{4\pi e_h^2} r \frac{\omega_{pi}^2}{\omega_{ci}^2}\left(1 - \frac{\Omega_G^2}{\omega^2}\right)\Psi + \\
&-r\int dv^3 \left[1 + \frac{(\delta r^2)_0}{2}\frac{\partial^2}{\partial r^2}\right]\frac{\partial F}{\partial E}\left[(\delta r^2)_0 - 2\omega^2 \sum_{p=1}^{\infty} \frac{(\delta r_p)^2}{\omega^2 - p^2\omega_b^2}\right]\Psi \\
&+ \frac{1}{2}\frac{d}{dr}\left[r\int dv^3 \frac{\partial F}{\partial E}\left((\delta r^3)_0 - 2\omega^2 \sum_{p=1}^{\infty} \frac{\delta r_p (\delta r^2)_p}{\omega^2 - p^2\omega_b^2}\right)\right]\Psi \\
&+ \frac{d}{dr}\left[r\int dv^3 \frac{\partial^2 F}{\partial E \partial r}\left((\delta r^2)_0^2 - 2\omega^2 \sum_{p=1}^{\infty} \frac{(\delta r^2)_0 (\delta r^2)_p}{\omega^2 - p^2\omega_b^2}\right)\right]\Psi \\
&+ \frac{d}{dr} r\int dv^3 \frac{\partial F}{\partial E}\left[\frac{1}{4}(\delta r^4)_0 + \frac{3}{4}(\delta r^2)_0^2 - 2\omega^2 \sum_{p=1}^{\infty} \frac{\frac{1}{4}(\delta r^2)_p^2 + (\delta r^2)_0(\delta r^2)_p}{\omega^2 - p^2\omega_b^2}\right]\frac{d\Psi}{dr} \\
&- \frac{d^2}{dr^2} r\int dv^3 \frac{\partial F}{\partial E}\left[\frac{1}{2}(\delta r^2)_0^2 + \frac{1}{6}(\delta r^4)_0 - 2\omega^2 \sum_{p=1}^{\infty} \frac{\frac{1}{2}(\delta r^2)_0(\delta r^2)_p + \frac{1}{6}(\delta r^3)_p \delta r_p}{\omega^2 - p^2\omega_b^2}\right]\Psi \\
&- r\int dv^3 \frac{\partial F}{\partial E}\left[\frac{1}{2}(\delta r^2)_0^2 + \frac{1}{6}(\delta r^4)_0 - 2\omega^2 \sum_{p=1}^{\infty} \frac{\frac{1}{2}(\delta r^2)_0(\delta r^2)_p + \frac{1}{6}(\delta r^3)_p \delta r_p}{\omega^2 - p^2\omega_b^2}\right]\frac{d^2\Psi}{dr^2}
\end{aligned}
$$

2^{nd} Order ODE

- Simple $F(x, \Lambda, \mu)$ is taken

$$F(x,\Lambda,\mu) = \frac{n_h(x)}{2\sqrt{2}\pi B_0^{3/2}} \frac{\Theta(\Lambda-\Lambda_1)\Theta(\Lambda_2-\Lambda)}{\sqrt{\Lambda_2}-\sqrt{\Lambda_1}} \frac{\Theta(\mu-\mu_1)\Theta(\mu_2-\mu)}{\sqrt{(\mu_2-\mu_1)\mu}}$$

- Take energetic particle density such that $n_h' = n_h'' = 0$

$$n_h(x) = n_{h0}(1-x^2)^3, \qquad x \equiv r/r_h.$$

- Simplified mode equation

$$\left(\frac{\Delta_b}{r_h}\right)^2 \left[\frac{d^2\psi}{dx^2} + A(x,\Omega)\frac{d\psi}{dx}\right] + B(x,\Omega)\psi = 0$$

$$A(x,\Omega,\Omega_{bm},\lambda,\eta) \equiv \frac{\frac{d}{dx}\left\{x\hat{n}_h(x)Q^2(x)\left[4\,g_1 - g_{\frac{1}{2}}\right]\right\}}{x\hat{n}_h(x)Q^2(x)\left[4\,g_1 - g_{\frac{1}{2}}\right]}$$

2^{nd} Order ODE–Continue

$$B(x,\Omega,\Omega_{bm},\alpha,\Delta_b/r_h,\lambda,\eta) \equiv \frac{32}{(4\,g_1 - g_{\frac{1}{2}})} \cdot \Big\{ \frac{(1-\sqrt{\lambda})(1-\eta)}{\alpha} \cdot$$

$$\Big\{ \frac{\hat{n}_i(x)}{\hat{n}_h(x)Q^2(x)} \cdot (1 - \frac{\hat{\Omega}_G^2}{\Omega^2}) + \alpha\big[1 + \frac{1}{(1-\sqrt{\lambda})(1-\eta)}$$

$$Q^2(x)\frac{\Omega^2}{\Omega_{bm}^2} \cdot \big(\ln \frac{Q^2(x)\Omega^2 - \Omega_{bm}^2}{Q^2(x)\Omega^2 - \eta\Omega_{bm}^2} - \frac{1}{\sqrt{\lambda}} \ln \frac{Q^2(x)\Omega^2 - \lambda\Omega_{bm}^2}{Q^2(x)\Omega^2 - \lambda\eta\Omega_{bm}^2} \big)\big]\Big\}$$

$$+ \quad (\frac{\Delta_b}{r_h})^2 \big[\frac{\hat{n}_h''(x)}{16\hat{n}_h(x)}(3\,g_1 - 2\,g_{\frac{1}{2}}) + \frac{\hat{n}_h'(x)}{8\hat{n}_h(x)xQ^2(x)} \frac{d}{dx}(xQ^2(x)g_1)$$

$$+ \frac{1}{16xQ^2(x)} \frac{d^2}{dx^2} xQ^2(x)(g_1 + 2\,g_{\frac{1}{2}}) \big] \Big\}$$

$\alpha \equiv \frac{n_h(x=0)}{n_i(x=0)} q_0^2$

$Q(x) \equiv q(x)/q_0$

$g_1 \equiv g(x,\Omega,\Omega_{bm},\lambda,\eta)$–outcome of integral over velocity space

$g_{\frac{1}{2}} \equiv g(x,\Omega/2,\Omega_{bm},\lambda,\eta)$

$\frac{1}{4}$-Wavelength Mode & Theory

- $\Psi(x) \sim J_0(\sqrt{\mathcal{B}(x=0,\Omega)}x)$ around $x=0$; $\mathcal{B} \equiv (\frac{\Delta_b}{r_h})^{-2} B(x,\Omega)$
- Mode equation tells
  global dispersion function $=$ local dispersion function $+ O((\frac{\Delta_b}{r_h})^2)$,
  so possible that
  1. $\Omega = \Omega_0 + \delta\Omega$, Ω_0–local root of $B(x=0,\Omega)=0$, $\delta\Omega$–small shift
  2. $\frac{1}{4}$-wavelength-like mode: maximum at the center and damps outwards with no oscillation.
- Theory
  1. Around the center $x=0$,
  $$A(x,\Omega) \to \frac{1}{x},\ \mathcal{B}(x,\Omega_0+\delta\Omega) = \mathcal{B}_\Omega \delta\Omega + \frac{1}{2}\mathcal{B}_{xx} x^2$$
  2. Mode equation
  $$\frac{d^2\Psi}{dy^2} + \frac{1}{y}\frac{d\Psi}{dy} + \left(\frac{\mathcal{B}_\Omega \delta\Omega}{\sqrt{-\frac{1}{2}\mathcal{B}_{xx}}} - y^2\right)\Psi = 0,\ \ y \equiv (-\frac{1}{2}\mathcal{B}_{xx})^{1/4} x$$
  Schroedinger equation for 2-D axially symmetrical harmonic oscillator in polar coordinates with m=0. The ground state has no oscillation if $\mathcal{B}_{xx}$ has a predominantly negative real part.

Theory of $\frac{1}{4}$-Wavelength Mode–Continue

- Solution

$$\delta\Omega_0 = 2\frac{\sqrt{-\frac{\mathcal{B}_{xx}}{2}}}{\mathcal{B}_\Omega} \propto \frac{\Delta_b}{r_h}$$

$$\Psi(x) \sim \exp\left(-\sqrt{\frac{-\mathcal{B}_{xx}}{8}}x^2\right)$$

- Validity of expansion condition check

$$k(x) = -\sqrt{-\frac{1}{2}\mathcal{B}_{xx}x}, \;\; x \sim \sqrt{2}(-\frac{1}{2}\mathcal{B}_{xx})^{-1/4}, \;\; \mathcal{B} \equiv \Delta_b^{-2}\mathcal{B}'$$

$$k(x)\Delta_b \sim (-\mathcal{B}'_{xx})^{1/4}\Delta_b^{1/2}$$

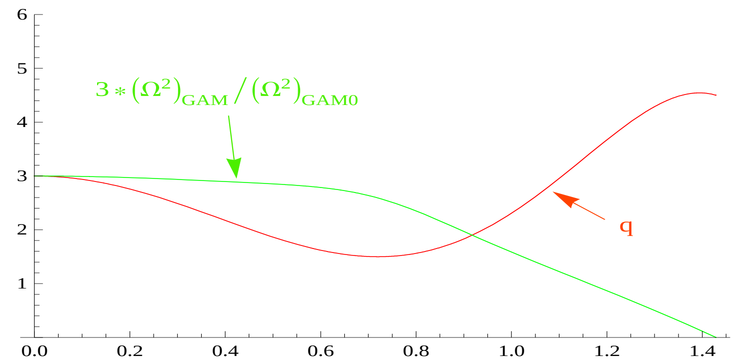
So $|k(x)|\Delta_b \ll 1$ can be guaranteed for small orbit width Δ_b if other parameters in $\mathcal{B}$ are properly chosen such that $|(-\mathcal{B}'_{xx})^{1/4}|$ is not large.

Comparison between Theory and Numerical Results

$\alpha = 1.056,\ \Delta_b/r_h = 1/20,\ \Omega_{bm} = 1.225$

$\Omega_{analytical} = 0.552005 + 0.146526i$

$\Omega_{numerical} = 0.551803 + 0.14621i$



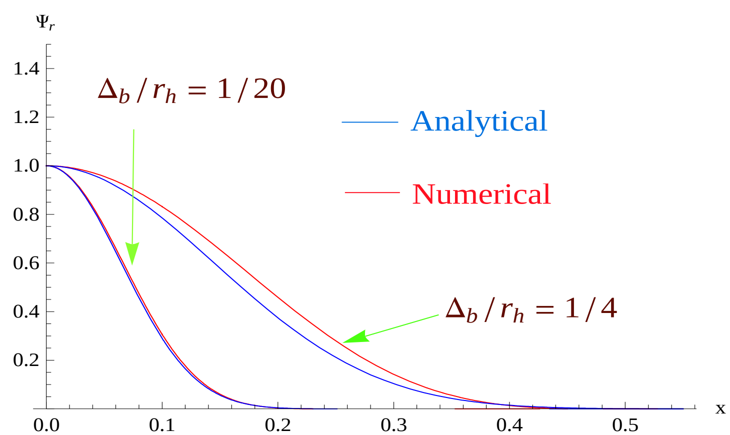


Figure: Real part of Ψ

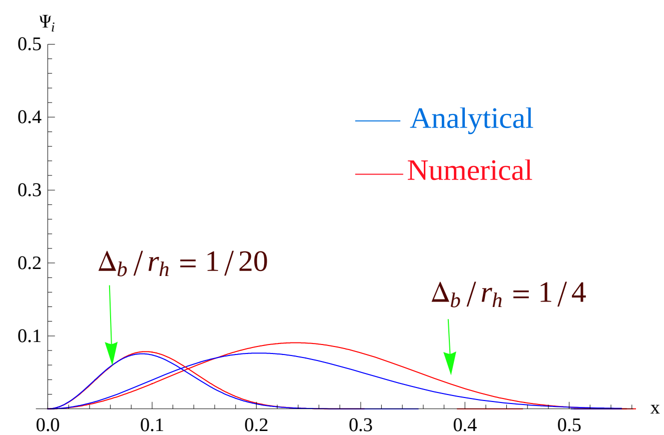


Figure: Imaginary part of Ψ

Comparison between Theory and Numerical Results– Continue

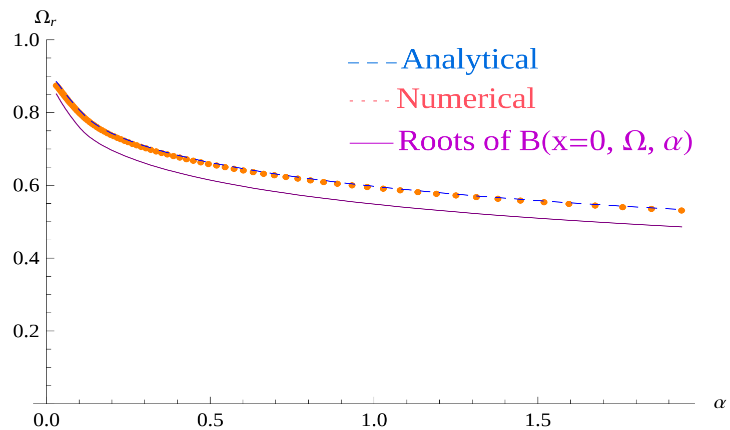


Figure: $(\Omega/\Omega_{GAM0})_{real} v.s.\alpha$

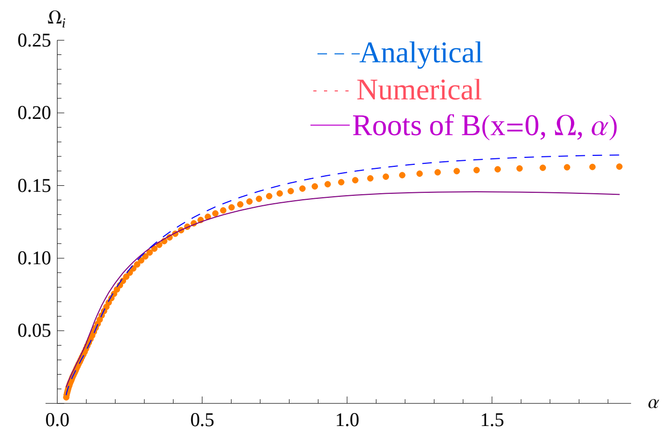


Figure: $(\Omega/\Omega_{GAM0})_{IM} v.s.\alpha$

Summary

- We treat thermal electrons, thermal ions and energetic ions differently due to the differences of the relative magnitudes of their bounce frequencies with respect to the mode frequency.
- Find out unstable EGAM driven by energetic trapped particles when their orbit widths are comparable to mode width.
- Find out unstable EGAM driven by energetic passing particles when their orbit widths are smaller than mode width.
- Hard to match experiment as differential equation breaks down at relative small Δ_b/r_h for $\alpha \sim 1$.