On the Noether Derivation of the Gyrokinetic Energy-Momentum Conservation Laws*

Alain J. Brizard
Saint Michael's College (Vermont)

Princeton Plasma Physics Laboratory

May 19, 2011

A.J. Brizard & N. Tronko, Exact momentum conservation laws for the gyrokinetic Vlasov-Poisson equations, submitted to Physics of Plasmas (2011)

Outline

- A Tutorial on the Calculus of Variations for Fields
- Gyrokinetic Variational Principles
- Gyrokinetic Energy-Momentum Conservation Laws
- Gyrokinetic Angular Momentum Conservation in Axisymmetric Tokamak Geometry
- Summary and Future Work

A Tutorial on the Calculus of Variations for Fields

- **Action functional → Variational Principle**

$$\mathcal{A}[\psi^a] = \int \mathcal{L}(\psi^a, \partial_\mu \psi^a; x^\mu)\, d^4x \quad \rightarrow \quad \delta\mathcal{A} = 0$$

◦ **Functional derivative**

$$\delta\mathcal{A} \equiv \left(\frac{d}{d\epsilon}\mathcal{A}[\psi^a + \epsilon\,\delta\psi^a]\right)_{\epsilon=0} \equiv \int \delta\psi^a \,\frac{\delta\mathcal{A}}{\delta\psi^a}\, d^4x$$

$$= \int \delta\psi^a \left[\frac{\partial\mathcal{L}}{\partial\psi^a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^a)}\right)\right] d^4x$$

◦ **Variation of Lagrangian density**

$$\delta\mathcal{L} \equiv \underbrace{\left(\delta\psi^a \,\frac{\delta\mathcal{A}}{\delta\psi^a}\right)}_{\text{Euler-Lagrange Equations}} + \underbrace{\frac{\partial}{\partial x^\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^a)}\,\delta\psi^a\right]}_{\text{Noether Equation}}$$

- **Noether Method**

Lagrangian Symmetry $\Leftrightarrow$ Conservation Law

- **Eulerian variation: Dynamical definition**

$$\boxed{\delta\psi^a \equiv \lim_{\Delta t\to 0}\left(\Delta t\,\frac{\partial\psi^a}{\partial t}\right)_{\dot{\mathbf{x}}\,\Delta t\,\to\,\delta\mathbf{x}}}$$

- **Lagrangian variation: Constraints!**

$$\Delta\psi^a \equiv \delta\psi^a + \delta x^\mu\,\partial_\mu\psi^a$$

- **Example: Fluid density** $(\delta\mathbf{x} = \boldsymbol{\xi})$

$$\partial_t n = -\nabla\cdot(n\,\mathbf{u}) \;\rightarrow\; \delta n = -\nabla\cdot(n\,\boldsymbol{\xi})$$

$$\Delta(n\,d^3x) \equiv 0 \;\rightarrow\; (\Delta n)\,d^3x = -\,n\,\Delta(d^3x) = -\,n\,[(\nabla\cdot\boldsymbol{\xi})\,d^3x]$$

- **Energy-momentum Conservation Law**

  - **Infinitesimal space-time translations** $x^\nu \;\longrightarrow\; x^\nu + \delta x^\nu$

$$\begin{aligned} \delta\psi^a &= \Delta\psi^a - \delta x^\nu\,\partial_\nu\psi^a \equiv -\,\delta x^\nu\,\partial_\nu\psi^a \\ \delta\mathcal{L} &= \underbrace{-\,\partial_\nu\Big(\delta x^\nu\,\mathcal{L}\Big)}_{\text{Density Variation}} + \underbrace{\delta x^\nu\,\partial'_\nu\mathcal{L}}_{\text{Explicit Gradient}} \end{aligned}$$

$$\boxed{\partial_\mu T^{\mu\nu} = \partial'^{\nu}\mathcal{L}}$$

  - **Canonical energy-momentum tensor**

$$T^{\mu\nu} \equiv g^{\mu\nu}\,\mathcal{L} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^a)}\,\frac{\partial\psi^a}{\partial x_\nu}$$

  - **Noether Theorem**

Symmetry : $\partial'_\lambda\mathcal{L} = 0 \;\Leftrightarrow\;$ Conservation Law : $\partial_\mu T^\mu_\lambda = 0$

Gyrokinetic Variational Principles

- **Gyrocenter Hamilton variational principle**

  Particle $\longrightarrow$ Guiding-center (gc) $\longrightarrow$ Gyrocenter (gy)

◦ **Guiding-center transformation**

$$\boldsymbol{\rho}_0 \equiv \text{guiding-center gyroradius}$$
$$\{\ ,\ \}_{\text{gc}} \equiv \text{Guiding-center Poisson bracket}$$

◦ **Noncanonical phase-space Lagrangian**

$$\boxed{\Gamma_{\text{gy}} \equiv \underbrace{\left[\left(\frac{e}{c}\,\mathbf{A} + p_{\|}\,\widehat{\mathsf{b}}\right)\cdot \mathsf{d}\mathbf{X} - W\,\mathsf{d}t\right]}_{\equiv\ \text{gc Symplectic Part}} - \left(H_{\text{gy}} - W\right)\,\mathsf{d}\tau}$$

$$\mathbf{B} \equiv \nabla\times\mathbf{A} = \text{Background magnetic field}$$
$$(\mathbf{X}, p_{\|}, W, t; \mu, \zeta) \equiv \text{Gyrocenter phase-space coordinates}$$

- **Gyrocenter Hamiltonian**

$$\boxed{H_{\mathsf{gy}}(\mathbf{X}, p_{\|}, \mu, t;\ \phi_1) \;\equiv\; \underbrace{\left(\mu\, B + \frac{p_{\|}^2}{2m}\right)}_{\equiv\, H_{\mathsf{gc}}} + e\,\Phi_{\mathsf{gy}}(\mathbf{X}, \mu, t)}$$

- **Effective gyrocenter potential**

$$\boxed{\Phi_{\mathsf{gy}} \;\equiv\; \epsilon\,\langle\phi_{1\mathsf{gc}}\rangle \;-\; \frac{\epsilon^2}{2}\,\left\langle\left\{S_1,\, \widetilde{\phi}_{1\mathsf{gc}}\right\}_{\mathsf{gc}}\right\rangle \;+\; \cdots}$$

- **Gyrocenter transformation** ($\langle\ \rangle \equiv$ gyroangle average)

$$\phi_{1\mathsf{gc}} \;\equiv\; \mathsf{T}_{\mathsf{gc}}^{-1}\phi_1 = \phi_1(\mathbf{X} + \boldsymbol{\rho}_0, t) \equiv \langle\phi_{1\mathsf{gc}}\rangle + \widetilde{\phi}_{1\mathsf{gc}}$$

$$S_1 \;\equiv\; (e/\Omega)\int \widetilde{\phi}_{1\mathsf{gc}}\, d\zeta$$

- **Gyrocenter Equations of Motion in Axisymmetric Tokamak Geometry**

◦ **Gyrocenter Euler-Lagrange equations**

$$0 \;=\; \delta \int \Gamma_{\rm gy} \;\equiv\; \int \;\delta Z^{a} \left(\omega_{ab}\, \frac{d_{\rm gy} Z^{b}}{d\tau} \;-\; \frac{\partial H_{\rm gy}}{\partial Z^{a}} \right)\; d\tau$$

$$\boxed{\frac{e}{c}\,\frac{d_{\rm gy}\mathbf{X}}{dt} \times \mathbf{B}^{*} - \frac{d_{\rm gy} p_{\|}}{dt}\,\widehat{\mathsf{b}} - \nabla H_{\rm gy} \;=\; 0 \;=\; \widehat{\mathsf{b}}\cdot \frac{d_{\rm gy}\mathbf{X}}{dt} - \frac{\partial H_{\rm gy}}{\partial p_{\|}}}$$

$$\mathbf{B}^{*} \;\equiv\; \nabla \times \left(\mathbf{A} \;+\; p_{\|}\, \frac{c\widehat{\mathsf{b}}}{e} \right) \;\rightarrow\; B^{*}_{\|} \equiv \widehat{\mathsf{b}} \cdot \mathbf{B}^{*}$$

- **Gyrocenter noncanonical Hamilton equations**

$$
\boxed{
\begin{aligned}
\frac{d_{\sf gy}\mathbf{X}}{dt} &= \frac{p_{\|}}{m}\,\frac{\mathbf{B}^{*}}{B_{\|}^{*}} + \frac{c\widehat{\sf b}}{eB_{\|}^{*}}\times\nabla H_{\sf gy} \\
\frac{d_{\sf gy}p_{\|}}{dt} &= -\,\frac{\mathbf{B}^{*}}{B_{\|}^{*}}\cdot\nabla H_{\sf gy}
\end{aligned}
}
$$

$$
\frac{1}{B_{\|}^{*}}\nabla\cdot\left(B_{\|}^{*}\,\frac{d_{\sf gy}\mathbf{X}}{dt}\right) + \frac{1}{B_{\|}^{*}}\frac{\partial}{\partial p_{\|}}\left(B_{\|}^{*}\,\frac{d_{\sf gy}p_{\|}}{dt}\right) \equiv 0
$$

- **Gyrocenter canonical Hamilton equations**

$$
\mathbf{p}_{\sf gy} \equiv \frac{e}{c}\mathbf{A} + p_{\|}\,\widehat{\sf b} \;\Rightarrow\; \frac{d_{\sf gy}\mathbf{p}_{\sf gy}}{dt} = -\,\nabla H_{\sf gy}
$$

- **Axisymmetric Tokamak Geometry**

$$\mathbf{B} \;\equiv\; \nabla\varphi\times\nabla\psi \;+\; q(\psi)\,\nabla\psi\times\nabla\vartheta$$

- **Toroidal canonical gyrocenter momentum**

$$p_{\mathrm{gy}\varphi} \;\equiv\; \frac{\partial\mathbf{X}}{\partial\varphi}\cdot\mathbf{p}_{\mathrm{gy}} \;=\; -\,\frac{e}{c}\,\psi \;+\; p_{\|}\,b_{\varphi}$$

- **Gyrocenter toroidal-momentum Euler-Lagrange equation**

$$\boxed{\begin{aligned} -\,\frac{\partial H_{\mathrm{gy}}}{\partial\varphi} \;&=\; \frac{\partial\mathbf{X}}{\partial\varphi}\cdot\left(\frac{d_{\mathrm{gy}}p_{\|}}{dt}\,\widehat{\mathsf{b}} \;-\; \frac{e}{c}\,\frac{d_{\mathrm{gy}}\mathbf{X}}{dt}\times\mathbf{B}^{*}\right) \\ &=\; \frac{d_{\mathrm{gy}}p_{\|}}{dt}\,b_{\varphi} \;-\; \left(\frac{e}{c}\,\frac{d_{\mathrm{gy}}\psi}{dt} \;-\; p_{\|}\,\frac{d_{\mathrm{gy}}b_{\varphi}}{dt}\right) \\ &\equiv\; \frac{d_{\mathrm{gy}}}{dt}p_{\mathrm{gy}\varphi} \end{aligned}}$$

- **Gyrokinetic Vlasov-Poisson Action Functional**

$$\boxed{\begin{aligned} \mathcal{A}_{\mathrm{gy}} &= \int \frac{d^4x}{8\pi} \left(\epsilon^2 \, |\mathbf{E}_1|^2 \, - |\mathbf{B}|^2 \right) \\ &\quad - \sum \int d^8\mathcal{Z} \, \mathcal{F}_{\mathrm{gy}}(\mathcal{Z}) \, \mathcal{H}_{\mathrm{gy}}(\mathcal{Z};\, \phi_1) \\ &\equiv \int \mathcal{L}_{\mathrm{gy}} \, d^4x \end{aligned}}$$

- **Extended gyrocenter Hamiltonian**

$$\mathcal{H}_{\mathrm{gy}}(\mathcal{Z};\, \phi_1) \;\equiv\; H_{\mathrm{gy}}(\mathbf{X}, p_{\|}, \mu, t;\, \phi_1) \;-\; W$$

- **Extended gyrocenter Vlasov distribution**

$$\mathcal{F}_{\mathrm{gy}}(\mathcal{Z}) \;\equiv\; c\,\delta(W - H_{\mathrm{gy}})\; F(\mathbf{X}, p_{\|}, \mu, t)$$

- **Gyrokinetic Vlasov-Poisson Variational principle**

$$\int \delta\mathcal{L}_{\rm gy}\, d^4x \;=\; 0$$

$$\boxed{\begin{aligned}\delta\mathcal{L}_{\rm gy} \;&=\; -\,\epsilon\,\delta\phi_1\left[\sum\int\left(\epsilon^{-1}\frac{\delta H_{\rm gy}}{\delta\phi_1}\right)\,\mathcal{F}_{\rm gy}\,d^4p\right] \\ &\quad+\,\frac{\epsilon^2}{4\pi}\,\left(\delta\mathbf{E}_1\cdot\mathbf{E}_1\right)\;-\;\sum\int\left[\,\delta\mathcal{F}_{\rm gy}\,\mathcal{H}_{\rm gy}\,\right]d^4p\end{aligned}}$$

- **Eulerian variation of Vlasov distribution**

$$\delta\mathcal{F}_{\rm gy}\;\equiv\;\{\mathcal{S}_{\rm gy},\;\mathcal{F}_{\rm gy}\}_{\rm gc}\;=\;\frac{1}{B_\|^*}\,\frac{\partial}{\partial Z^a}\left(B_\|^*\,\mathcal{S}_{\rm gy}\,\{Z^a,\;\mathcal{F}_{\rm gy}\}_{\rm gc}\right)$$

- **Functional derivative** $[\delta^3_{\rm gc}\;\equiv\;\delta^3(\mathbf{X}+\boldsymbol{\rho}_0-\mathbf{x})]$

$$\epsilon^{-1}\,\frac{\delta H_{\rm gy}}{\delta\phi_1(\mathbf{x})}\;=\;e\,\left\langle\delta^3_{\rm gc}\;-\;\epsilon\,\{S_1,\;\delta^3_{\rm gc}\}_{\rm gc}\;+\;\cdots\right\rangle\;\equiv\;e\,\left\langle\mathsf{T}_{\rm gy}^{-1}\,\delta^3_{\rm gc}\right\rangle$$

- **Total Gyrocenter Reduced Displacement**

$$\mathsf{T}_{\mathrm{gy}}^{-1}\delta_{\mathrm{gc}}^{3} \;\equiv\; \mathsf{T}_{\epsilon}^{-1}\delta^{3}(\mathbf{X}-\mathbf{x}) \;\equiv\; \delta^{3}(\mathbf{X}+\boldsymbol{\rho}_{\epsilon}-\mathbf{x})$$

$$\boxed{\begin{aligned} \boldsymbol{\rho}_{\epsilon} \;&\equiv\; \boldsymbol{\rho}_{0} + \epsilon\,\boldsymbol{\rho}_{1} + \cdots \\ &=\; (\boldsymbol{\rho}_{00} + \epsilon_B\,\boldsymbol{\rho}_{01} + \cdots) \;\rightarrow\; \textbf{Guiding-center} \\ &\quad\; + \epsilon\,(\boldsymbol{\rho}_{10} + \epsilon_B\,\boldsymbol{\rho}_{11} + \cdots) + \cdots \;\rightarrow\; \textbf{Gyrocenter} \end{aligned}}$$

- **First-order gyrocenter displacement**

$$\boxed{\boldsymbol{\rho}_{10} \;\equiv\; \left\{\mathbf{X} + \boldsymbol{\rho}_{00},\, S_{10}\right\}_{\mathrm{gc}} \;=\; \langle\boldsymbol{\rho}_{10}\rangle + \widetilde{\boldsymbol{\rho}}_{10}}$$

- **Gyrokinetic polarization = Guiding-center** $(\epsilon_B\,\langle\boldsymbol{\rho}_{01}\rangle)$
  + Gyrocenter $(\epsilon\,\langle\boldsymbol{\rho}_{10}\rangle + \cdots)$

- **Final form of Gyrokinetic Vlasov-Poisson Variational principle**

$$\int \delta \mathcal{L}_{\mathsf{gy}}\, d^4x \;=\; \int \epsilon\, \delta\phi_1 \left[\frac{\epsilon\, \nabla \cdot \mathbf{E}_1}{4\pi} - \sum e \int F \left\langle \mathsf{T}_{\mathsf{gy}}^{-1}\, \delta_{\mathsf{gc}}^3 \right\rangle d^6z \right] d^4x$$

$$- \sum \int \mathcal{S}_{\mathsf{gy}} \left\{ \mathcal{F}_{\mathsf{gy}},\, \mathcal{H}_{\mathsf{gy}} \right\}_{\mathsf{gc}} d^8Z$$

$$+ \int \left(\frac{\partial \Lambda}{\partial t} + \nabla \cdot \mathbf{\Gamma} \right) d^4x \;\rightarrow\; 0$$

- **Noether fields**

$$\Lambda \;\equiv\; \sum \int \mathcal{S}_{\mathsf{gy}}\, \mathcal{F}_{\mathsf{gy}}\, d^4p$$

$$\mathbf{\Gamma} \;\equiv\; -\frac{\epsilon^2\, \delta\phi_1}{4\pi}\, \mathbf{E}_1 + \sum \int \left(\mathcal{S}_{\mathsf{gy}}\, \mathcal{F}_{\mathsf{gy}} \right) \frac{d_{\mathsf{gy}}\mathbf{X}}{dt}\, d^4p$$

- **Gyrokinetic Vlasov Equation**

$$\{\mathcal{F}_{\mathsf{gy}},\, \mathcal{H}_{\mathsf{gy}}\}_{\mathsf{gc}} \;=\; 0$$

$$\boxed{\begin{aligned} 0 \;&=\; \frac{\partial F}{\partial t} \;+\; \left\{F,\, H_{\mathsf{gy}}\right\}_{\mathsf{gc}} \\ &\equiv\; \frac{\partial F}{\partial t} \;+\; \frac{d_{\mathsf{gy}}\mathbf{X}}{dt}\cdot\nabla F \;+\; \frac{d_{\mathsf{gy}}p_{\|}}{dt}\,\frac{\partial F}{\partial p_{\|}} \end{aligned}}$$

- **Iterative solution of (Particle) Vlasov equation**

$$f \;\equiv\; \mathsf{T}_{\epsilon}F \;=\; \mathsf{T}_{\mathsf{gc}}(\mathsf{T}_{\mathsf{gy}}F) \;\simeq\; e^{-\boldsymbol{\rho}_0\cdot\nabla}\left(F \;+\; \epsilon\,\{S_1,\; F\}_{\mathsf{gc}} + \cdots\right)$$

$$\boxed{0 \;=\; \frac{df}{dt} \quad\rightarrow\quad 0 \;=\; \mathsf{T}_{\epsilon}^{-1}\left[\frac{d}{dt}\left(\mathsf{T}_{\epsilon}\,F\right)\right] \;\equiv\; \frac{d_{\epsilon}F}{dt}}$$

- **Gyrokinetic Poisson Equation**

$$\boxed{\begin{aligned} \epsilon\,\nabla\cdot\mathbf{E}_1 &= 4\pi\sum e\int F\,\left\langle \mathsf{T}_{\mathsf{gy}}^{-1}\,\delta^3_{\mathsf{gc}}\right\rangle\,d^6z \\ &\equiv 4\pi\left(\varrho - \nabla\cdot\boldsymbol{\mathcal{P}}\right) \end{aligned}}$$

- **Gyrocenter charge density**

$$\varrho \equiv \sum e\int F\,d^3p$$

- **Gyrocenter polarization** $\rightarrow$ multipole expansion

$$\boldsymbol{\mathcal{P}} \equiv \underbrace{\sum e\int F\,\langle\boldsymbol{\rho}_\epsilon\rangle\,d^3p}_{\textbf{Linear}} - \underbrace{\nabla\cdot\left(\sum\frac{e}{2}\int F\,\left\langle\boldsymbol{\rho}_\epsilon\boldsymbol{\rho}_\epsilon\right\rangle\,d^3p\right)}_{\textbf{Quadratic (with FLR)}} + \cdots$$

Gyrokinetic Energy-Momentum Conservation Laws

- **Noether Equation**

$$\delta\mathcal{L}_{\mathsf{gy}} \;=\; \frac{\partial\Lambda}{\partial t} \;+\; \nabla\cdot\mathbf{\Gamma}$$

- **Eulerian variations associated with space-time translations**

$$\boxed{\begin{aligned} \mathcal{S}_{\mathsf{gy}} \;&\equiv\; -\,W\,\delta t \;+\; \mathbf{p}_{\mathsf{gy}}\cdot\delta\mathbf{x} \\ \delta\phi_1 \;&\equiv\; -\,\delta t\;\partial_t\phi_1 \;+\; \delta\mathbf{x}\cdot\mathbf{E}_1 \\ \delta\mathcal{L}_{\mathsf{gy}} \;&=\; -\,\partial_\nu\left(\delta x^\nu\,\mathcal{L}_{\mathsf{gy}}\right) \;+\; \delta x^\nu\;\partial'_\nu\mathcal{L}_{\mathsf{gy}} \end{aligned}}$$

- **Explicit gradient in time-independent** $(\partial'_t\mathcal{L}_{\mathsf{gy}} \equiv 0)$ **nonuniform background plasma**

$$\nabla'\mathcal{L}_{\mathsf{gy}} = -\;\left(\frac{B}{4\pi} + \sum\int F\;\mu\;d^3p\right)\nabla B - \sum\;e\;\int\;F\;\nabla'\Phi_{\mathsf{gy}}\;d^3p$$

- **Gyrokinetic Energy Conservation Law**

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \nabla\cdot\mathbf{S} = 0}$$

- **Gyrokinetic energy density**

$$\mathcal{E} = \sum \int F \left(H_{\rm gy} - \epsilon\, e \left\langle {\sf T}_{\rm gy}^{-1} \phi_{1\rm gc} \right\rangle \right) d^3p + \frac{\epsilon^2\, |\mathbf{E}_1|^2}{8\pi}$$

- **Gyrokinetic energy-density flux**

$$\mathbf{S} = \sum \int F\, H_{\rm gy}\, \frac{d_{\rm gy}\mathbf{X}}{dt}\, d^3p + \frac{\epsilon^2\, \phi_1}{4\pi}\, \frac{\partial \mathbf{E}_1}{\partial t}$$

- **Gyrokinetic Linear Momentum Conservation Law**

$$\boxed{\frac{\partial \mathbf{P}}{\partial t} + \nabla\cdot\mathbf{\Pi} = -\sum\int F\,\nabla' H_{\mathrm{gy}}\,d^3p}$$

◦ **Gyrokinetic momentum density**

$$\mathbf{P} = \sum\int F\,\mathbf{p}_{\mathrm{gy}}\,d^3p$$

◦ **Gyrokinetic momentum-stress tensor**

$$\mathbf{\Pi} = \frac{\epsilon^2}{4\pi}\left(|\mathbf{E}_1|^2\,\frac{\mathbf{I}}{2} - \mathbf{E}_1\,\mathbf{E}_1\right) + \sum\int F\,\frac{d_{\mathrm{gy}}\mathbf{X}}{dt}\,\mathbf{p}_{\mathrm{gy}}\,d^3p$$

- **Proofs of Energy-Momentum Conservation**

  - **Gyrokinetic Energy Conservation**

$$\frac{\partial \mathcal{E}}{\partial t} = -\nabla\cdot\mathbf{S} + \sum \epsilon\, e \int F\left[\left\langle \mathsf{T}_{\mathsf{gy}}^{-1}\left(\frac{\partial \phi_{1\mathsf{gc}}}{\partial t}\right)\right\rangle - \left\langle \frac{\partial}{\partial t}\left(\mathsf{T}_{\mathsf{gy}}^{-1}\,\phi_{1\mathsf{gc}}\right)\right\rangle\right] + \frac{\epsilon^2\phi_1}{4\pi}\,\nabla\cdot\frac{\partial \mathbf{E}_1}{\partial t} - \sum \epsilon\, e \int \frac{\partial F}{\partial t}\,\left\langle \mathsf{T}_{\mathsf{gy}}^{-1}\,\phi_{1\mathsf{gc}}\right\rangle \equiv -\nabla\cdot\mathbf{S}$$

  - **Gyrokinetic Momentum Conservation**

$$\begin{aligned}\frac{\partial \mathbf{P}}{\partial t} &= -\nabla\cdot\mathbf{\Pi} + \sum \int F\left(\frac{d_{\mathsf{gy}}\mathbf{p}_{\mathsf{gy}}}{dt} - \epsilon\, e\,\left\langle \mathsf{T}_{\mathsf{gy}}^{-1}\,\mathbf{E}_{1\mathsf{gc}}\right\rangle\right) \\ &= -\nabla\cdot\mathbf{\Pi} + \sum \int F\left[\left(-\mu\,\nabla B - \epsilon\, e\,\nabla\Phi_{\mathsf{gy}}\right)\right. \\ &\qquad\qquad \left. + \epsilon\, e\,\left(\nabla\Phi_{\mathsf{gy}} - \nabla'\Phi_{\mathsf{gy}}\right)\right] \\ &\equiv -\nabla\cdot\mathbf{\Pi} - \sum \int F\,\nabla' H_{\mathsf{gy}}\end{aligned}$$

- **Gyrokinetic Angular Momentum Conservation in Axisymmetric Tokamak Geometry**

  - **Infinitesimal toroidal rotation**

$$\delta\mathbf{x} \;\equiv\; \frac{\partial\mathbf{x}}{\partial\varphi}\,\delta\varphi \;=\; \delta\varphi\,\widehat{\mathsf{z}}\times\mathbf{x}$$

  - **Gyrokinetic angular-momentum conservation law**

$$\frac{\partial P_{\varphi}}{\partial t} \;+\; \nabla\cdot\mathbf{\Pi}_{\varphi} \;=\; \frac{\partial' H_{\mathrm{gy}}}{\partial\varphi} \;\equiv\; 0 \;\; (\text{axisymmetry})$$

  - **Gyrokinetic angular-momentum density**

$$P_{\varphi} \;\equiv\; \sum \int F\, p_{\mathrm{gy}\varphi}\; d^3p$$

  - **Gyrokinetic angular-momentum flux**

$$\mathbf{\Pi}_{\varphi} \;\equiv\; \left(\frac{\epsilon^2}{8\pi}\,|\mathbf{E}_1|^2\right)\frac{\partial\mathbf{X}}{\partial\varphi} \;-\; \frac{\epsilon^2}{4\pi}\,\mathbf{E}_1\left(\mathbf{E}_1\cdot\frac{\partial\mathbf{X}}{\partial\varphi}\right) \;+\; \sum\int F\,\frac{d_{\mathrm{gy}}\mathbf{X}}{dt}\,p_{\mathrm{gy}\varphi}\; d^3p$$

∘ **Gyrokinetic Poisson equation** $\Rightarrow$

$$\nabla\cdot\left[\left(\frac{\epsilon^2}{8\pi}\,|\mathbf{E}_1|^2\right)\,\frac{\partial\mathbf{X}}{\partial\varphi}-\frac{\epsilon^2}{4\pi}\mathbf{E}_1\left(\mathbf{E}_1\cdot\frac{\partial\mathbf{X}}{\partial\varphi}\right)\right]=\sum\int F\,\frac{\partial H_{\mathrm{gy}}}{\partial\varphi}\,d^3p$$

$$\boxed{\begin{aligned}\frac{\partial H_{\mathrm{gy}}}{\partial\varphi}&=\epsilon\,e\left\langle\frac{\partial\phi_{1\mathrm{gc}}}{\partial\varphi}-\epsilon\left\{S_1,\,\frac{\partial\phi_{1\mathrm{gc}}}{\partial\varphi}\right\}_{\mathrm{gc}}+\cdots\right\rangle\\&\equiv\epsilon\,e\left\langle\mathsf{T}_{\mathrm{gy}}^{-1}\left(\frac{\partial\phi_{1\mathrm{gc}}}{\partial\varphi}\right)\right\rangle\end{aligned}}$$

∘ **Scott-Smirnov equation**

(Gyrokinetic Vlasov moment approach)

$$\boxed{\frac{\partial P_\varphi}{\partial t}=-\nabla\cdot\mathbf{Q}_\varphi-\sum\int F\left(\frac{\partial H_{\mathrm{gy}}}{\partial\varphi}\right)d^3p}$$

$$\mathbf{Q}_\varphi\equiv\sum\int F\,\frac{d_{\mathrm{gy}}\mathbf{X}}{dt}\,p_{\mathrm{gy}\varphi}\,d^3p$$

Gyrokinetic Momentum Conservation in Axisymmetric Tokamak Geometry

- **Magnetic-surface average**

$$\llbracket \cdots \rrbracket \;\equiv\; \frac{1}{\mathcal{V}} \oint (\cdots)\,\mathcal{J}\,d\vartheta\,d\varphi$$

- **Jacobian for magnetic coordinates (ψ, θ, φ)**

$$\begin{aligned} \mathcal{J}^{-1} \;&\equiv\; \nabla\psi\times\nabla\theta\cdot\nabla\varphi \;=\; B^{\theta} \\ \mathcal{V}(\psi) \;&\equiv\; \oint \mathcal{J}\,d\vartheta\,d\varphi \end{aligned}$$

- **Important identity for arbitrary vector field $\mathbf{C}$**

$$\llbracket \nabla\cdot\mathbf{C} \rrbracket \;\equiv\; \frac{1}{\mathcal{V}}\,\frac{\partial}{\partial\psi}\left(\mathcal{V}\;\left\llbracket \mathbf{C}\cdot\nabla\psi \right\rrbracket\right)$$

- **Gyrokinetic parallel-toroidal momentum equation**

  - **Gyrokinetic parallel-toroidal momentum**

$$\boxed{P_{\|\varphi} \;\equiv\; P_{\varphi} + \frac{\psi}{c}\,\varrho \;=\; \left(\sum \int F\, p_{\|}\, d^3p\right) b_{\varphi}}$$

  - **Gyrokinetic parallel-toroidal momentum equation**

$$\boxed{\frac{\partial P_{\|\varphi}}{\partial t} + \nabla\cdot\mathbf{Q}_{\varphi} \;=\; \frac{\psi}{c}\,\frac{\partial \varrho}{\partial t} - \sum \int F\,\frac{\partial H_{\mathsf{gy}}}{\partial \varphi}\,d^3p}$$

- **Surface-averaged gyrokinetic parallel-toroidal momentum equation**

$$\boxed{\begin{aligned}\frac{\partial [\![P_{\|\varphi}]\!]}{\partial t} \;=\; & -\frac{1}{\mathcal{V}}\,\frac{\partial}{\partial\psi}\left(\mathcal{V}\;\left[\!\!\left[Q_{\varphi}^{\psi}\right]\!\!\right]\right) \;+\; \frac{\psi}{c}\,\frac{\partial [\![\varrho]\!]}{\partial t} \\ & -\sum\left[\!\!\left[\int F\,\frac{\partial H_{\mathrm{gy}}}{\partial\varphi}\,d^{3}p\right]\!\!\right]\end{aligned}}$$

$$\begin{aligned}[\![Q_{\varphi}^{\psi}]\!] \;&=\; \sum\left[\!\!\left[\int F\,\frac{d_{\mathrm{gy}}\psi}{dt}\,p_{\mathrm{gy}\varphi}\,d^{3}p\right]\!\!\right] \\ &\equiv\; [\![Q_{\|\varphi}^{\psi}]\!] \;-\; \frac{\psi}{c}\;\underbrace{\left(\sum e\left[\!\!\left[\int F\,\frac{d_{\mathrm{gy}}\psi}{dt}\,d^{3}p\right]\!\!\right]\right)}_{\text{Gyrocenter current } [\![\nabla\psi\,\cdot\,\mathbf{J}]\!]}\end{aligned}$$

- **Gyrocenter charge conservation law**

$$\frac{\partial [\![\varrho]\!]}{\partial t} = -\,[\![\nabla\cdot\mathbf{J}]\!] \equiv -\,\frac{1}{\mathcal{V}}\,\frac{\partial}{\partial\psi}\left(\mathcal{V}\sum e\,\left[\!\!\left[\int F\,\frac{d_{\mathrm{gy}}\psi}{dt}\,d^3p\right]\!\!\right]\right)$$

$$\boxed{\begin{aligned}\frac{\partial [\![P_{\|\varphi}]\!]}{\partial t} = {} & -\,\frac{1}{\mathcal{V}}\,\frac{\partial}{\partial\psi}\left(\mathcal{V}\,[\![Q^{\psi}_{\|\varphi}]\!]\right) \\ & + \sum e\,\left[\!\!\left[\int F\,\left(\frac{1}{c}\,\frac{d_{\mathrm{gy}}\psi}{dt} - \frac{\partial\Phi_{\mathrm{gy}}}{\partial\varphi}\right)\,d^3p\right]\!\!\right]\end{aligned}}$$

$$\frac{1}{c}\,\frac{d_{\mathrm{gy}}\psi}{dt} - \frac{\partial\Phi_{\mathrm{gy}}}{\partial\varphi} \equiv \text{Generalized toroidal electric field}$$

- **Gyrocenter quasineutrality condition**

$$\boxed{\mathbf{E}_1 + 4\pi\,\boldsymbol{\mathcal{P}} \simeq 4\pi\,\boldsymbol{\mathcal{P}} \;\;\Rightarrow\;\; \varrho \;\equiv\; \nabla\cdot\boldsymbol{\mathcal{P}}}$$

$$\frac{\partial[\![\varrho]\!]}{\partial t} \;\equiv\; \left[\!\!\left[\nabla\cdot\frac{\partial\boldsymbol{\mathcal{P}}}{\partial t}\right]\!\!\right] \;=\; \frac{1}{\mathcal{V}}\,\frac{\partial}{\partial\psi}\left(\mathcal{V}\,\frac{\partial[\![\mathcal{P}^{\psi}]\!]}{\partial t}\right)$$

- **Gyrocenter charge conservation law** $\Rightarrow$

$$\frac{\partial[\![\mathcal{P}^{\psi}]\!]}{\partial t} + \sum e\left[\!\!\left[\int F\,\frac{d_{\rm gy}\psi}{dt}\,d^3p\right]\!\!\right] \;\equiv\; 0$$

- **Parallel-toroidal momentum equation**

$$\frac{\partial}{\partial t}\left(\llbracket P_{\|\varphi}\rrbracket + \frac{1}{c}\,\llbracket \mathcal{P}^{\psi}\rrbracket\right) \;=\; -\,\frac{1}{\mathcal{V}}\frac{\partial}{\partial\psi}\left(\mathcal{V}\,\llbracket Q^{\psi}_{\|\varphi}\rrbracket\right) - \sum e\,\left\llbracket \int F\,\frac{\partial \Phi_{\rm gy}}{\partial\varphi}\,d^3p\right\rrbracket$$

- **Physics of total toroidal-momentum density**

$$\boxed{\underbrace{\left(\frac{\partial \mathbf{X}}{\partial\varphi}\cdot p_{\|}\,\widehat{\mathsf{b}}\right)}_{\to\; P_{\|\varphi}} + \underbrace{\left[\frac{\partial \mathbf{X}}{\partial\varphi}\times\frac{\widehat{\mathsf{b}}}{\Omega}\cdot\left(\mu\,\nabla B + \frac{p_{\|}^2}{m}\,\widehat{\mathsf{b}}\cdot\nabla\widehat{\mathsf{b}} + \epsilon\,e\nabla\phi_1\right)\right]}_{\to\; c^{-1}\,\mathcal{P}^{\psi}}}$$

- **Gyrokinetic Parallel-toroidal Momentum Conservation Law**

∘ **Multipole expansion + Quasineutrality condition**

$$\frac{\partial H_{\mathrm{gy}}}{\partial \varphi} \;\equiv\; \epsilon\, e\left(\frac{\partial \phi_1}{\partial \varphi} + \langle \boldsymbol{\rho}_\epsilon \rangle \cdot \nabla \frac{\partial \phi_1}{\partial \varphi} + \frac{1}{2} \langle \boldsymbol{\rho}_\epsilon \boldsymbol{\rho}_\epsilon \rangle : \nabla\nabla \frac{\partial \phi_1}{\partial \varphi} + \cdots \right)$$

$$\boxed{\begin{aligned} &\sum \int F\, \frac{\partial H_{\mathrm{gy}}}{\partial \varphi}\, d^3p \;= \\ &\nabla \cdot \left[\epsilon\, \boldsymbol{\mathcal{P}}\, \frac{\partial \phi_1}{\partial \varphi} + \epsilon \left(\sum \frac{e}{2} \int F\, \langle \boldsymbol{\rho}_\epsilon \boldsymbol{\rho}_\epsilon \rangle\, d^3p\right) \cdot \nabla \frac{\partial \phi_1}{\partial \varphi} + \cdots \right] \end{aligned}}$$

∘ **ZLR model of Scott and Smirnov**

$$\boldsymbol{\mathcal{P}} \;\simeq\; \epsilon\, (mnc^2/B^2) \nabla_\perp \phi_1$$

- **Gyrokinetic parallel-toroidal momentum conservation law**

$$\boxed{\begin{aligned} 0 \;=\; & \frac{\partial}{\partial t}\left([\![P_{\|\varphi}]\!] \;+\; \frac{1}{c}\,[\![\mathcal{P}^{\psi}]\!] \right) \;+\; \frac{1}{\mathcal{V}}\frac{\partial}{\partial \psi}\left(\mathcal{V}\,[\![Q^{\psi}_{\|\varphi}]\!] \right) \\ & + \frac{1}{\mathcal{V}}\frac{\partial}{\partial \psi}\left[\mathcal{V}\left(\epsilon\left[\!\!\left[\mathcal{P}^{\psi}\,\frac{\partial \phi_1}{\partial \varphi} \;+\; \mathbf{R}^{\psi}\cdot\nabla\frac{\partial \phi_1}{\partial \varphi} \;+\; \cdots \right]\!\!\right] \right) \right] \end{aligned}}$$

$$\mathbf{R}^{\psi} \;\equiv\; \nabla\psi\cdot\left(\sum \frac{e}{2}\int F\,\langle \boldsymbol{\rho}_{\epsilon}\boldsymbol{\rho}_{\epsilon}\rangle\, d^3p \right) \;=\; \mathbf{R}^{\psi}_0 \;+\; \epsilon\,\mathbf{R}^{\psi}_1 \;+\; \cdots$$

- **Residual-stress toroidal-momentum transport**

$$\mathcal{P}^{\psi}\,\frac{\partial \phi_1}{\partial \varphi} \;\equiv\; \text{Turbulent contribution}$$

$$\mathbf{R}^{\psi}\cdot\nabla\frac{\partial \phi_1}{\partial \varphi} \;\equiv\; \text{Electric-field shear-induced contribution}$$

Summary and Future Work

- **Noether Method**

Frontload Physics into Lagrangian $\Rightarrow$ Exact Conservation Laws

- **Dynamical Reduction**

Conservation of Toroidal Canonical Momentum

$$p_{\mathrm{gy}\varphi} \;=\; -\,\frac{e}{c}\,\psi \;+\; p_{\|}\,b_{\varphi}$$

$$\Downarrow$$

Conservation of Toroidal Momentum Density

$$\sum \int \, F\, \left(m\,\frac{d_{\mathrm{gy}}\varphi}{dt} \right)\, d^3p$$

- **Future Work: Electromagnetic Case**