Review of Phase Space Trapping Saturation

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Purpose of Talk

- Need for nonlinear benchmarks
- Dominant Nonlinearity is due to dynamics of phase space structures
- Model simulations and analysis have already been performed; results will be discussed here
- Probing for resonant phase space areas

  an interesting diagnostic to develop
- New direction for simulation is to fold in fluid equations with dynamics of resonant particles
- Following just resonant particle dynamics, which may lead to alternate to combine fluid and kinetic phenomena.

Basic assumptions for predicting saturation level of resonant particle modes

1. Separation of resonances; $\omega_{bi} < \omega_i - \omega_j$
2. Low saturation level, with a background plasma wave described linearly; resonant region nonlinearly
3. Presence of source, sink and extrinsic dissipation mechanisms

Further simplifying assumptions

1. Perturbative mode (non-perturbative mode insight not as complete)
2. Closeness to marginal stability (a universal characteristic of a steady plasma discharge)
3. Treatment of 2-D phase space problem

   (extension to 6-D phase space is approximated by 2- D surfaces covering 6-D space as will be discussed)

Natural variables for problem

- Trapping frequency: ω_b radian wave trapping frequency of deepest trapped particle in fixed wave field (for sufficiently small amplitude $\omega_b \propto \left[E^{1/2}, B^{1/2} \right]$

  e.g. electrostatic waves $\omega_b = \sqrt{\dfrac{ekE}{m}}$

- Linear growth and damping rate: γ_L, γ_d
- Effective collisionality: $\nu_{eff,D} \approx (\nu_{90}\omega^2)^{1/3} \approx \nu_{\text{anh}}$

Saturated amplitude values can be expressed in terms of these parameters

Saturation in 2-d phase space scenarios

1. Initially smooth function, without dissipation,

   grows until local gradient of resonant particle distribution is flattened (Sagdeev, Fried, Liu): $\omega_b \approx 3.3\gamma_L$
2. saturation occurs when excited wave momentum equals wave momentum released by resonant particles:
   obtained from simulation
3. Two extreme assumptions bound this result:

   a. Slow adiabatic evolution of wave:

$$\frac{d\omega_b}{dt} << \gamma_L^2 \rightarrow \omega_b = 2.9\gamma_L$$

   b. Impulsive approximation (saturated mode amplitude instantaneously reached and particles then absorb released wave momentum): $\omega_b = 3.5\gamma_L$

4. Natural Saturation Level: $\omega_b \sim \gamma_L$

Sink and Source Maintain Steady Saturation Level

1. Dissipation, $\gamma_d \leq \gamma_L$ is taken into account
2. There is a balance between the rate energy is extracted from the system via dissipation and energy is fed into the resonant region from the source.
3. Two limits analyzed:

   a. significantly above marginal stability: $\gamma_L >> \gamma_d$

$$\frac{\omega_b}{\nu_{eff}} = .85\left(\frac{\gamma_L}{\gamma_d}\right)^{1/3}$$

   b just above marginal stability threshold: $\gamma_L - \gamma_d << \gamma_L$

$$\frac{\omega_b}{\nu_{eff}} = 1.4\sqrt{\frac{\gamma_L - \gamma_d}{\gamma_L}}$$

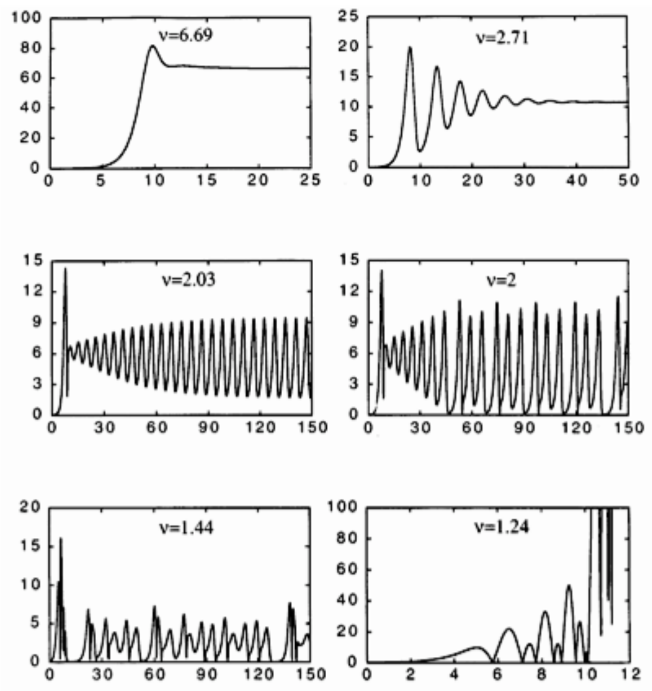
Bursting

1. When saturation level is below 'natural saturation level' steady sustained level cannot be maintained, as follows from energy considerations (more power dissipated that supplied by beam) $\frac{\nu_{eff}}{\omega_b} < 1$

2. Significantly above threshold and burst saturate at $3\gamma_L \geq \omega_b \geq \gamma_L$ and expected to recur within a period $T \approx \gamma_L^2 / \nu_{eff}^3 \approx \gamma_L^2 / \nu_{col}\omega^2$

3. Closer to threshold, pitchfork bifurcations arise giving rise to ever more complex response, terminating an explosive instability, which is the signature to frequency sweeping events

"Signature" for Formation of Phase Space Structure (single resonance)



Cubic nonlinear equation near marginal stability for perturbative ($\phi \approx 0$) and non-perturbative systems ($\phi \approx 1$)

$$\frac{dA}{dt} = A - e^{i\phi}\int_0^{t/2}\tau^2\, d\tau \int_0^{t-2\tau} d\tau_1\, e^{-\nu^3\tau^2(2\tau/3+\tau_1)}A(t-\tau) \times A(t-\tau-\tau_1)A^*(t-2\tau-\tau_1),$$

$$\exp(i\phi) = -\frac{G_\omega}{|G_\omega|};\ \text{WE}=\text{Re}G_\omega\,|A|^2\ \text{(prtb wave)}$$

Explosive response leads formation of phase space structure

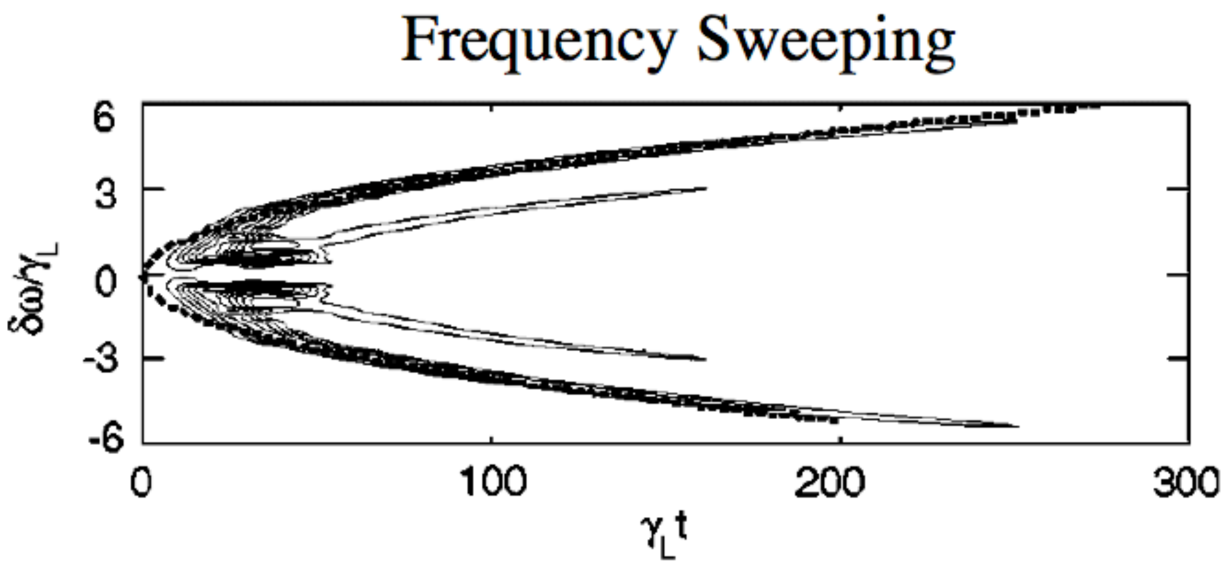
Explosive self-similar solution $\rightarrow A \approx \frac{\exp\left(i\alpha\ln(t_0-t)\right)}{(t_0-t)^{5/2}} \rightarrow$ Valid as long as amplitude sufficiently small, must then fail and then what?

Hole-clump pair $t^{1/2}$ frequency sweeping

- Simulation of near-threshold bump-on-tail instability *(N. Petviashvili, 1997)* reveals spontaneous formation of phase space structures locked to the chirping frequency

- Chirp extends the mode lifetime as phase space structures seek lower energy states to compensate wave energy losses due to background dissipation

- Clumps move to lower energy regions and holes move to higher energy regions



Under assumption $\omega_b << \delta\omega << \omega_L$

How can phase space structure evolution be explained?

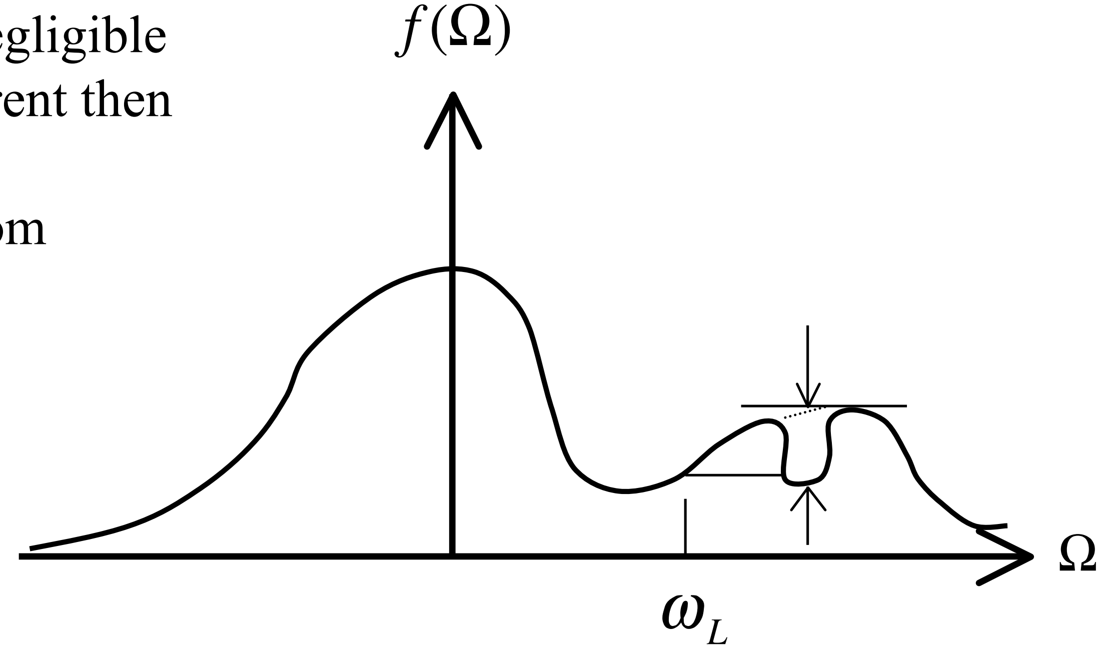
We need equation that reflects mode in presence of phase space structure.

a. We assume an ideal phase space structure is established where distribution in trapping region is the same as it was when mode was first created and distribution outside was the original unperturbed distribution $\Omega \equiv kv = \omega(t)$

b. Outside trapping region, contribution to resonant particle current negligible

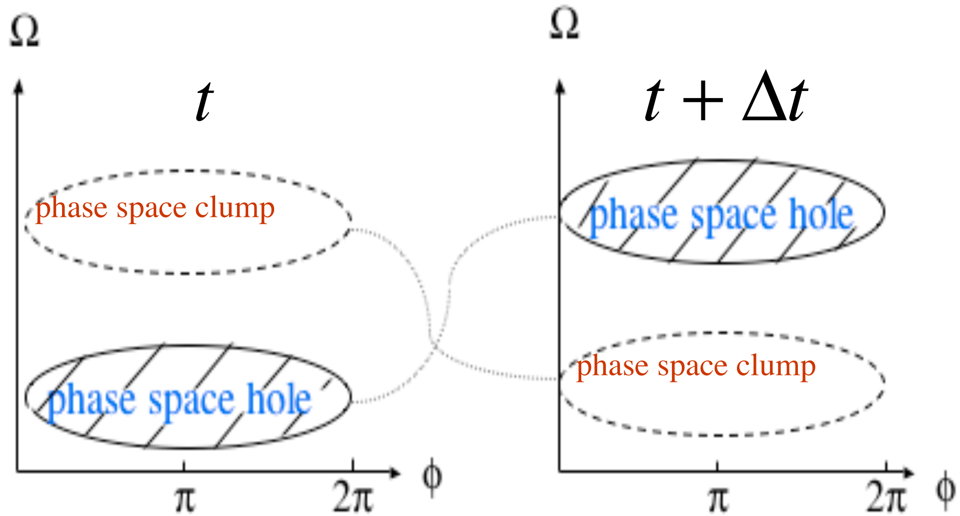
c. Integration of resonant current then straight-forward to obtain extra reactive contribution from trapped particles.

d. Equilibrium of plasma is is then altered to account for non-linear BGK from a single resonance



Role of dissipation

We need account for extrinsic dissipation that is extracting energy from energetic particles

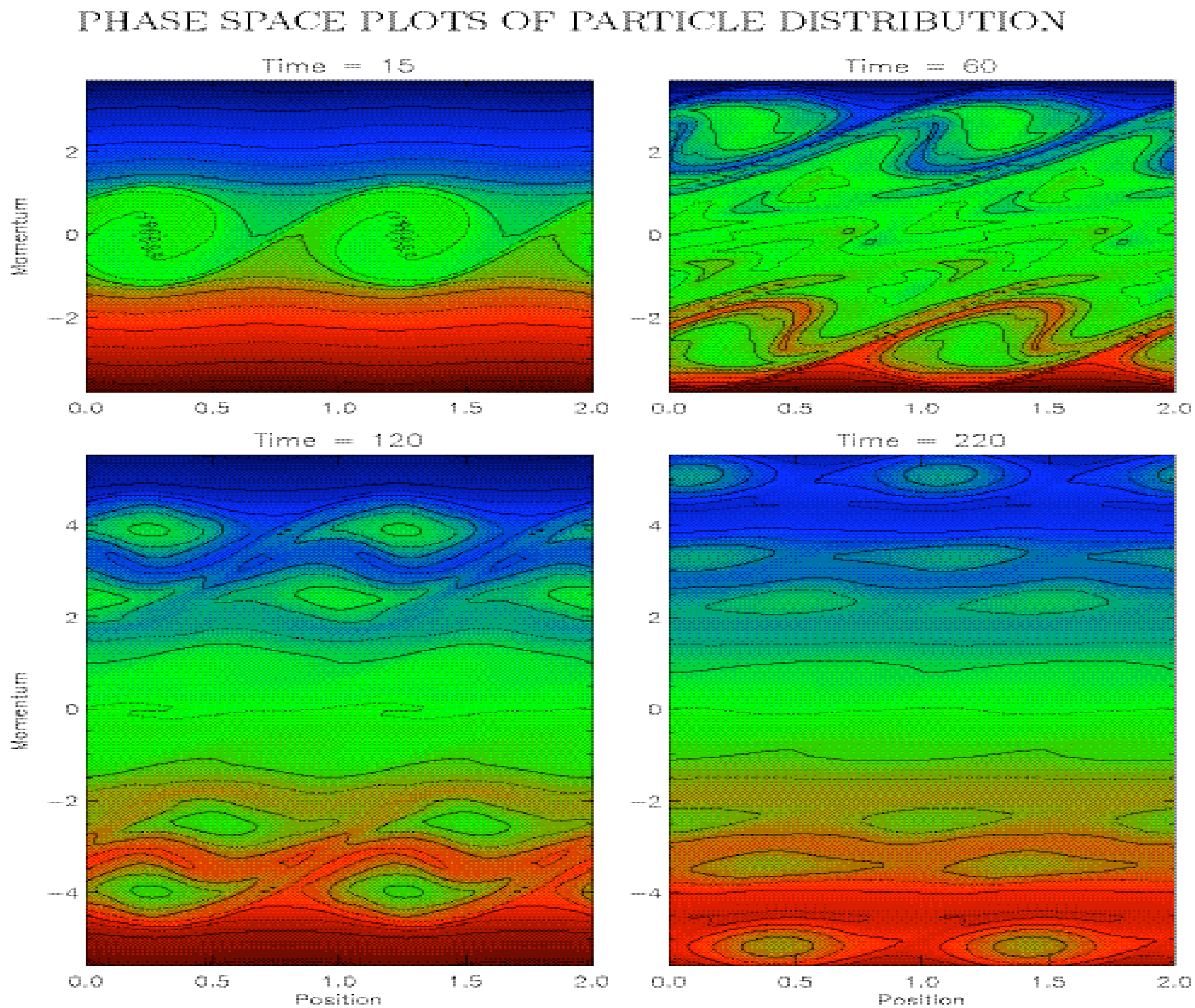


- Phase space structure has to sweep in the resonant frequency in order to balance the energy dissipated by extrinsic dissipation!
- Phase space structure moves to lower energy states (holes rise & clumps fall in electrostatic two stream instability)
- The essential theoretical aspects of model have now been explained and will now be compared with numerical simulation.
- Using BGK and power transfer conditions yields saturated amplitude

$$\omega_b = .54\gamma_L; \quad \omega - \omega_L = .41\gamma_L\left(\gamma_d t\right)^{1/2}$$

Convective Transport in Phase

- **Explosive nonlinear dynamics produces coherent structures**
  - Convective transport of trapped ("green") particles
- **Phase space "holes and clumps" are ubiquitous to near-threshold single mode instabilities**
  - Examples: bump-on-tail, TAE's, etc.



N. Petviashvili et al., Phys. Lett. A ***234***, *213 (1997)*

From 2-D phase space to 6-D phase space:
View as ‘parallel’ 2-dimensional phase spaces

1. In tokamks and stellarators nearly all the unperturbed orbits are ‘integrable’ with three adiabatic constants of motion defining the orbit. $\mu \equiv$ magnetic moment very robust constant of motion nearly every confinement device. In tokamaks two constants of motion are exact; $P_\phi \equiv$ angular momentum (also an adiabatic invariant) and $E \equiv$ energy from which third adiabatic invariant is determined.
2. One then constructs perturbed interaction using action angle variables for the Hamiltonian

$$H = H_0(J_j) + \left(ieC(t)\frac{\mathbf{e}(\vec{r})\cdot\vec{v}}{\omega(t)}\exp\left(-i\int_0^t dt'\omega(t')\right) + cc \right),\ (j = 1-3)$$

with

$$C(t)\mathbf{e}(\vec{r})\exp\left(-i\int_0^t dt'\omega(t')\right) + cc$$

Reduction to 2-dimensional phase space (continued-2)

3. Expand interaction term in Fourier Series of action angles

$$C(t)e\frac{\mathbf{e}(\vec{r})\cdot\vec{\upsilon}}{\omega(t)} = C(t)\sum_{\ell_i} e\frac{<\mathbf{e}\cdot\vec{\upsilon};J_j>_{l_i}}{\omega(t)}\exp\left[i\left(\ell_1\vartheta_1 + \ell_2\vartheta_2 + \ell_3\vartheta_3 - i\int_0^t \omega(t')dt'\right)\right]$$

4. In resonant interaction, most terms gives rise to rapidly oscillating terms in frame of wave, but only particles near resonant condition,

$$\Omega(E,\mu,P_\phi) \equiv \ell_{10}\Omega_1 + \ell_{20}\Omega_2 + \ell_{30}\Omega_3 - \omega(t) \approx 0$$

give rise to non-oscillatory (secular terms).

5. Keep only this single resonant Hamiltonian, and account for other particles from reactive linear response

$$H_1 = eC(t)\frac{<\mathbf{e}\cdot\vec{\upsilon};J_j>_{l_{i0}}}{\omega(t)}\exp\left[i\left(\ell_{10}\vartheta_1 + \ell_{20}\vartheta_2 + \ell_{30}\vartheta_3 - i\int_0^t \omega(t')dt'\right)\right]$$

$$\chi = \ell_{10}\vartheta_1 + \ell_{20}\vartheta_2 + \ell_{30}\vartheta_3 - i\int_0^t \omega(t')dt';\ \dot{\chi} = \Omega(E,\mu,P_\phi)$$

Reduction to 2-dimensional phase space (continued-4)

6. Reduced Hamiltonian gives rise to Vlasov equation is 2-D Vlasov equation (with a frequency sweeping term and we add collisions

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\chi}+\left[\left(\frac{\omega_b^2(J(t),t)}{2}\exp\left(i\chi\right)+cc\right)+\dot{\omega}\right]\frac{\partial}{\partial\Omega}\right)f(\chi,\Omega,t)=-\hat{v}\left(f(\chi,\Omega,t)-f_{00}\right)$$

$$\omega_b^2(J,t)=2C(t)\frac{e<\mathbf{e}\cdot\vec{v};J(t)>_{l_0}}{\omega(t)\,\partial J(t)/\partial\Omega}=C(t)\hat{\omega}_b^2(J,t)$$

$\equiv$ wave trapping frequency of deeply trapped particles

7. We thus we may hope to tract '2-D sheets' in phase space that in principle covers the entire phase space region of resonant particles
8 . If in a simulation we can track frequency of resonant structure through dynamic feedback, only regions of phase space near moving resonance needs to be tracked using a time step determined by $1/\omega_b$ rather than $1/\omega$. This can be a diagnostic tool and perhaps a way to speed up the treatment of resonances particles, such for resonances of background electrons or ions giving rise to damping

Wave Evolution Equation

1. Maxwell's wave equation can in principle be written for a medium with a spatially non-local linear <u>reactive</u> conductivity.

$$\vec{j}_{Rc}(t) = \int_{-\infty}^{t} dt' \overleftrightarrow{\sigma}_{Rc}(t-t',\vec{r},\vec{r}\,') \cdot \vec{E}(t',\vec{r}\,')$$

   and a nonlinear source of current from resonant particles

2. Electric field taken in the form,

$$\vec{E}(t,\vec{r}) = C(t)\mathbf{e}(\vec{r})\exp[-i\int_{0}^{t} dt'\omega(t')] + cc, \quad \frac{1}{\omega(t)C(t)}\frac{dC(t)}{dt} << 1$$

3. $\mathbf{e}(\vec{r})$ is best guess for eigenfunction's spatial structure
4. Quadratic form constructed from wave equation in standard way to form wave evolution equation

Wave Evolution Equation -2

Basic Wave Equation for long time scale time steps: $1/\omega << \Delta t << 1/\omega_b$

$$G\big(\omega(t)\big)+i\left(\frac{\partial G\big(\omega(t)\big)}{\partial\omega}\right)^{1/2}\frac{d}{dt}\left(C(t)\frac{\partial G\big(\omega(t)\big)}{\partial\omega}\right)^{1/2}=-i\int d\vec{r}j_{Rs}(t)\cdot\mathbf{e}^*(\vec{r})\exp\left(i\int^{t}\omega(t')dt'\right)$$

$$\cong -i(2\pi)^2\ \mathrm{e}\iint dJ_2'dJ_3'\int_{\Delta\Omega}d\Omega\int_0^{2\pi}d\chi\frac{<\vec{\upsilon}\cdot\mathbf{e}^*(\vec{r})>|_{l_0}}{\omega(t)}\big(f(\Omega,\chi;J)-f_0(\Omega,\chi;J)\big)\exp(i\chi)$$

$$\text{where, } G\big(\omega(t)\big)=\iint\frac{d\vec{r}d\vec{r}\,'}{4\pi}\begin{pmatrix}-\frac{c^2}{\omega}\delta(\vec{r}-\vec{r}\,')\nabla'\times\mathbf{e}^*(\vec{r}\,')\cdot\nabla\times\mathbf{e}(\vec{r})+\omega\delta(\vec{r}-\vec{r}\,')\mathbf{e}^*(\vec{r}\,')\cdot\mathbf{e}(\vec{r})\\ +4\pi\Big(i\mathbf{e}^*(\vec{r}\,')\cdot\mathbf{e}(\vec{r})\sigma_{Ex}-\mathbf{e}^*(\vec{r}\,')\cdot\overleftrightarrow{\Sigma}_{Rc}(\omega,\vec{r}\,',\vec{r})\cdot\mathbf{e}(\vec{r}\,')\Big)\end{pmatrix}$$

note $G_\omega(\omega=\omega_L)|C|^2$ wave energy for perturbative mode at $\omega=\omega_L$

Reactive conductivity tensor in frequency space:

$\overleftrightarrow{\Sigma}_{Rc}(\omega,\vec{r}\,',\vec{r}) \equiv$ Laplace transform of $\overleftrightarrow{\sigma}_{Rc}(\omega,\vec{r}\,',\vec{r})$;

$\rightarrow$ usual calculatation of linear theory without resonant particles

Saturation Levels in Multi-Dimensions

1. Saturation levels very similar to 2-D phase space levels when expressed in terms of appropriate variables
2. For example, for the sweeping problem, close to resonance the saturation level and sweeping rate is given by:

$$|2C|^{1/2} = \frac{16 <|\hat{\omega}_b(J(t);J'_i)|^3>}{3\pi^2 <|\hat{\omega}_b(J(t);J'_i)|^4>} \gamma_L(\omega = \omega_L)$$

$$\omega - \omega_0 = \pm \frac{\pm 16\sqrt{2}\gamma_L\left(\sigma\gamma_d t\right)^{1/2}}{3\pi^2\sqrt{3}}, \text{ with } \sigma = \frac{<\hat{\omega}_b^3>^{3/2}}{<\hat{\omega}_b^4><\hat{\omega}_b>^{1/2}} \approx 1$$

$$<|\hat{\omega}_b(J(t);J'_i)|^p> = \omega(t)\int d\Gamma \frac{\partial f_o(\Omega(J);J'_i)}{\partial\Omega}\frac{\partial E(\Omega;J'_i)}{\partial\Omega}|\hat{\omega}_b(J(t);J'_i)|^p \,\delta(\omega(t)-\Omega(J))$$

3. Saturation level for constant amplitude saturation regime where there is a relatively high collisionality has been programmed into theNOVA-K code by Nicolai.

Summary

1. Considerable theory available to verify predictions and understand dynamics of kinetic simulation codes

2. It would be very interesting to attempt to track phase space of resonant region

   a. Resonant region could be tracked with passive tracers

   b. Tracers can feed back to the overall dynamics through reduce Hamiltonian that move particles on 'sheets' of phase space

   c. Can such simulation be done efficiently? Long time steps, processing is complex (will it be fast or slow?)

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