Generalized Fjørtoft argument for the gyrokinetic dual cascade

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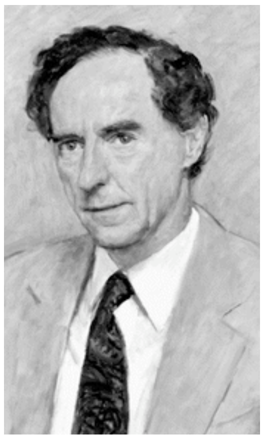
Outline

- The inverse cascade and Fjørtoft's argument for fluid turbulence
  - Some history, some context
  - Thought experiment: Transfer among 3 scales, and among arbitrarily many
  - Centroids
- Two dimensional gyrokinetics
  - Phase-space spectrum
  - Generalised Fjørtoft argument
  - Three flavors of cascade behavior
- Zonal flows and the inverse cascade
  - Generalized Hasegawa Mima
  - Gyrokinetics

Early history of the "inverse cascade"

Onsager, L. (1949), *Nuovo Cimento, Suppl. 6, 249*:
Statistical equilibrium of point vortices – negative temperature states and explanation of persistent large-scale motions in 2D fluid flow



Batchelor, G. K. (1953), The Theory of Homogeneous Turbulence:
Identifies the cause of inverse energy transfer: simultaneous conservation of enstrophy and energy. Predicts that the motion of the energy "centroid" will be toward progressively larger scales.

Fjortoft, R., (1953), *Tellus, 5, 225 (Also see Merilees and H. Warn, 1975)*:
Precise and general limits on the spectral redistribution of energy. **Does not** present a theory of cascade. **Does not** make predictions or assumptions about equilibrium or non-equilibrium stationary states.

Kraichnan, R. H. (1967), Phys. Fluids, 10, 1417 (Also Leith, Batchelor):
Calculates statistical equilibrium (following T.D. Lee), revisits Fjortoft argument and advances the concept of a "dual cascade" with two inertial subranges with distinct power-laws.

Fjørtoft: three-scale energy transfer

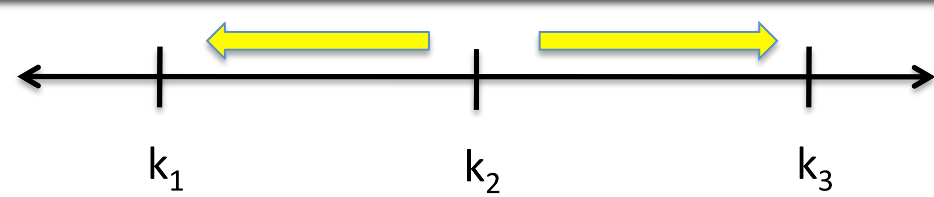
Relationship between spectra: $Z(k) = k^2 E(k)$

$$\Delta E = 0 \quad \Rightarrow \quad \Delta E_1 + \Delta E_2 + \Delta E_3 = 0$$

$$\Delta Z = 0 \quad \Rightarrow \quad k_1^2 \Delta E_1 + k_2^2 \Delta E_2 + k_3^2 \Delta E_3 = 0$$

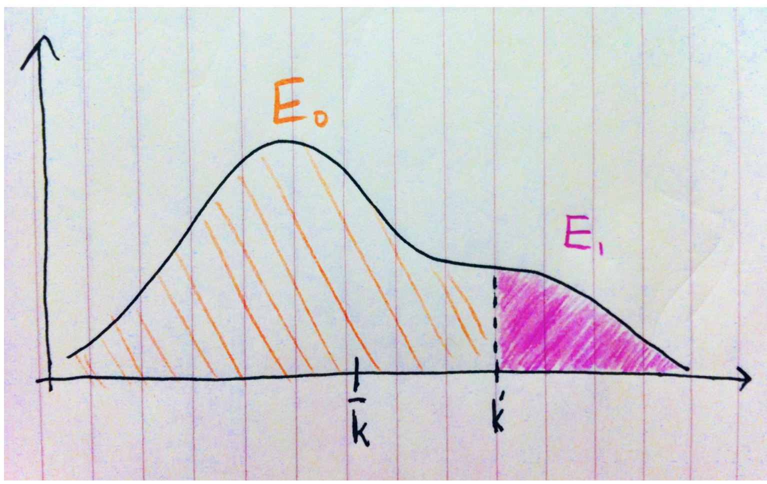
$$\boxed{\Delta E_1 = -\frac{k_3^2 - k_2^2}{k_3^2 - k_1^2} \Delta E_2} \qquad \boxed{\Delta E_3 = -\frac{k_2^2 - k_1^2}{k_3^2 - k_1^2} \Delta E_2}$$

> follows: No single of the three components can in this case represent a source or a sink for the *both* two remaining ones unless this is represented by a scale intermediate between the scales of the two other components.



Arbitrarily many scales

$$E = \int E(k)dk \qquad Z = \int k^2 E(k)dk$$



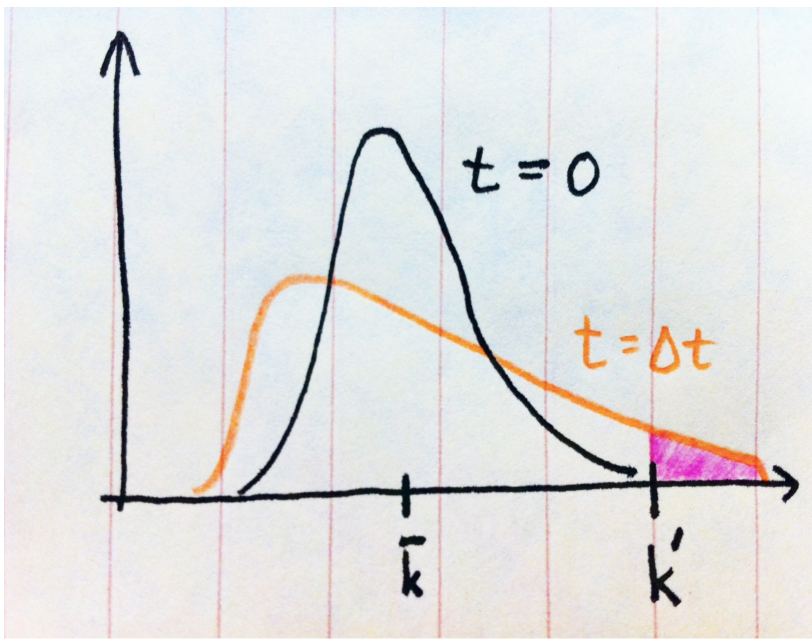
$$E_0 + E_1 = E$$
$$Z_0 + Z_1 = Z$$
$$\Downarrow$$
$$k_0^2 E_0 + k_1^2 E_1 = \bar{k}^2 E$$

$$k_0 \le (\bar{k}, k') \le k_1$$

$$F = E_1/E \qquad F = \frac{\bar{k}^2 - k_0^2}{k_1^2 - k_0^2} < \frac{\bar{k}^2}{k_1^2 - k_0^2} < \frac{\bar{k}^2}{(k')^2 - k_0^2}$$

A statement of Fjortoft's general result

- The fraction of energy, F(t), that can be found above some $k' \gg \bar{k}$ is bounded:



$$F(t) < \left(\frac{k'}{\bar{k}}\right)^2$$

Centroid Inequalities

Energy centroid:

$$k_E = \frac{\int kE(k)dk}{\int E(k)dk}$$

Enstrophy centroid:

$$k_Z = \frac{\int k^3E(k)dk}{\int k^2E(k)dk}$$

Invariant wavenumber:

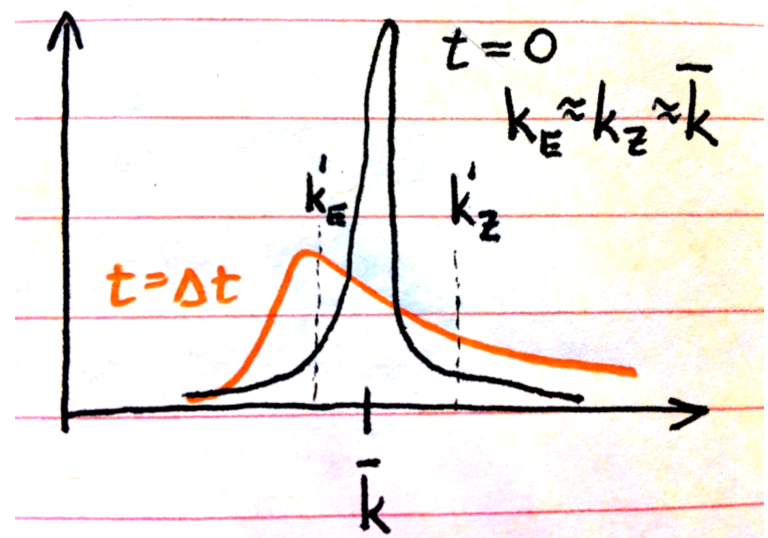
$$\bar{k} = \sqrt{Z/E}$$

[Nazarenko, Quinn, 2010. IUTAM Symposium on Turbulence in the Atmosphere and Oceans, pp. 265]

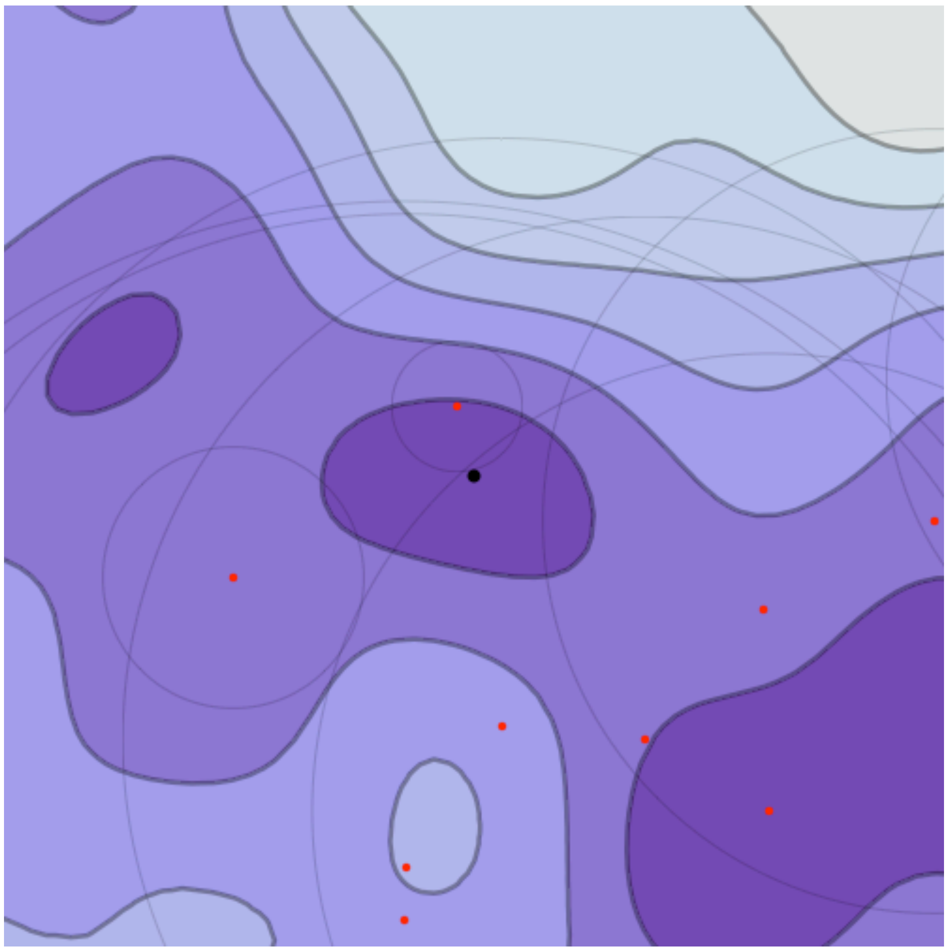
Cauchy-Schwarz Inequality:

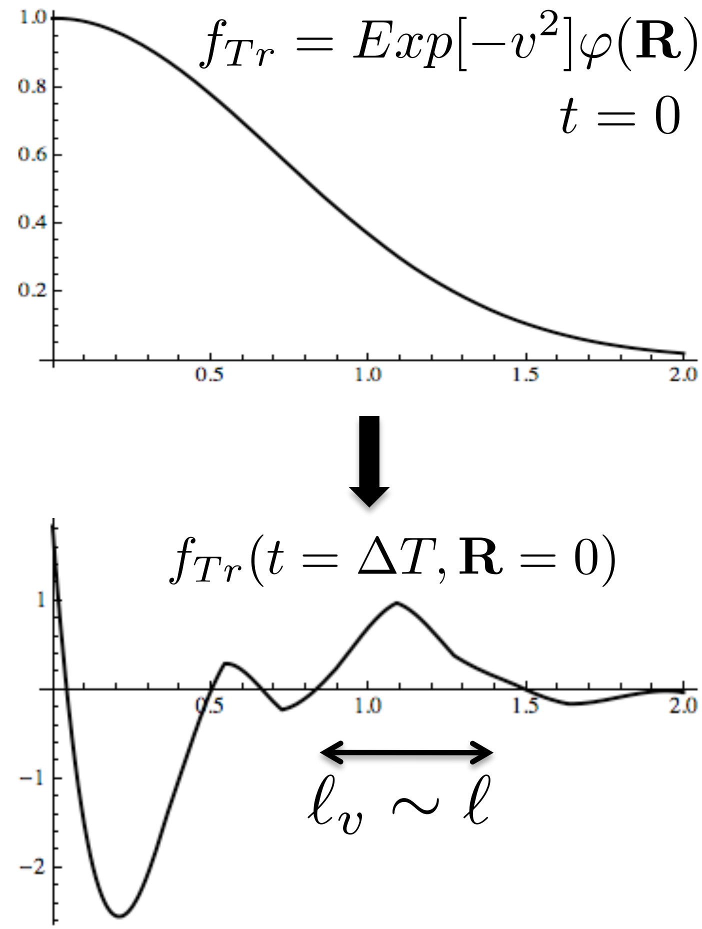
$$\left|\int f(x)g(x)dx\right| \leq \left|\int f(x)^2dx\right|^{1/2}\left|\int g(x)^2dx\right|^{1/2}$$

$$\boxed{k_E \leq \bar{k} \leq k_Z \qquad k_E k_Z \geq \bar{k}^2}$$



Gyrokinetic "phase space cascade": Physics of nonlinear phase mixing





2D Gyrokinetics: Nonlinear phase-mixing and not much else.

$$\boxed{\frac{\partial g}{\partial t} + \{\langle \phi \rangle_{\mathbf{R}}\,,\ g\} = \langle C \rangle_{\mathbf{R}}}$$

$$\boxed{\int d^3\mathbf{v}\,\langle g \rangle_{\mathbf{r}} = (1+\tau)\varphi - \Gamma_0\varphi}$$

"Gen. Free Energy": $$W_g = 2\pi \int v dv \int \frac{d^2\mathbf{R}}{V}\frac{g^2}{2F_0}$$

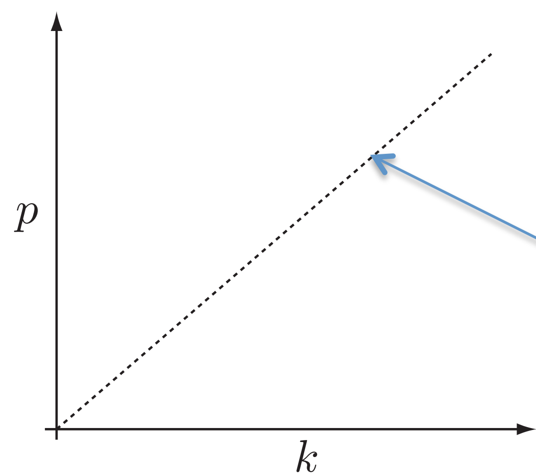
"Electrostatic Energy": $$E = \frac{1}{2}\int \frac{d^2\mathbf{r}}{V}\left[(1+\tau)\varphi^2 - \varphi\Gamma_0\varphi\right]$$

* [G. G. Plunk, et al., (2010). J. Fluid Mech., 664, pp 407-435]

Phase-space spectrum

Hankel & Fourier Transform:

$$\hat{g}(\mathbf{k}, p) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} d^2\mathbf{R} \int_0^{\infty} v dv J_0(pv) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{R}} g(\mathbf{R}, v)$$



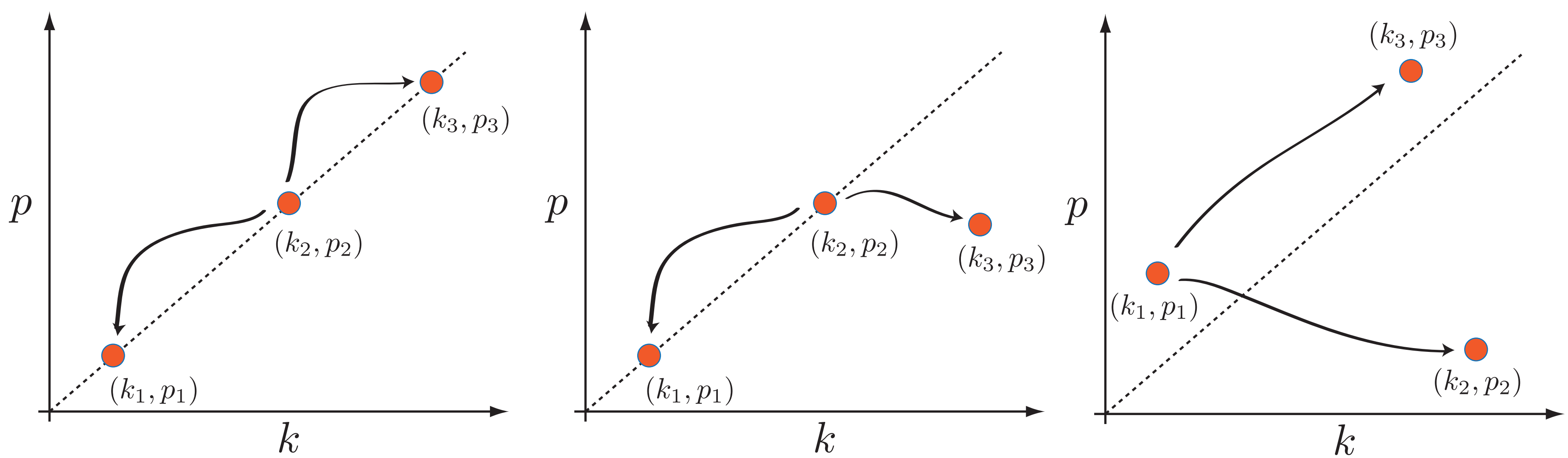
$$\hat{\varphi}(\mathbf{k}) = \beta(\mathbf{k}) \hat{g}(\mathbf{k}, k)$$

Spectral Transfer

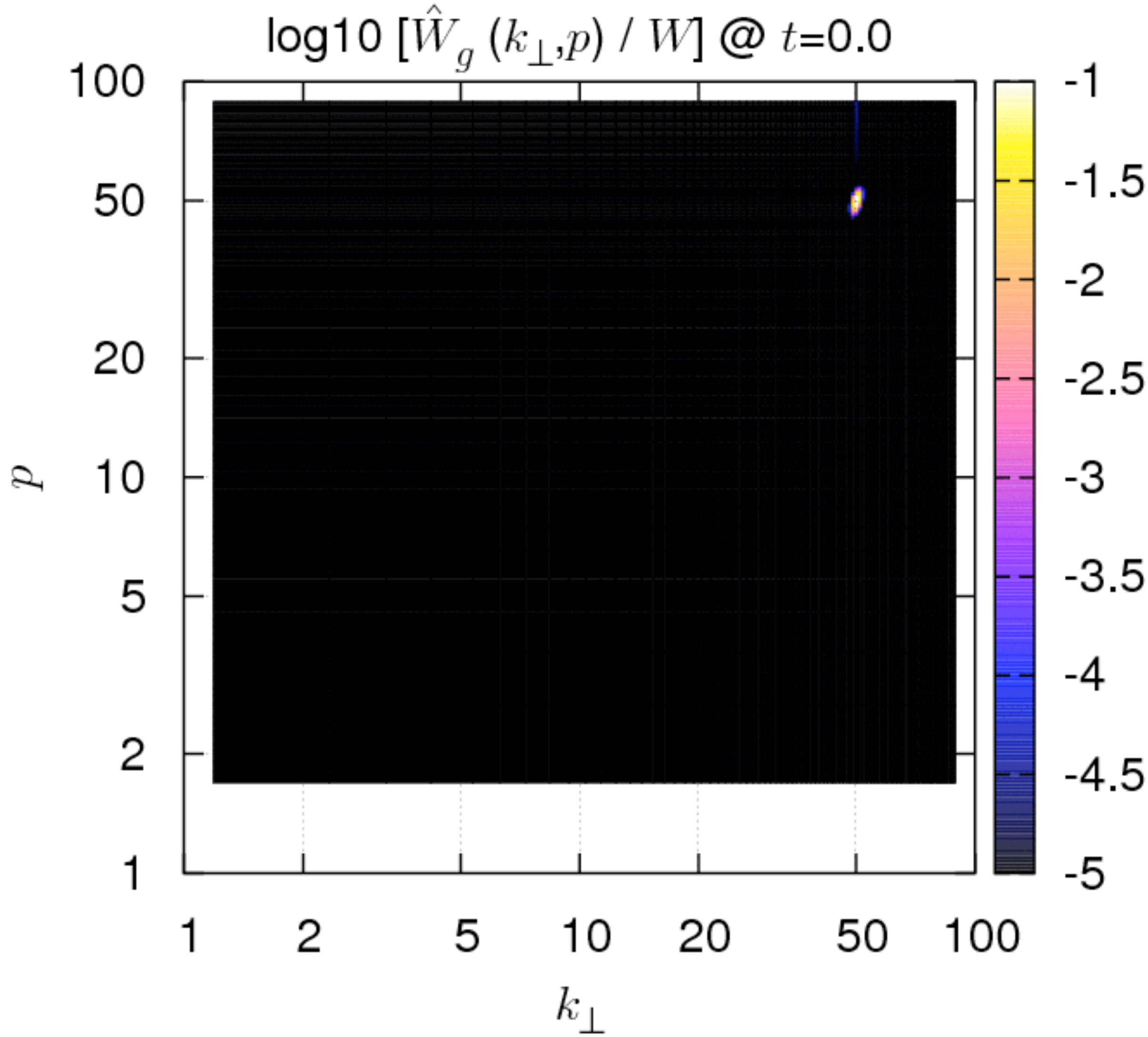
"Free Energy": $$W_g = 2\pi \int v dv \int \frac{d^2\mathbf{R}}{V} \frac{g^2}{2F_0}$$

"Electrostatic Energy": $$E = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} \left[(1+\tau)\varphi^2 - \varphi\Gamma_0\varphi\right]$$

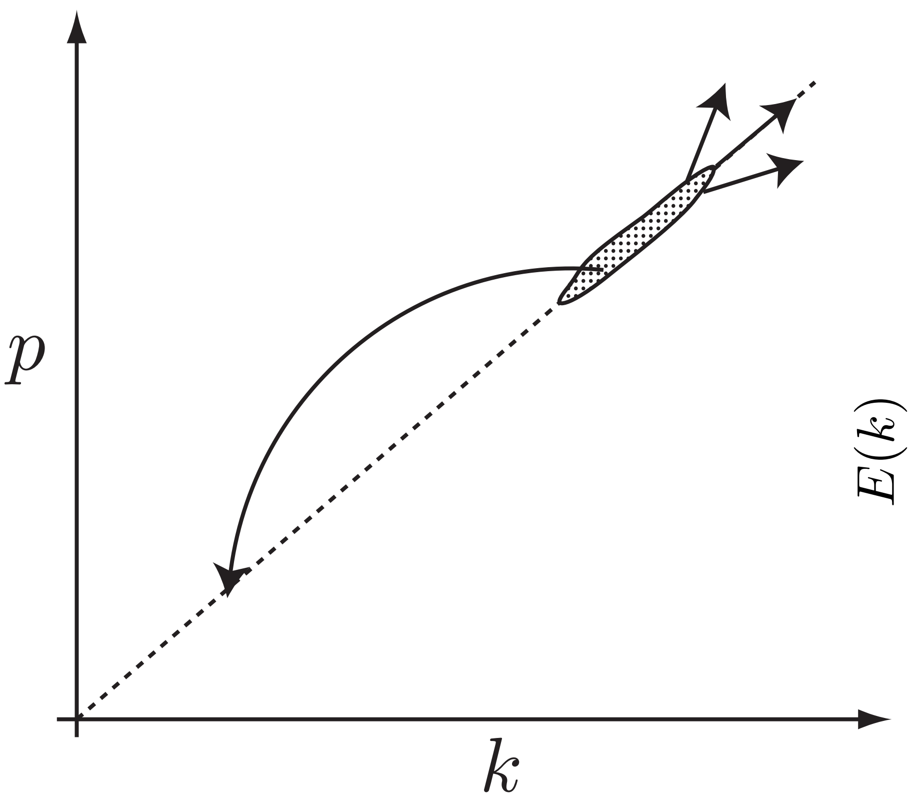
Constraint: $$E(k) = \frac{\beta(k)}{k} W_g(k,k) \sim W_g(k,k)/k$$

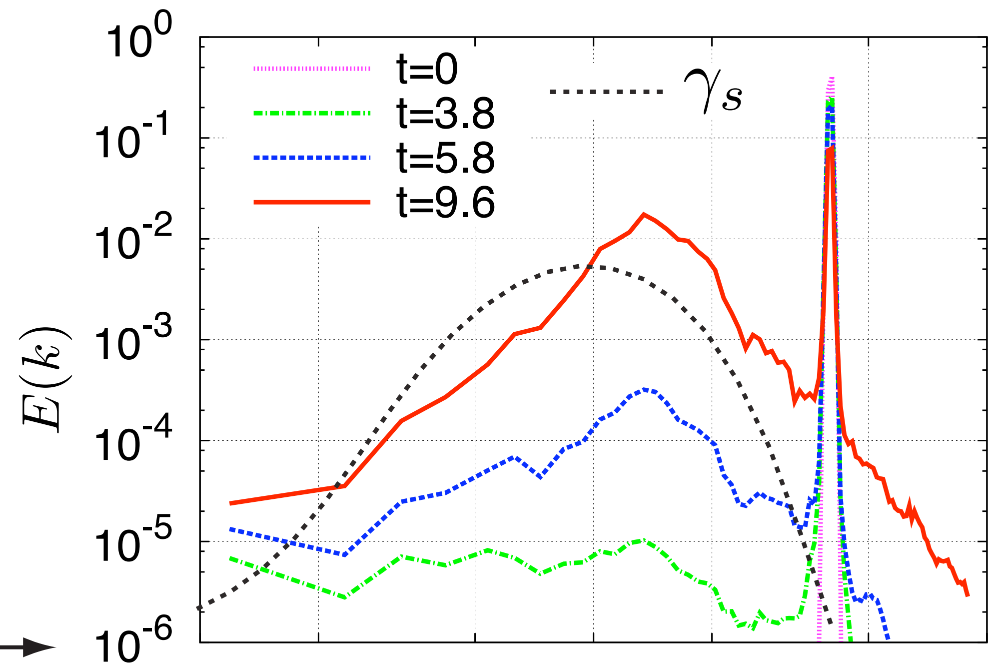


Inverse cascade of E

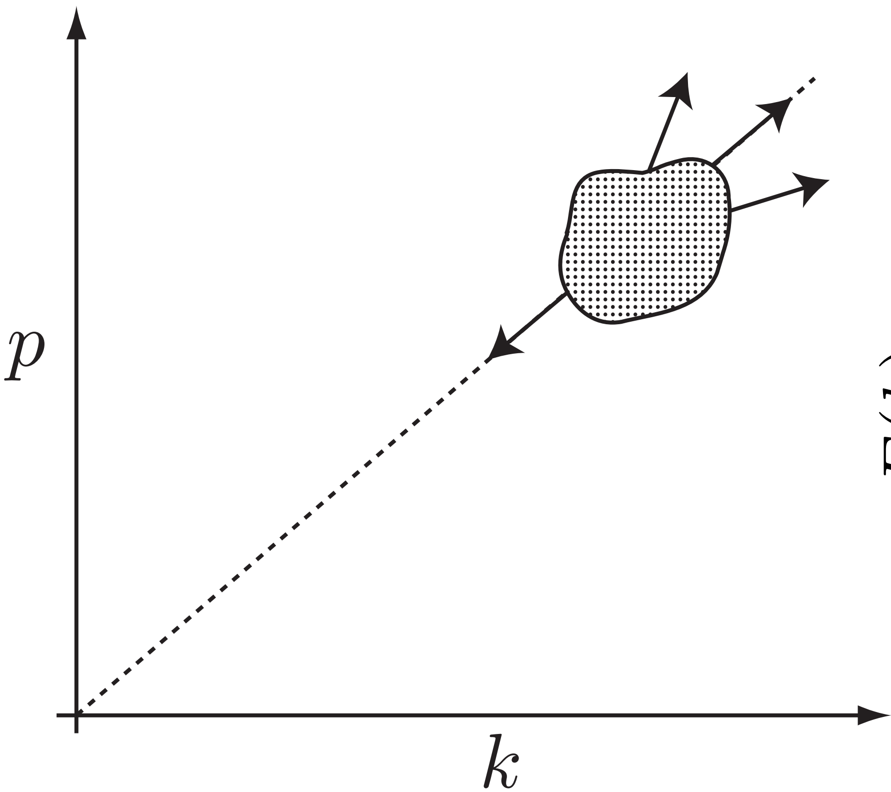


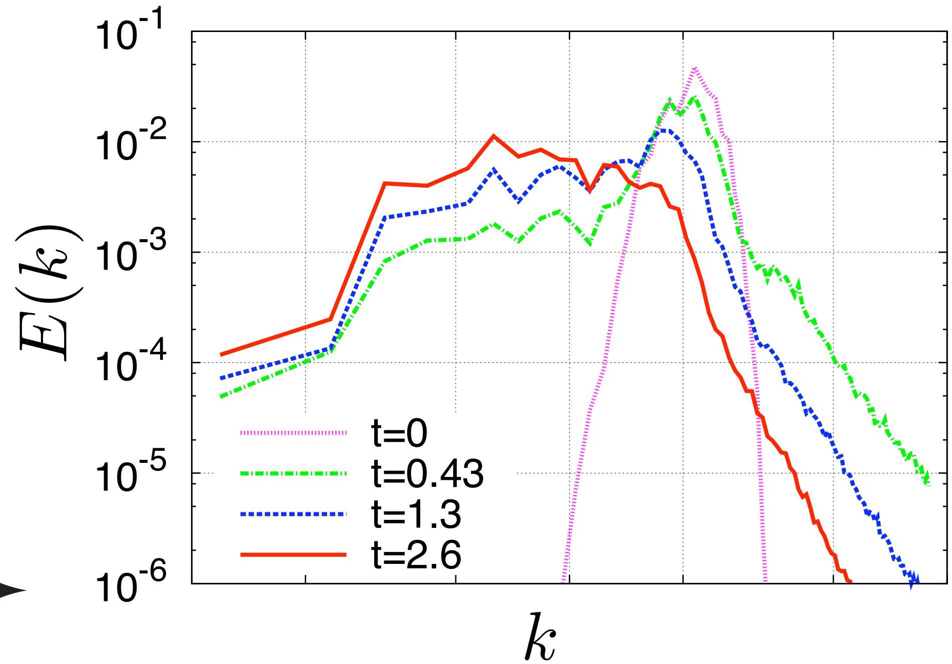
Flavors of dual cascade:
Local forward, Nonlocal inverse



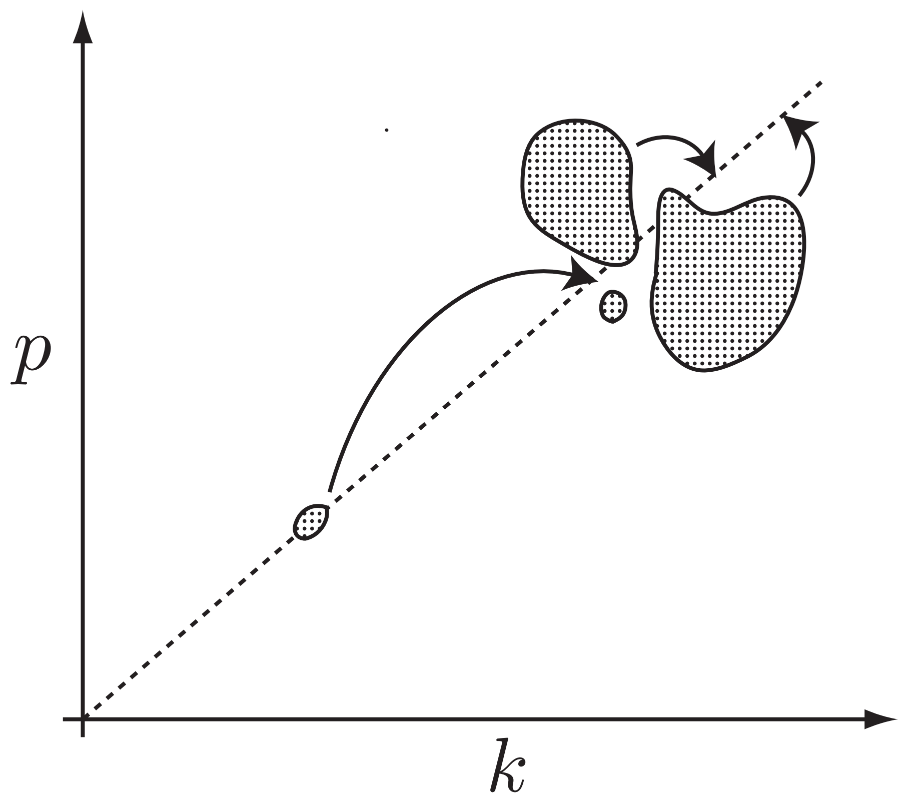


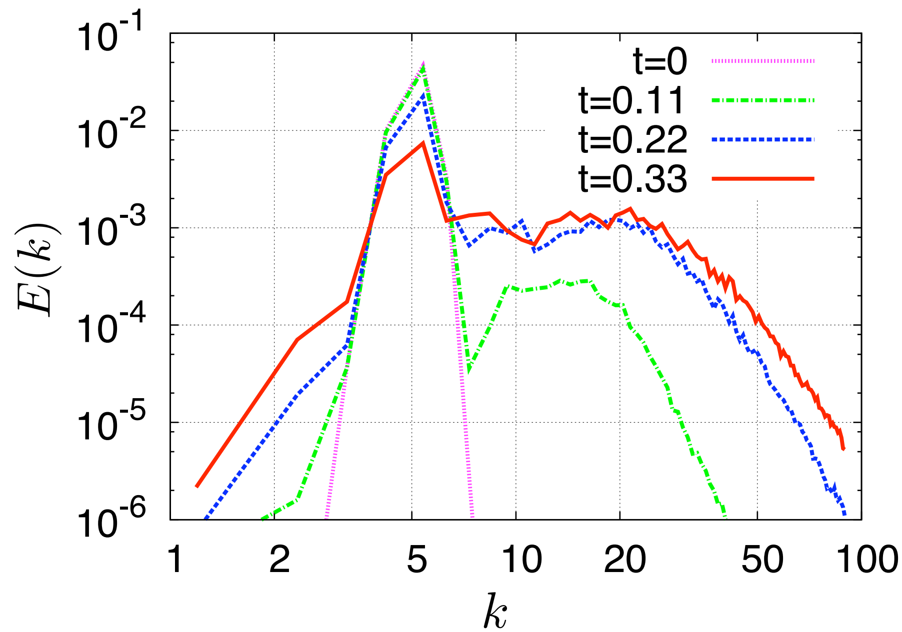
Flavors of dual cascade:
Local forward, Local inverse





Flavors of dual cascade:
Dual forward





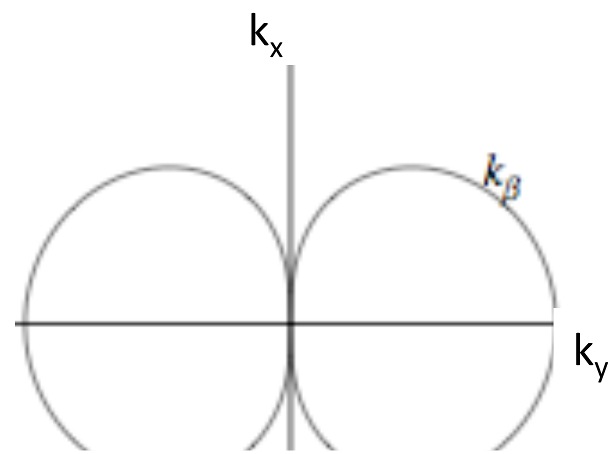
"Sub-Larmor damping"

Zonal flows

Anisotropic inverse cascade by the linear "B-effect" or "critical balance" in the inverse cascade

Williams, G. P. 1978. *J. Atmos. Sci.* ***35, 1399–1426.***
See also: G. K. Vallis., Atmospheric and Oceanic Fluid Dynamics, pp. 381

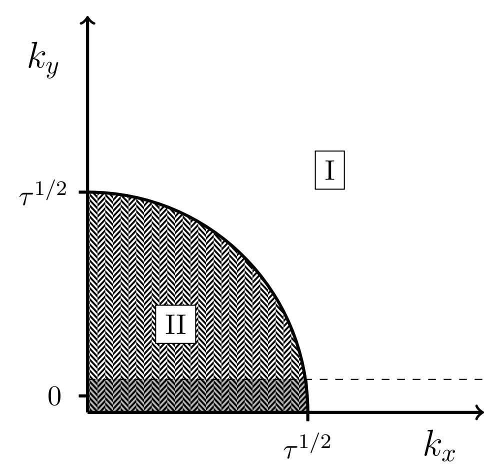
Hasegawa, A. & Mima, K. 1978



Generalized Hasegawa Mima (GHM)

$$\frac{Z(\mathbf{k})}{E(\mathbf{k})} = q^2 = \tilde{\tau} + k^2$$

$$\tilde{\tau} = \tau(1 - \delta(k_y))$$



* [Plunk, et al., *in preparation* (2011)]

Zonal flow regulation by dual "cascade" ($k^2\rho^2 << 1$)

Orthogonalize: $\hat{\varphi} = a\hat{g}_0 + b\hat{g}_1$ $\hat{\psi} = b\hat{g}_0 - a\hat{g}_1$

Free energy can be re-expressed:

$$W_g = W_0 + W_1 + W_2 + \ldots$$
$$= \boxed{W_0' + W_1'} + W_2 + \ldots$$

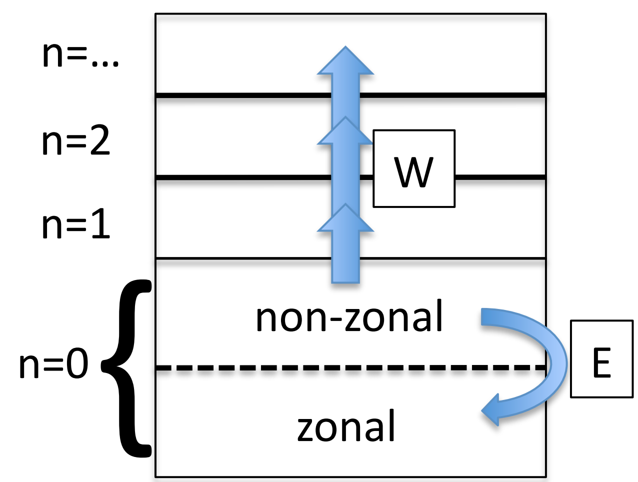
$$W_0' = W_\varphi, \; W_1' = W_\psi, \; W_2' = W_2, \ldots$$

Effective wavenumber that governs dual cascade:

$$q^2 \equiv W_\varphi(\mathbf{k})/E(\mathbf{k})$$

$$q^2 \approx \begin{cases} k^2 & \textit{for zonal flows} \\ \tau & \textit{for non-zonal fluctuations} \end{cases}$$

Fjortoft Argument:



[Plunk, Tatsuno, PRL **106**, 165003 (2011) arXiv:1007.4787]

Two-field gyrofluid model of ITG turbulence

$$\frac{\partial \varphi}{\partial t} + B^{-1}\{\varphi,\ \tau\tilde{\varphi} - \nabla^2\varphi\} + B^{-1}N_2[\varphi, T_\perp] = A_{11}\tilde{\varphi} + A_{12}\tilde{T_\perp}$$

$$\frac{\partial T_\perp}{\partial t} + \{\varphi,\ T_\perp\} + \{\nabla^2\varphi,\ 2T_\perp - \tau\tilde{\varphi}/2\} = A_{22}\tilde{T_\perp} + A_{21}\tilde{\varphi}$$

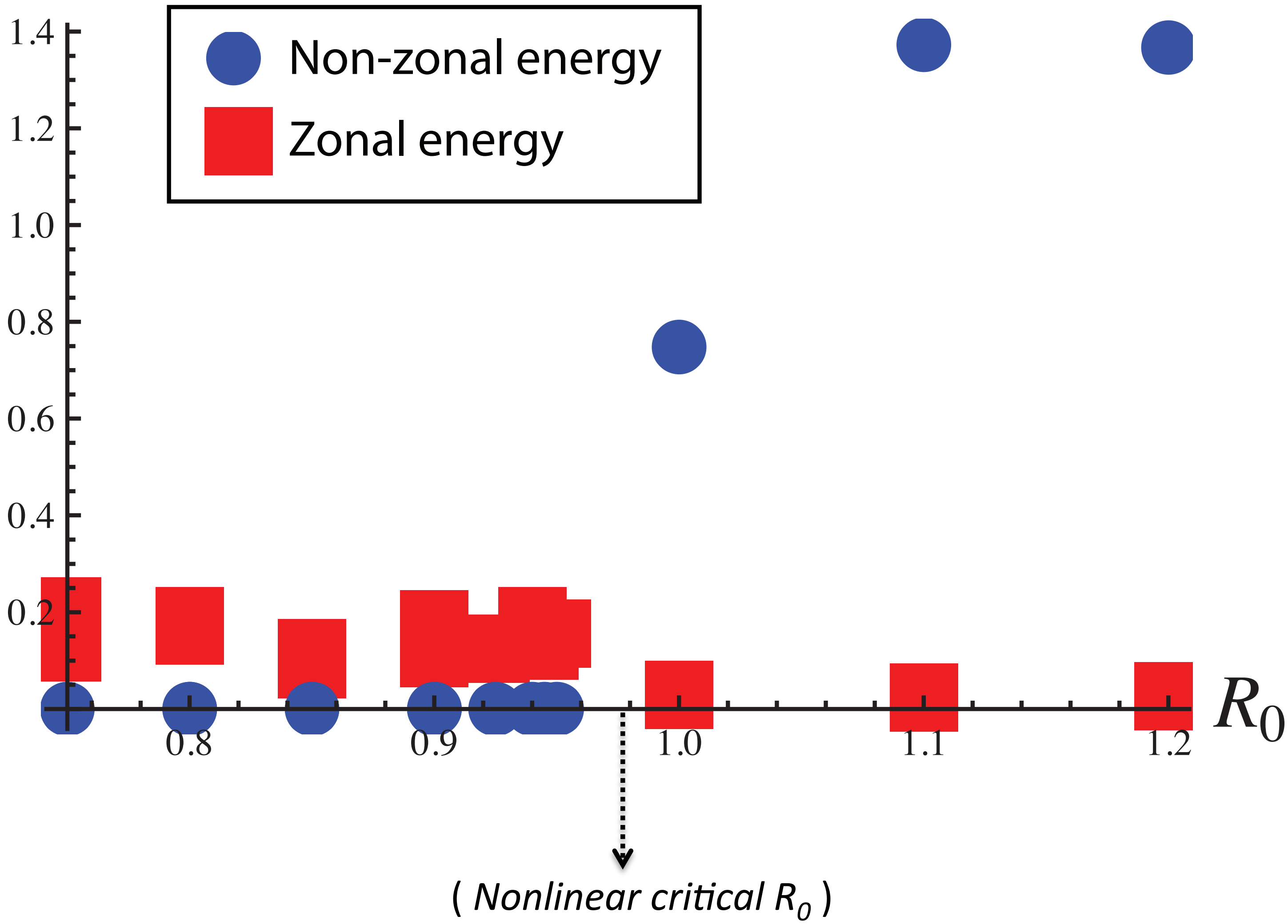
$$B = \tilde{\tau} - \nabla^2$$

$$N_2[\varphi, T_\perp] = \nabla^2\{\varphi,\ T_\perp\} + \{\nabla^2\varphi,\ T_\perp\} - \{\varphi,\ \nabla^2 T_\perp\}$$

These linear operators model ITG instability, diamagnetic frequency, models of kinetic damping and collisional dissipation.

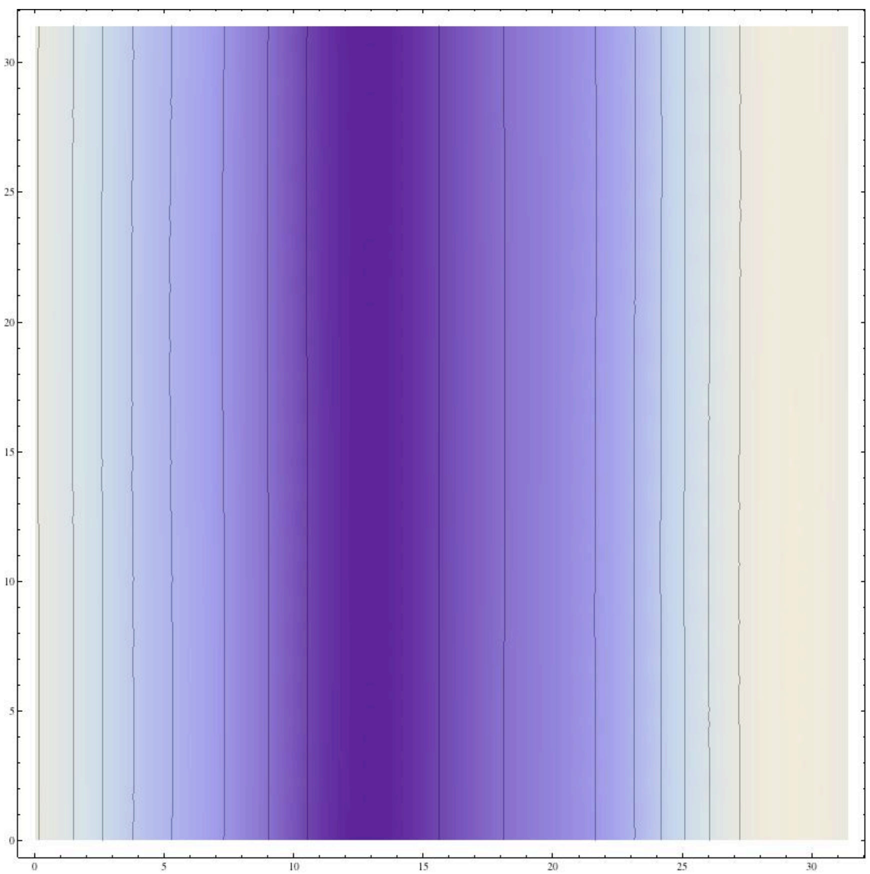
Hand-tuned linear model:
$$\left\{ \begin{aligned} \omega &= \frac{k_y}{2}\left(v_* \pm G\sqrt{(k/k_w)^2 - 1}\right) - i\nu_D(\mathbf{k}) \\ \frac{\hat{T}_\perp}{\hat{\varphi}} &= R_0[\sin(\phi_0)\sqrt{(k/k_w)^2 - 1} + \cos(\phi_0)] \end{aligned} \right\}$$

Steady State Energies vs. R_0



Electrostatic Potential

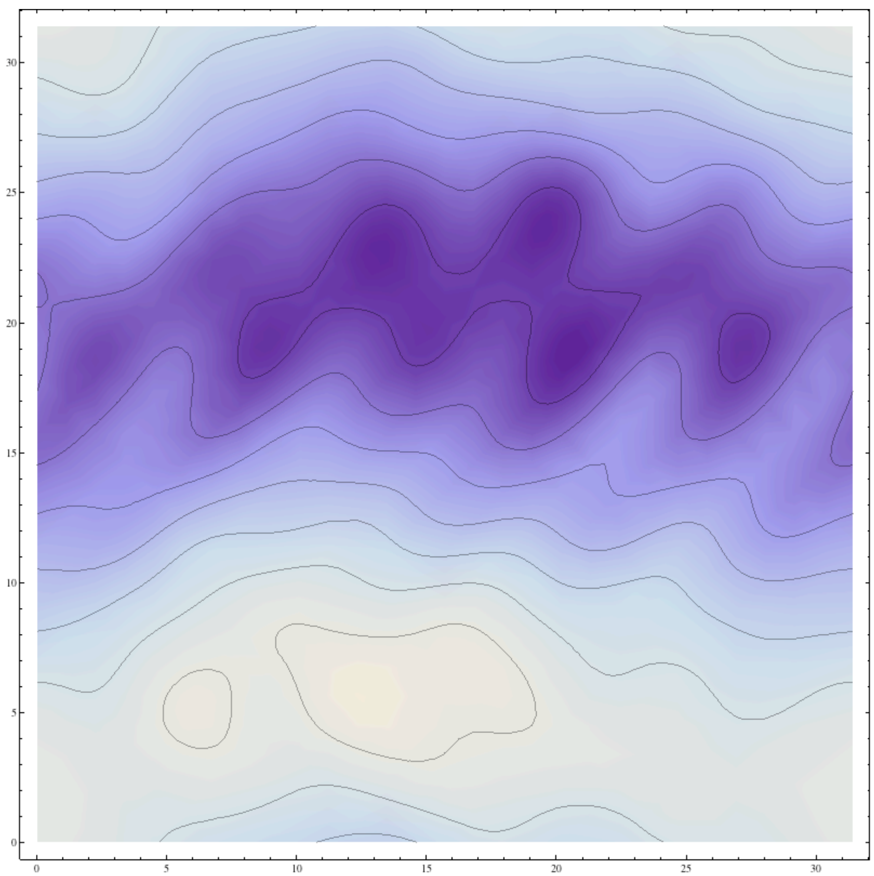
$R_0 = 0.85$



y (poloidal direction)

x (radial direction)

$R_0 = 1.2$



x (radial direction)

Concluding Remarks

- Nonlinearity in Gyrokinetics conserves two quantities
- Dual cascade can induce upscale or downscale transfer of (electrostatic) energy, depending on initial excitation
  - Distinguishes GK turbulence from fluid turbulence
- Nonlinear zonal flow regulation by dual cascade
  - Appears by simple arguments in GHM turbulence
  - More sophisticated in gyrokinetics – direction of energy flow can be reversed!
- Open question: How do we tailor the drive to control the dual cascade (control transport)?