# Edge Simulation Laboratory (ESL)

# Progress and Plans (June 2008)

**Presented by: R.H. Cohen**

edge
simulation
laboratory

PSACI Meeting, PPPL
June 5-6, 2008

Collaborators:

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 Work at LLNL performed under the auspices of the U.S. Department of Energy under contract No. DE-AC52-07NA27344

# OUTLINE

edge simulation laboratory

- ESL at a glance
- ESL components: recent progress
- Plans

Recent publications/submissions:

- X.Q. Xu, E. Belli, K. Bodi, J. Candy, C.S. Chang, R. H. Cohen, P. Colella, A.M. Dimits, M. R. Dorr, J. A. Hittinger, Z. Gao, J. Hittinger, S. Krasheninnikov, S. Ko, S. Krasheninnikov, G.R. McKee, W. M. Nevins, T. D. Rognlien, P. B. Snyder, J. Suh, M. V. Umansky,, "Fully Nonlinear Simulations of Kinetic Geodesic Acoustic Modes and Boundary Flows", to be presented at 2008 IAEA Fusion Energy Conference.
- X.Q. Xu , Z. Xiong, M.R. Dorr, J.A. Hittinger, et al., "Edge Gyrokinetic Theory and Continuum Simulations", Nuclear Fusion **47**, 809 (2007).
- Z.Xiong, R. Cohen, T. Rognlien and X. Xu, "A high-order finite-volume algorithm for Fokker-Planck collisions in magnetized plasmas", J. Comp. Phys., published online as http://dx.doi.org/10.1016/j.jcp.2008.04.004
- X.Q. Xu, A. Xiong, W. Nevins and G. McKee, "TEMPEST simulations of collisionless damping of GAM and neoclassical residual in edge plasma pedestal", Phys. Rev. Lett **100**, 215001 (2008)
- R.H.Cohen and X.Q.Xu, (2007), "Progress in Kinetic Simulation of Edge Plasmas" , Contrib. Plasma Phys. **48**, 212 (2008).
- X. Xu, "Neoclassical Simulation of Tokamak Plasmas using Continuum Gyrokinetic Code TEMPEST", submitted to Phys. Rev. E.
- E. Belli & J. Candy, 'Kinetic Calculation of Neoclassical Transport Including Self-Consistent Electron and Impurity Dynamics',submitted to PPCF.
- P. Colella and M.D. Sekora, "A Limiter for PPM that Preserves Accuracy at Smooth Extrema, to be published in JCP.

# What is the ESL?

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- Edge Simulation Laboratory (ESL): a project to develop gyrokinetic simulation for MFE edge plasmas based on continuum (Eulerian) techniques
- Why continuum?
  - Concerns about PIC noise in environment where there are large density variations and where full f is required
  - Exploit advanced numerical methods from fluids community
  - Build on successes of continuum core codes (GYRO, GS2, GENE)
- ESL is a collaboration: LLNL, GA, UCSD, LBNL, PPPL, Lodestar, CompX. Others welcome.
- Present funding: **NOT SCIDAC**. Funded from OFES and OASCR base programs
  - OFES
    - 0.8 FTE LLNL
    - Postdoc + $\varepsilon$ FTE GA
    - Grad student at UCSD
    - New: 1/3 FTE CS support, allocation to be determined
  - Associated math activity in algorithm research, promised 1.8 FTE split between LBL and LLNL finally achieved spring ’08.
  - Question for PAC: is this balance appropriate?

# ESL is different than the other projects being reviewed

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- Not a SciDAC; base program funding
- We are more exploratory:
  - Relatively uncharted terrain
  - Heavy algorithmic emphasis
  - No immediate mandate to exploit HPC resources, though we will need these as we move to 5D (have tested scaling of one of our codes up to 500 processors)
- Not funded as a project, especially on OASCR side where funding is explicitly for algorithmic development (and where there have been several funding gaps)

- Historical note: funding began Dec. 2005-Sept 2006 depending on institution so project is between 1.75 and 2.5 years old.

# ESL has three funded components

- TEMPEST code (outgrowth of LLNL LDRD project; full geometry, full-f, $\varepsilon$ -$\mu$ finite difference.)
- GA rapid-prototype codes, EGK ($v_{||}$-$\mu$), NEO ($v_{||}/v$, $\varepsilon_k$), simple geometry (no separatrix); presently linear
- Math component: develops and implements algorithms for a next-generation code
- Previously unfunded, needed, and now getting 1/3 FTE from OFES: a computer science component to enhance/maintain software infrastructure, provide user support, and address needs for data handing and analysis

# TEMPEST is a full-f, full-geometry edge kinetic code

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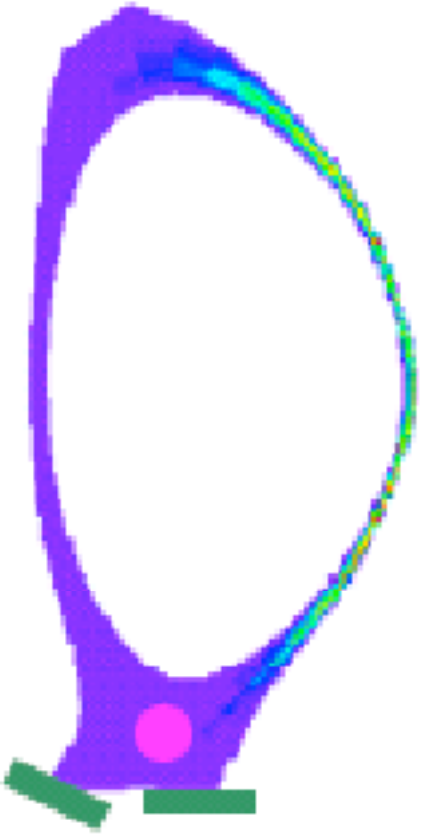
- 5D ($\psi,\theta,\zeta,E_0,\mu$); results here 4D
  - $E_0$-$\mu$ choice for accurate || streaming
- Full f, but also $\delta$f option
- Electrostatic (EM deferred to next gen. code)
- Geometry options:
  - Shifted circle core
  - Full single-null diverted, closed-flux-surface + SOL
- Implicit backward-differencing time advance; Newton-Krylov iteration
- 4th-order upwinded finite-difference spatial discretization, and Weno
- Low-order finite-volume discretization for collisions
- Collision options
  - Krook
  - Lorentz with full v dependence
  - Full collision op. with test-particle or fully nonlinear Rosenbluth potentials



# EGK and Neo are simple-geometry rapid-prototype codes

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- "Rapid-prototype codes" to explore physics, coordinate, and algorithm issues associated with edge simulation
- Last year you heard about EGK, a $\delta f$ code in $v_{||}$-$\mu$ coordinates used to explore the use of this representation for drift-wave and neoclassical phyiscs.
- This year: NEO:
  - Core geometry but general shape
  - $v_{||}/v$-$\varepsilon_k$ coordinates represented by basis functions
  - Choice of linearized collision operators (Connor, zero-order and full Hirshman-Sgmar)
  - Electrostatic with poloidal correction to potential
  - Multi-species (kinetic electrons, multiple ions)
- Goals:
  - Develop practical tool for high-accuracy neoclassical calculations
  - Framework for exploration of new formulations for calculation of neoclassical $E_r^0$

# Next-generation ESL code builds on experiences from TEMPEST, EGK, Neo, and core codes

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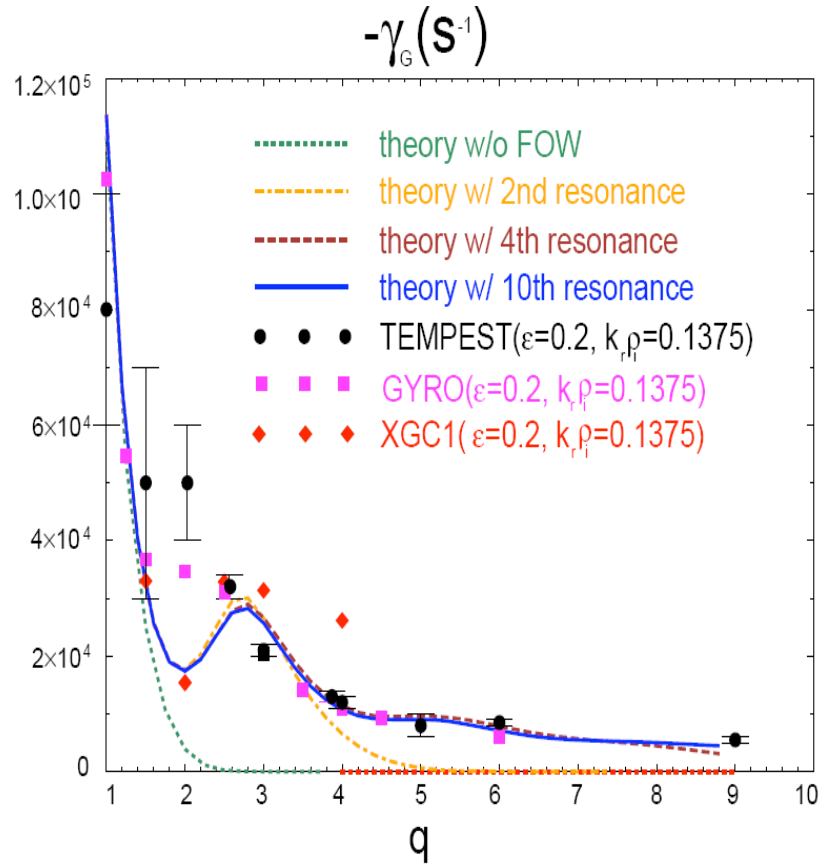
- Conservative form of GK equations
- 4th order finite-volume (conservative) discretization
- $v_{||}$-μ coordinates (tentatively)
- Arbitrarily mapped multiblock grids, field-aligned on block (allowing for shifts at any box boundary), to handle magnetic shear
- AMR capability
- Electrostatic initially; subsequently EM
- Math team has been developing algorithms to enable this, and initial implementation of a code

# TEMPEST simulation of GAMs motivate extension to theory and cross-code comparison

- Geodesic acoustic modes (GAMs): a coherent poloidal flow oscillation
- Why we are interested:
  - A good test problem
  - Clearly identified experimentally
  - May dominate in edge
- Tempest results indicated more damping at high-q than traditional 0-orbit-width theory.
- With recent extension of theory calculation to higher bounce-harmonic resonances agreement is now good between TEMPEST, theory, and other codes
  - Using theory formulation from Gao, Itoh et al PoP 06




**X.Q. Xu, W.M. Nevins, Z. Gao, G.R. McKee, A.M. Dimits, E. Belli, J. Candy, P. Snyder, C.S. Chang, S. Ko, and J. Suh, TTF 2008**

# TEMPEST shows that GAMs radially propagate outward in edge pedestal plasma as in expt.

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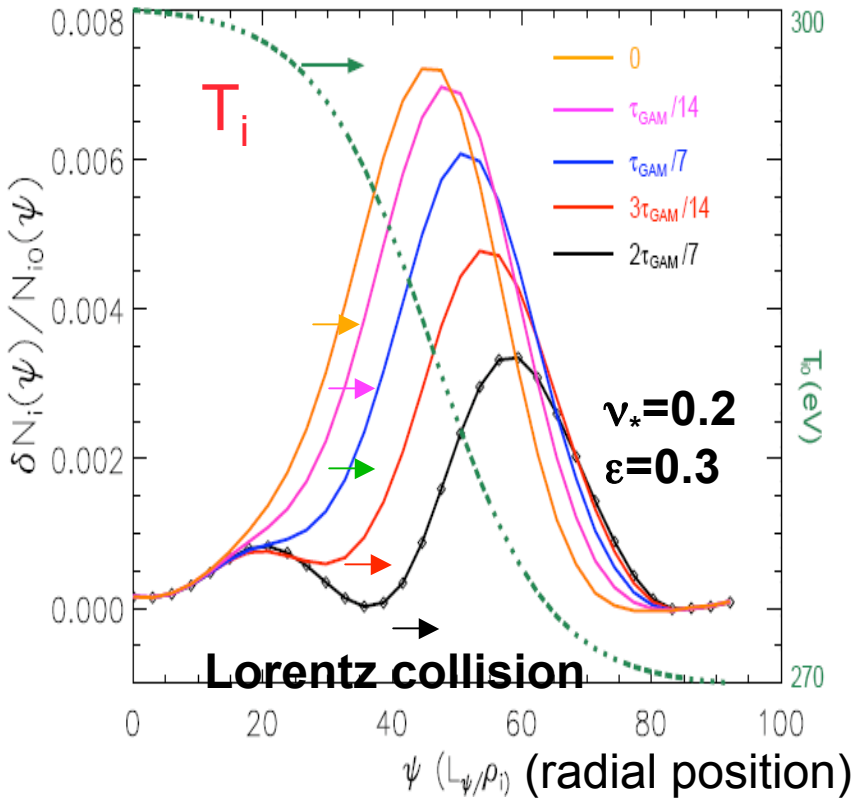
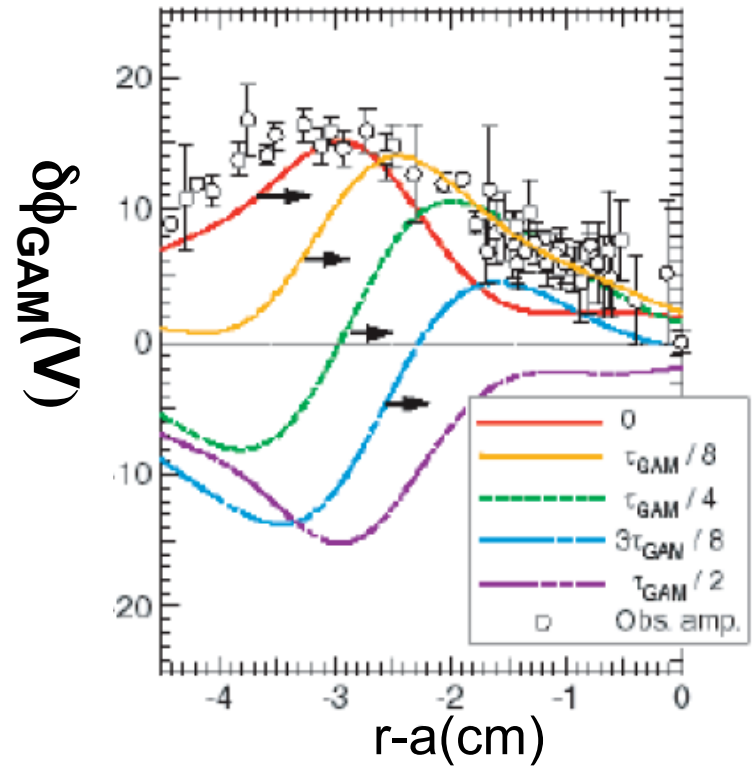
## TEMPEST simulation

($\nabla T_i$ required for propagation)




**Evidence of outward propagating GAM In JFT-2M [ Ido et al. PPCF 2006]**




$$\omega^2=\omega^2_{GAM}(\psi)+C(\psi,\omega)k^2_\psi\rho_i^2$$

Itoh, Itoh, Diamond, et al, Plasma and Fusion Res., 1, 037 (2006)

L Chen & Zonca, Europ. Phys. Lett, 2007

$\lambda_{GAM}^{Itoh}\sim\pi\rho_i(L_{Ti}/\rho_i)^{1/3}$

$V_{pr}^{Itoh}\sim\omega_{GAM}/\lambda_{GAM}^{Itoh}=15.95v_{Ti}\rho_i/R$

$V_{pr}^{Sim}=12.08v_{Ti}\rho_i/R$

# TEMPEST GAM simulations shows long-time behavior consistent with theory

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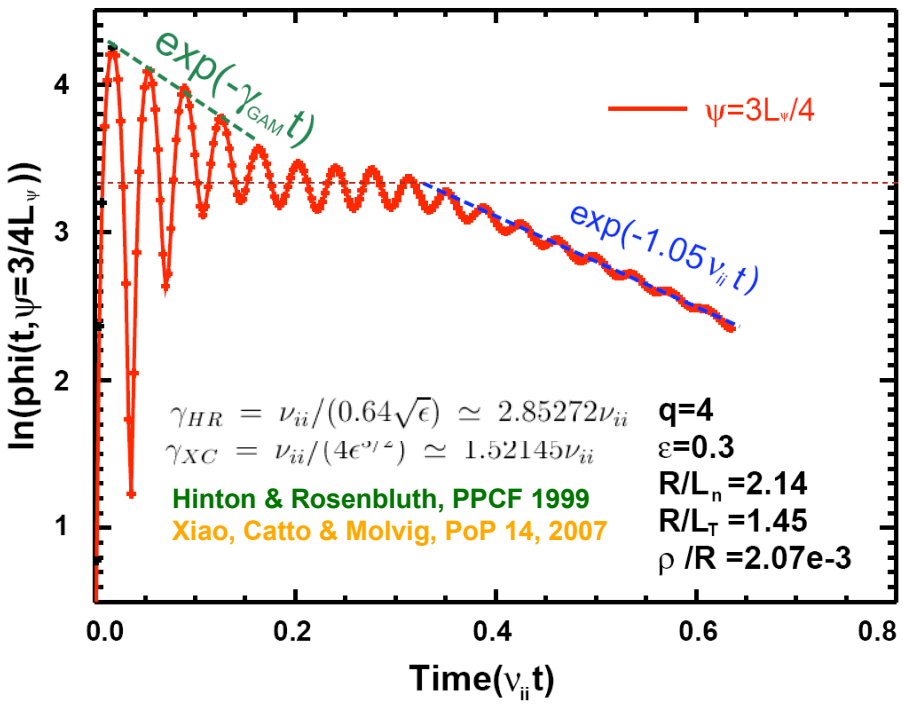
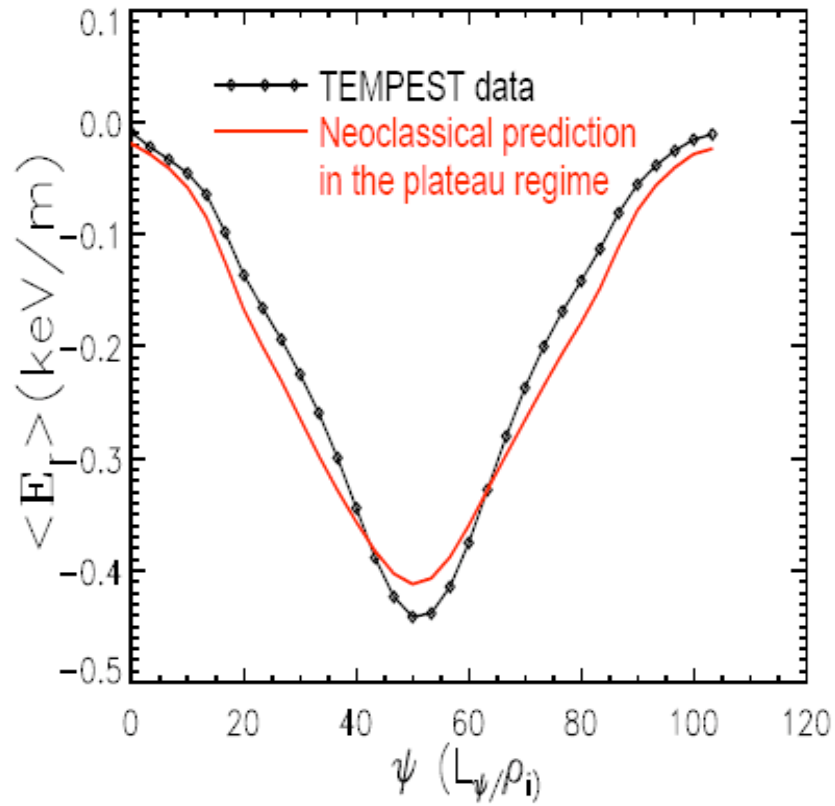
## Long-time collisional decay of zonal flow with Lorentz collisions is consistent with theories




## Neoclassical $E_r$ obeys the standard neoclassical relationship to the parallel flow in the plateau regime



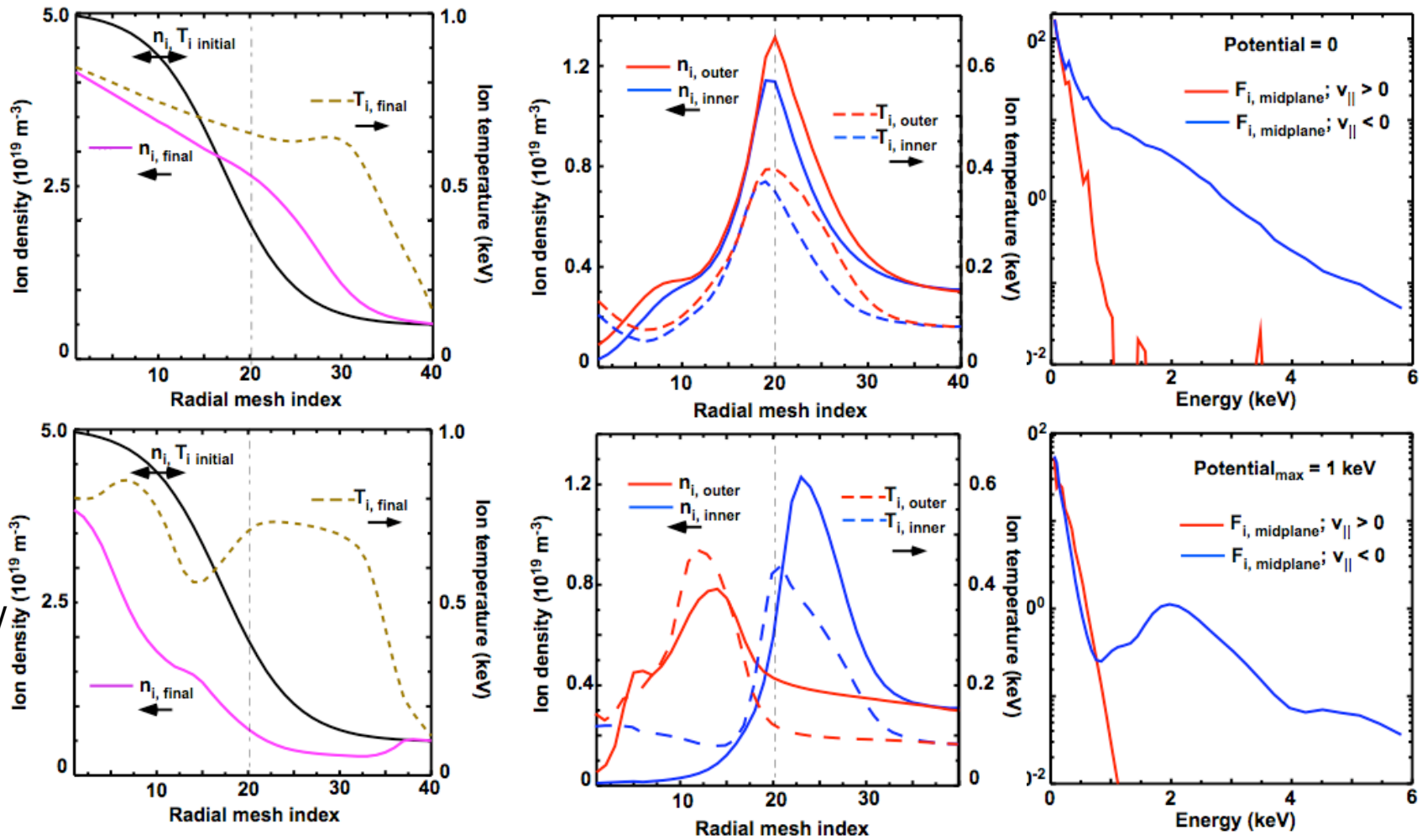
# TEMPEST simulations in divertor geometry with model $E_r$ indicate expected asymmetries and non-Maxwellian f

Outer midplane | Divertor plate | f at outer midplane

$\Phi$=0

DIII-D

$\Phi_{sep}$=1 keV




- Obtaining these results required proper accounting for direction of total flow at divertor plates in b.c.'s

# Anomalous Radial Transport in 4D TEMPEST

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## Goal: kinetic transport code ("kinetic UEDGE") for long-time simulation

### Transport Model

$$\frac{\partial f}{\partial t} = -\frac{1}{V}\frac{\partial\left(V\,\Gamma_a/h_\psi\right)}{\partial\psi}\bigg|_{\theta,v}$$

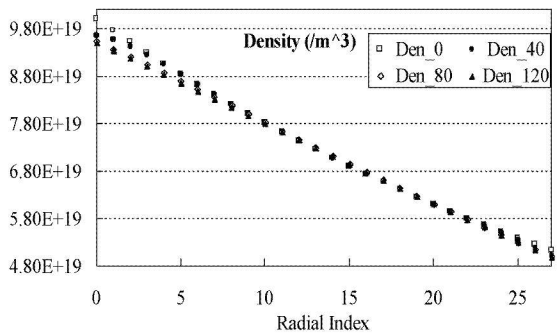
$$\Gamma_a = U_a f - \frac{D}{h_\psi}\frac{\partial f}{\partial\psi}\bigg|_{\theta,v}$$

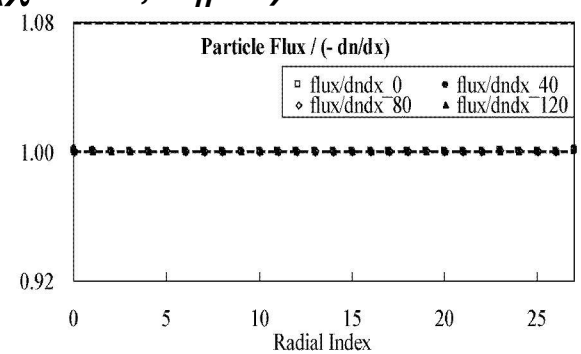
**Diagonal transport matrix for comparison with fluid models**

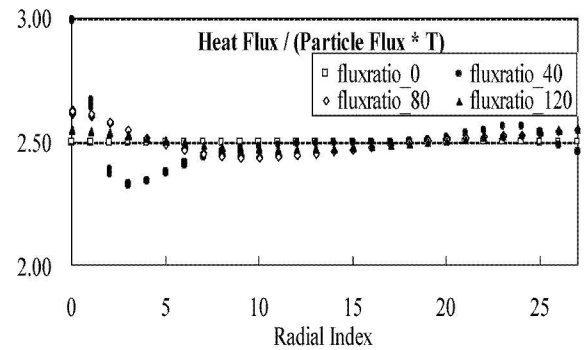
$$h_\psi \Gamma_n = -D_n \frac{\partial n}{\partial \psi}$$

$$h_\psi Q = -\frac{5}{2} D_n T \frac{\partial n}{\partial \psi} - \chi n \frac{\partial T}{\partial \psi}$$

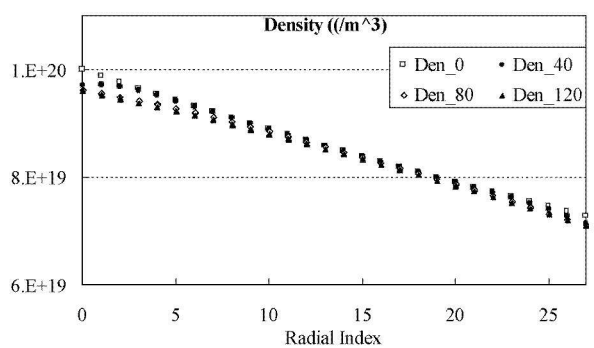
**Test I: *Gradient in Density* ($\chi$ = 2.5, $D_n$= 1)**

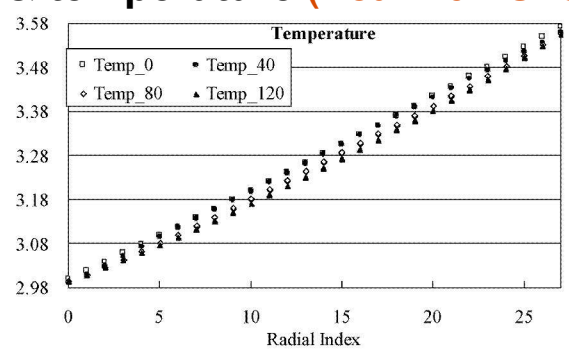


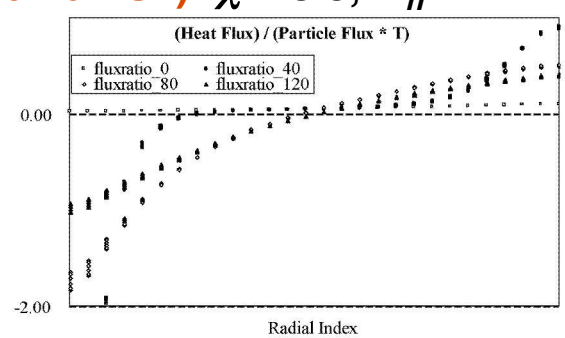



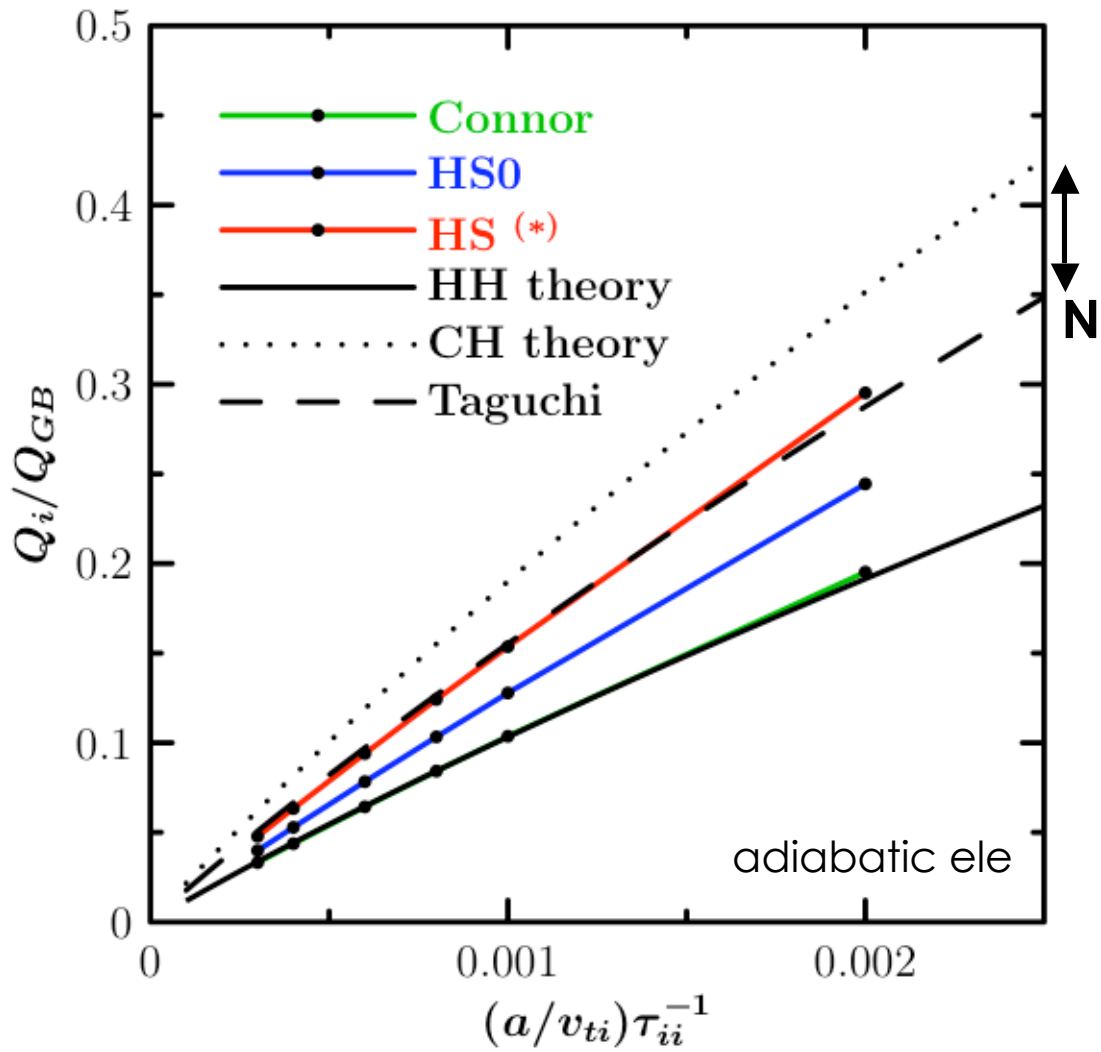

**Test II: Gradients in density & temperature (heat flux should vanish): $\chi$ = 3.5, $D_n$= 1**



# NEO has been verified by comparison with analytic theories in banana regime




**Only the full HS op recovers Taguchi's theory.**

**Note: Chang-Hinton theory overestimates $Q_i$.**

GA standard parameters:
(s-α geometry)
R/a=3 $a/L_n$=1
r/a=0.5 $a/L_T$=3
q=2 $T_i$=$T_e$


Hinton & Hazeltine, RMP 48, 239 (1976).
Chang & Hinton, PF 25, 1493 (1982).
Taguchi, PPCF 30, 1897 (1988).

# NEO accurately calculates bootstrap current, important for edge stability calculations

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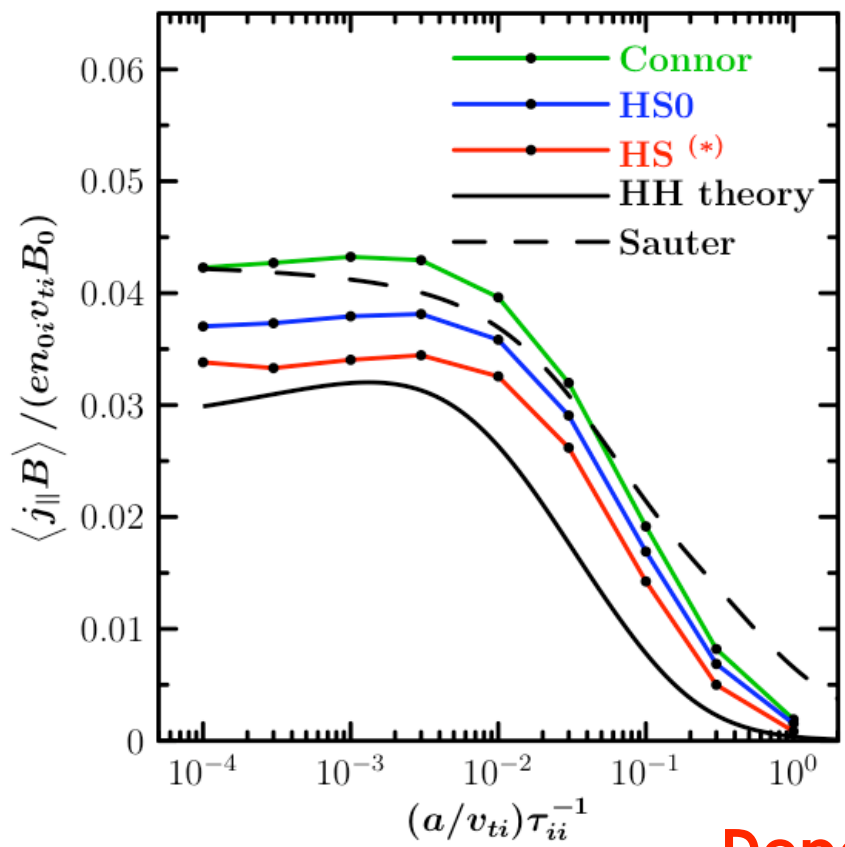
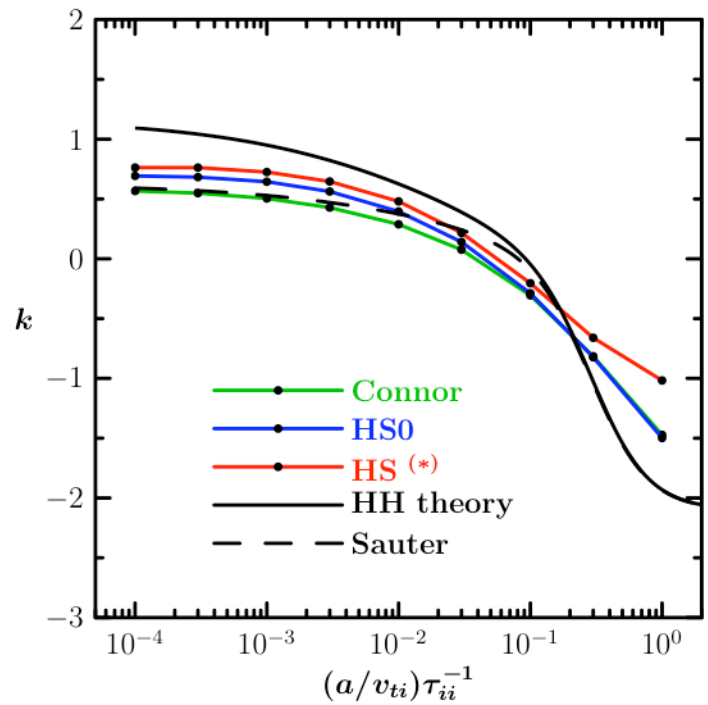



## Standard neoclassical relation:

$$\langle u_\parallel B \rangle_i = \frac{cT_{0i}}{Z_i e}\frac{I}{\psi'}\left(-\frac{Z_i e}{T_{0i}}\frac{\partial \Phi}{\partial r} + \frac{1}{L_n} + (1-k)\frac{1}{L_T}\right)$$




**Dependence of k on ε & $\nu_{*i}$ is coupled and difficult to predict analytically.**

Hinton & Hazeltine, RMP 48, 239 (1976).

Sauter et al, PoP 6, 2834 (1999).

# NEO simulations with impurities illustrate inadequacy of Connor model and multi-species analytic theory

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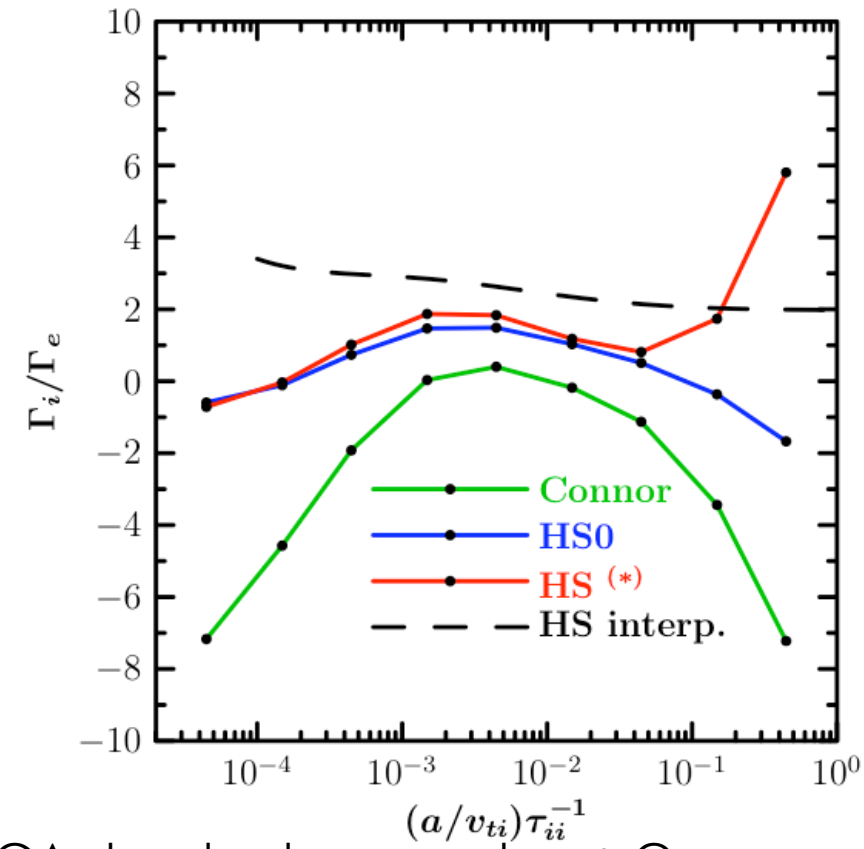
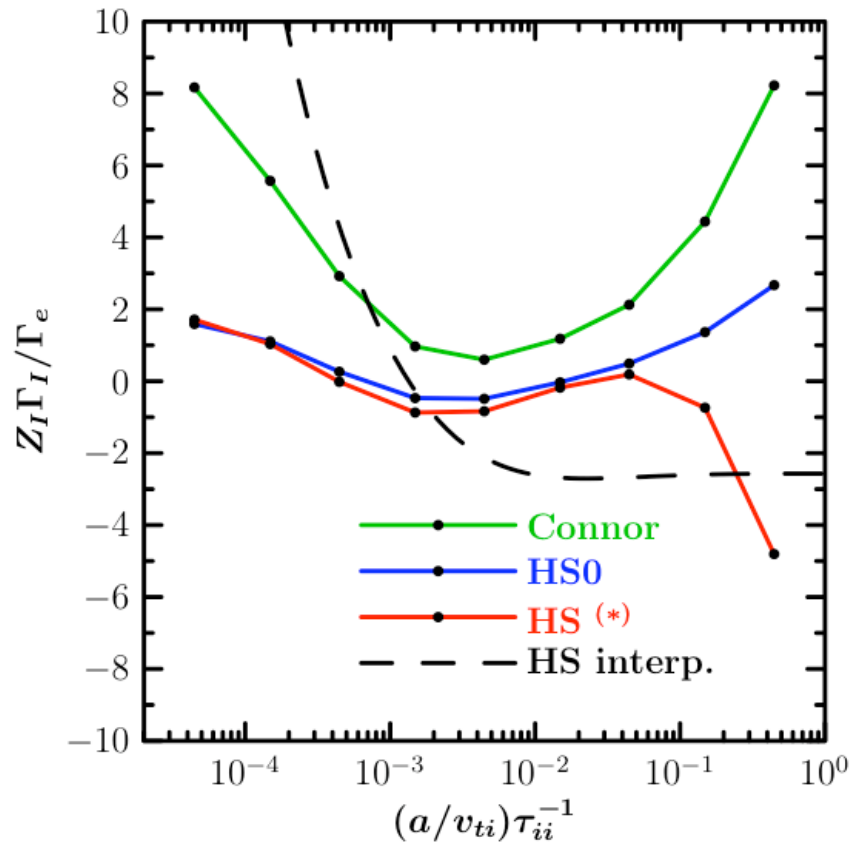
- Simulations with ions + electrons + carbon impurity




GA standard parameters + C:
$f_I=Z_I(n_{0I}/n_{0e})=0.1$ $a/L_{nI}=a/L_{ni}$ $T_I=T_i$

Hirshman & Sigmar, PF 20, 418 (1977).

# Summary of NEO Work

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NEO provides a first-principles DKP-based calculation of the neoclassical transport coefficients.

- Verification/analytical comparisons: **Demonstrated agreement with the Taguchi model for the full HS collision op. Confirmed that C-H theory overestimates $Q_i$ for intermediate $\varepsilon$.**
- Ambipolarity: **Confirmed for multi-species plasmas.**
- Impurity transport **(heavy-ions): Showed that the H-S theory gives a poor prediction of ion and impurity fluxes.**
- Shaping studies: **Found a weak effect in $\kappa$ (stronger in $\delta$). Showed that modified C-H theory based on $B(\theta)$ is not generally valid.**
- FOW effects: **Higher-order solution identified the break-down of the $\delta f$ formalism near $r_{potato} \Rightarrow$ requires full F.**

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- Strong rotation effects: **presently being added.**

# A practical edge model has been formulated based on the gyrokinetic equations (Dimits)

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- Key finding: If $\delta f$ is not small, it must be included in the GK Poisson-equation operator. Equations of motion must then be evaluated to second order in $\varepsilon_{GK}$ in order for system to satisfy energy conservation
- The specific model:
  - allows for large amplitude perturbations
  - is consistent with energy conservation
  - minimizes number of (difficult to implement) 2nd-order terms
  - electromagnetic ion terms retained in equations of motion mostly to first order (one quadratic $A_{\parallel}$ term needed for positivity in the conserved energy)
  - electrostatic terms retained to second order
  - No non-identity guiding- to gyro-center terms in GK Ampere's law.
- Gyrokinetic collision operator - need to keep first order "pullback" terms for perturbations with $k\rho \approx 1$; implementation otherwise straightforward but potentially expensive to compute.
- An accurate Padé approximation to full-f GK Poisson equation has been developed, but this is compatible (for energy conservation) only with specific approximated equations of motion

# The math team is developing algorithms and a basic code to deal with the challenges of edge GK

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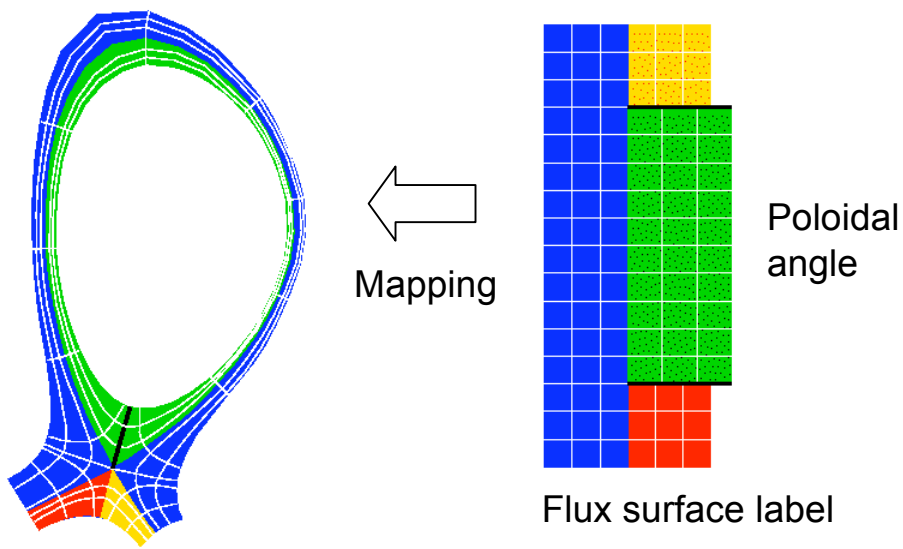
*Requirements:*

- Conservation
- Low-dissipation advection
- Preservation of distribution function positivity
- Efficient resolution of a large and complicated phase space
- Robust for high anisotropy
- Efficient implicit solves

*Numerical methodologies:*

- Finite volume discretizations applied to conservative formulations
- High-order discretization
- Mapped, multiblock grids
- Preconditioned iterative methods



# We are developing high-resolution discretizations for the gyrokinetic Vlasov equation

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- The gyrokinetic Vlasov equation describes advection by a phase space velocity field that is a nonlocal function of the distribution function f :

$$\frac{\partial f}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{R}} \cdot \left( \dot{\boldsymbol{R}}(f) f \right) + \frac{\partial}{\partial v_{\|}} \left( \dot{v}_{\|}(f) f \right) = 0$$

- Dependence of the phase space velocities $\dot{\boldsymbol{R}}$ and $\dot{\boldsymbol{v}}_{\|}$ on $f$ is through the Poisson solve
- To obtain a high-order discretization that is robust for this highly nonlinear system, we combine
  - fourth-order, multidimensional, flux-corrected transport (FCT) spatial discretization
  - fourth-order Runge Kutta time integration
- Based on a new PPM limiter
  - Preserves fourth-order accuracy where solution is smooth (does not reduce accuracy at smooth extrema like classical FCT and PPM)
  - Can be combined with an FCT multidimensional limiter (Zalesak) to preserve distribution function positivity

# We have developed a systematic strategy for 4th-order finite volume discretization on mapped grids

Cartesian computational grid control volumes:

$$V_{\mathbf{i}} = [(\mathbf{i} - \frac{1}{2}\mathbf{u})h, (\mathbf{i} + \frac{1}{2}\mathbf{u})h] \ , \ \mathbf{i} \in \mathbf{Z}^D \ , \ \mathbf{u} = (1, 1, \ldots, 1)$$

Smooth mapping to physical coordinates:

$$\mathbf{X} = \mathbf{X}(\xi) \ , \ \mathbf{X} : [0,1]^D \rightarrow \mathbf{R}^D$$

$$\int_{X(V_{\mathbf{i}})} \nabla_{\mathbf{x}} \cdot \vec{F} d\mathbf{x} = \sum_{\pm=+,-} \sum_{d=1}^{D} \pm \int_{A_d} (\mathbf{N}^T \vec{F})_d dA_\xi$$

where $N_{p,q} = det((\nabla_\xi \mathbf{X})(p|e^q))$, in which $A(p|v)$ denotes the matrix obtained by replacing the $p^{th}$ row of the matrix A by the vector v.

*Key elements:* The average of products is obtained by the repeated application of

$$\langle fg \rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = \langle f \rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} \langle g \rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + \frac{h^2}{12} G_0^{\perp,d}\left(\langle f \rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d}\right) \cdot G_0^{\perp,d}\left(\langle g \rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d}\right) + O(h^4).$$

where $G_0^{\perp,d} =$ second-order accurate centered difference of $\nabla_\xi - \mathbf{e}^d \frac{\partial}{\partial \xi_d}$ and $\langle q \rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = \frac{1}{h^{D-1}} \int_{A_d} q(\xi) dA_\xi + O(h^4)$

and fourth-order averages are computed using face-centered pointwise values via

$$\langle f \rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} = f_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + \frac{h^2}{24} \sum_{d' \neq d} \frac{\partial^2 f}{\partial \xi_{d'}^2}\Big|_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d} + O(h^4).$$

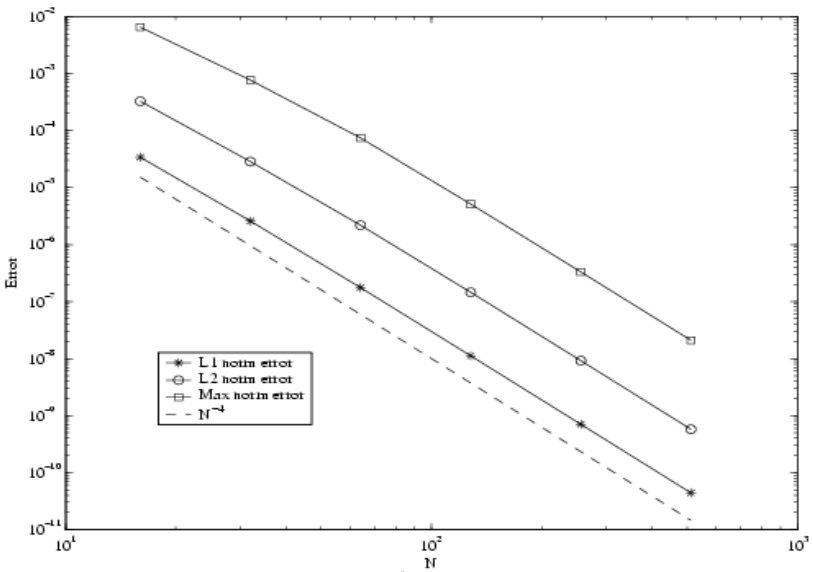
- Use to compute the face averages $\left\langle \mathbf{N}^T \vec{F} \right\rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d}$
- Use Poincaré Lemma to derive high-order estimates of $\left\langle \mathbf{N}^T \right\rangle_{\mathbf{i}+\frac{1}{2}\mathbf{e}^d}$ such that the cell-averaged $\nabla_{\mathbf{x}} \cdot \vec{F} = 0$ for constant $\vec{F}$

# 4th-order accuracy of the GK Poisson discretization has been obtained on core equilibrium geometries

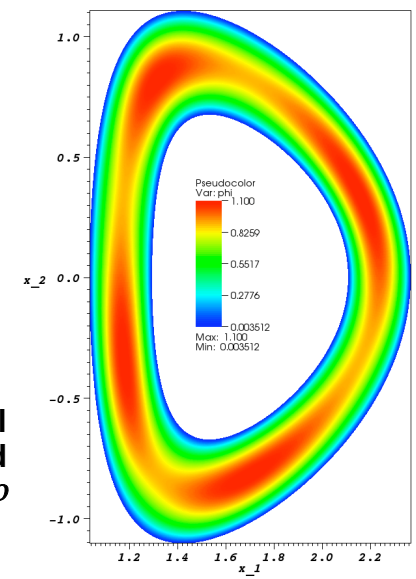
Given $\Phi$, use high accuracy quadrature to manufacture $\rho$ such that

$$\nabla \cdot \left\{ \left[ \epsilon_0 \boldsymbol{I} + \frac{n_i}{B^2} \left( \boldsymbol{I} - \vec{\boldsymbol{b}}\vec{\boldsymbol{b}}^T \right) \right] \nabla \Phi \right\} = \rho$$

using a prescribed density profile $n_i$ and a magnetic field from an analytically specified equilibrium model*.




**Verification of 4th order convergence**




**Potential computed from exact $\rho$**

**Convergence of *Hypre* CG solver preconditioned with multigrid solution of second-order operator**

| iter | Relative residual |
|---|---|
| 1 | 6.62e-03 |
| 2 | 1.19e-03 |
| 3 | 2.24e-04 |
| 4 | 1.17e-04 |
| 5 | 4.59e-05 |
| 6 | 1.18e-05 |

*Miller et al., "Noncircular, finite aspect ratio, local equilibrium model", Phys. Plasmas, Vol. 5, No. 4 (1998).

# Plans for next year:

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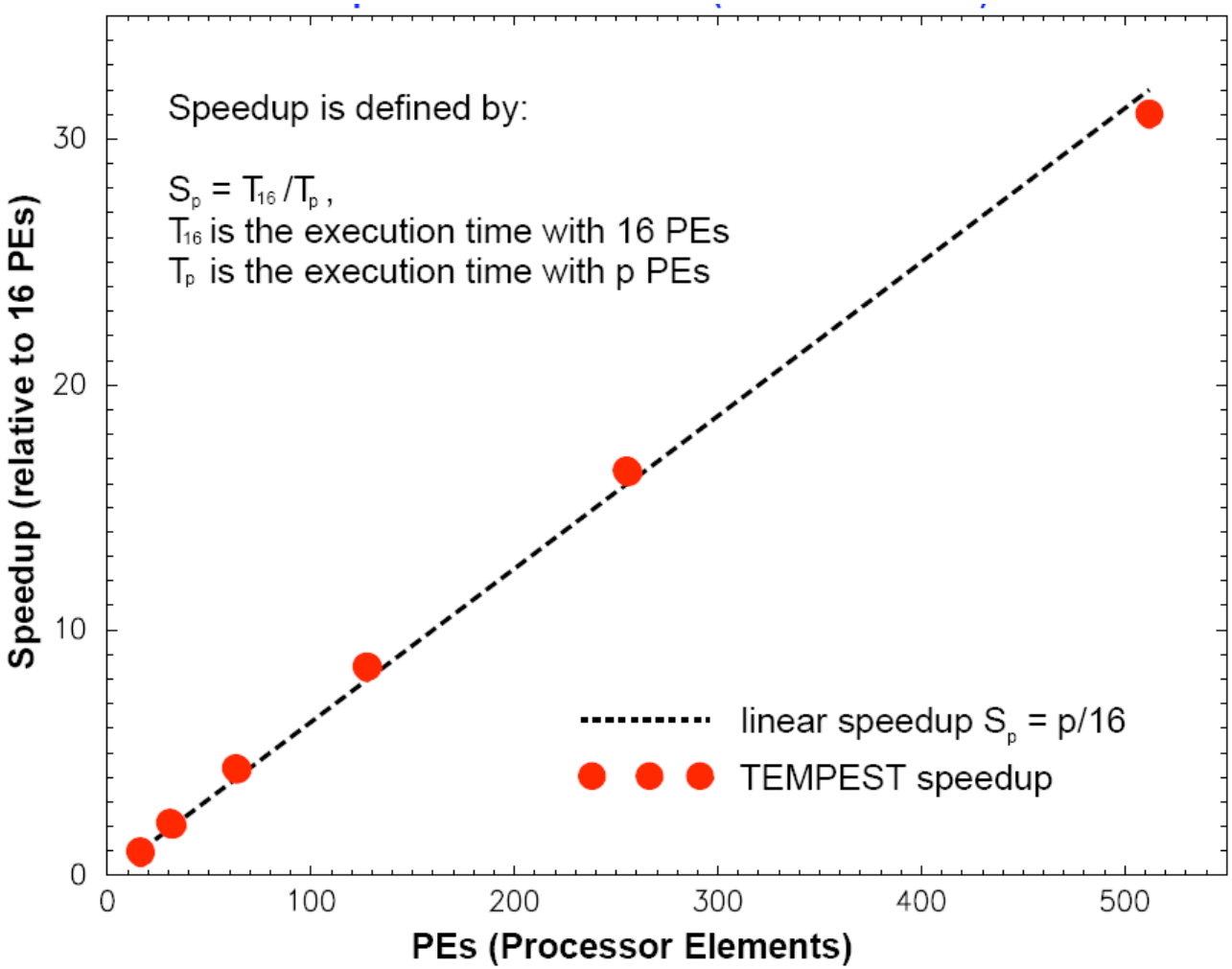
- New ESL code released to physics team:
  - 4D, core: summer ’08.
  - 4D, divertor: winter ’09
  - 5D, core: summer ’09
  - 5D, divertor, start summer ’09
- Physics teams will:
  - Perform V&V tests of new code (GAMs initially), including comparisons with our existing codes
  - Formulate, add collisions, sources and diagnostics
  - Collaborate with math team on code improvements to address issues that arise in V&V
  - Collaborate with math team on specification of design for follow-on EM code
  - Continue (at reduced level) physics and numerical studies with existing codes

# BACKUP SLIDES

# TEMPEST performance onATLAS

Problem size is fixed (128x128x30x60; GAM simulation)



# Hierarchy of Equations

## Based on expansion of the DKP eqns in powers of $\rho_{*i}$

edge simulation laboratory

$O(1):$

$$v_\parallel \hat{b}\cdot\nabla f_{0a} - \frac{Z_a e}{m_a} v_\parallel \hat{b}\cdot\nabla\Phi_0 \frac{\partial f_{0a}}{\partial\varepsilon} - C_{aa}\left(f_{0a}, f_{0a}\right) = 0$$

$$\sum_a Z_a e n_{0a} = 0$$

**Not solved**

$$f_{0a} = F_{Ma}$$

$O(\rho_{*i}):$

$$v_\parallel \hat{b}\cdot\nabla f_{1a} - \frac{Z_a e}{m_a} v_\parallel \hat{b}\cdot\nabla\Phi_1 \frac{\partial f_{0a}}{\partial\varepsilon} - \sum_b C^L_{ab}\left(f_{1a}, f_{1b}\right) = -\vec{v}_D\cdot\nabla f_{0a} + \frac{Z_a e}{m_a}\vec{v}_D\cdot\nabla\Phi_0 \frac{\partial f_{0a}}{\partial\varepsilon}$$

$$0 = \sum_a Z_a e \int d^3v f_{1a}$$

**Standard neoclassical**

$$\Gamma_{2a} = \left\langle \int d^3v \left( f_{0a}\vec{v}_E^{(1)}\cdot\nabla r + f_{1a}\vec{v}_D\cdot\nabla r \right)\right\rangle$$

$$Q_{2a} = \left\langle \int d^3v m_a \varepsilon \left( f_{0a}\vec{v}_E^{(1)}\cdot\nabla r + f_{1a}\vec{v}_D\cdot\nabla r \right)\right\rangle$$

$O(\rho_{*i}^2):$

$$v_\parallel \hat{b}\cdot\nabla f_{2a} - \frac{Z_a e}{m_a} v_\parallel \hat{b}\cdot\nabla\Phi_2 \frac{\partial f_{0a}}{\partial\varepsilon} - \sum_b C^L_{ab}\left(f_{2a}, f_{2b}\right) = -\vec{v}_D\cdot\nabla f_{1a} + \frac{Z_a e}{m_a}\vec{v}_D\cdot\nabla\Phi_1 \frac{\partial f_{0a}}{\partial\varepsilon} + \vec{v}_E^{(0)}\cdot\nabla f_{1a} + \vec{v}_E^{(1)}\cdot\nabla f_{0a}$$

$$+ \frac{Z_a e}{m_a}\left( v_\parallel \hat{b}\cdot\nabla\Phi_1 + \vec{v}_D\cdot\nabla\Phi_0 \right)\frac{\partial f_{1a}}{\partial\varepsilon} + \dot{\mu}\frac{\partial f_{1a}}{\partial\mu} + S_{2a} \; *$$

$$-\sum_a \frac{n_{0a} Z_s^2 e^2}{T_{0a}} \rho_a^2 |\nabla r|^2 \frac{\partial^2\Phi_0}{\partial r^2} = \sum_a Z_a e \int d^3v f_{2a}$$

*** required for solvability**

**FOW correction**

$$\Gamma_{3a} = \left\langle \int d^3v \left( f_{0a}\vec{v}_E^{(2)}\cdot\nabla r + f_{2a}\vec{v}_D\cdot\nabla r + f_{1a}\vec{v}_E^{(1)}\cdot\nabla r \right)\right\rangle$$

$$Q_{3a} = \left\langle \int d^3v m_a \varepsilon \left( f_{0a}\vec{v}_E^{(2)}\cdot\nabla r + f_{2a}\vec{v}_D\cdot\nabla r + f_{1a}\vec{v}_E^{(1)}\cdot\nabla r \right)\right\rangle$$

# Model forms of the linearized collision operator

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- Connor Model

$$C_{ab}^{L} = \nu_{ab} L f_{1a} + P_1(\mathrm{v}_{\parallel}/\mathrm{v})\left(\nu_{ab}\mathrm{v}\frac{r_{ba}}{\mathrm{v}_{ta}^2}\right) f_{0a}$$

- Zeroth Order Hirshman-Sigmar Operator

$$C_{ab}^{L} = \nu_{ab}^{D} L f_{1a} + P_1(\mathrm{v}_{\parallel}/\mathrm{v})\left[\nu_{ab}^{s}\mathrm{v}\frac{r_{ba}}{\mathrm{v}_{ta}^2} + \left(\nu_{ab}^{D} - \nu_{ab}^{s}\right)\frac{u_{a1}(\mathrm{v})}{\mathrm{v}}\right] f_{0a}$$

- Full Hirshman-Sigmar Operator

$$C_{ab}^{L} = \nu_{ab}^{D} L f_{1a} + P_1(\mathrm{v}_{\parallel}/\mathrm{v})\left[\nu_{ab}^{s}\mathrm{v}\frac{r_{ba}}{\mathrm{v}_{ta}^2} + \left(\nu_{ab}^{D} - \nu_{ab}^{s}\right)\frac{u_{a1}(\mathrm{v})}{\mathrm{v}} + \left(\nu_{ab}^{h} h_{ba} + \nu_{ab}^{k} k_{ab}\right)\frac{\mathrm{v}^3}{\mathrm{v}_{ta}^4}\right] f_{0a}$$

$$+ \frac{1}{\mathrm{v}^2}\frac{\partial}{\partial \mathrm{v}}\left[\frac{1}{2}\nu_{\mathrm{ab}}^{\parallel}\mathrm{v}^4\left(\frac{\partial n_{1a}}{\partial \mathrm{v}} - \frac{\mathrm{v}}{\mathrm{v}_{\mathrm{ta}}^2} q_{ba}\right) f_{0a}\right] + P_2(\mathrm{v}_{\parallel}/\mathrm{v})\left[\nu_{ab}^{p}\mathrm{v}^2\frac{5\pi_{ba}}{4\mathrm{v}_{ta}^4} - \nu_{ab}^{E}\frac{\pi_{1a}(\mathrm{v})}{\mathrm{v}^2}\right] f_{0a}$$

Connor, PP 15, 765 (1973); Hirshman & Sigmar, NF 21, 1079 (1981).