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SELF-CONSISTENT ELASTIC MODULI OF A CRACKED SOLID

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Abstract. A self-consistent theory for the determination of elastic moduli of cracked solids is presented, and worked out for an isotropic distribution of cracks. A missing ingredient of the previous theory of O'Connell and Budiansky is the correct accounting of crack interaction energy. The new theory leads to a set of differential equations for the effective elastic moduli which are easily solved. The solutions always lie within the physical range, and show that the influence of the cracks on the effective moduli is considerably less than has been previously calculated.

Introduction

We consider the self-consistent determination of the effective elastic moduli of a homogeneous rock "matrix" within which is embedded a random distribution of thin ellipsoidal cracks (with semiaxes $a > b >> c$).

Early theoretical work considered solids with a low concentration of cracks. The effect of the cracks on elastic properties of the uncracked solid was calculated by treating the cracks on an individual basis. Bruggeman [1937] and Dewey [1947] determined the elastic field of a spherical inclusion. Eshelby [1957]determined the same for an ellipsoidal geometry, and described how to calculate the effective elastic moduli in the presence of a low density of inclusions. Walpole [1972] performed the calculation for a sparse distribution of spheres. Garbin and Knopoff [1973,1975] calculated effective elastic moduli for randomly located, planar or randomly oriented, dry or wet thin circular cracks ($a = b$, $c/a \to 0$).

An alternative approach to these perturbation methods was developed by Hershey [1954] and Kröner [1958] for polycrystalline materials, and by Hill [1965] and Budiansky [1965, 1970] for multiphase composites. This is the "self-consistent" approach. Here one approximates the elastic field of one member of a (possibly) dense number of inclusions by the field of an isolated inclusion in an infinite isotropic medium which is assumed to have the same elastic properties as the composite as a whole. Clearly, the perturbation results must be obtained as a limit of the self-consistent calculations when the density of inclusions becomes vanishingly small. On the other hand, interactions between the inclusions are approximately taken into account, so the self-consistent method can be expected to be an improvement over first order perturbation theory.

O'Connell and Budiansky [1974] and Budiansky and O'Connell [1976] pioneered the application of self-consistent techniques to an isotropic network of dry or wet cracks. The strain field is significantly distorted by the presence of a crack out to a distance of order the width of the crack (Length > Width >> Thickness). The volume of the distorted strain field is therefore of order $(\text{Width})^2(\text{Length})$. The relevant expansion parameter ε is, up to a constant, the volume of the distorted field

$$\sum_{\text{cracks}} (\text{Width})^2\ (\text{Length}) \quad ,$$

divided by the total volume. O'Connell and Budiansky's results for effective S and P wave velocity and for effective Poisson ratio, $\bar{\sigma}$, differ very little from those of perturbation theory. In particular, $\bar{\sigma}$ leaves the physical range $0 < \bar{\sigma} < 1/2$ at moderate values of crack density ($\varepsilon = 9/16$ for dry, and $\varepsilon = 45/32$ for saturated cracks). The region of validity of the results is therefore severely limited.

In atomic physics the method of self-consistent fields was introduced by Hartree [1927] as an accurate means of solving the wave equation. The development of the O'Connell and Budiansky (OB) theory for cracks is in good analogy with the Hartree theory except for one important feature. OB calculate the energy of a single crack, where its interaction with other cracks is approximately taken into account by using the effective elastic moduli. They then write the total energy as the sum over cracks of the individual crack energies, obtaining an equation from which the effective moduli can be read off. It is known in atomic physics, however, that this step is not correct [e.g. see Schiff [1968]]; the interaction between cracks is included in the energy of each interacting crack. The purpose of this note is to put in the correct relationship between the single crack energy and the total energy. Differential equations for the effective elastic moduli are derived whose solutions always lie within the physical range. (These equations were first presented by Bruner [1976], but with less discussion). A significant difference is found between the new self-consistent results and those of OB and of perturbation theory which overestimate the effect of the cracks.

Corrected Self-Consistent Calculation

The energy that is approximately calculated by OB is the energy required to eliminate one crack. If the cracks are eliminated one by one, and the removal energy of each is added, the sum is the total energy of the crack system. As the cracks are being removed, the effective elastic moduli are changed, and the removal energy of a crack is no longer identical to what is was originally.

A single crack has energy $d\Phi$ and changes the crack density ε by $d\varepsilon$. First order perturbation theory estimates $d\Phi$ by using unperturbed elastic moduli C: $d\Phi/d\varepsilon|_{PT} = F(C)$. O'Connell and Budiansky estimate $d\Phi$ by using the effective

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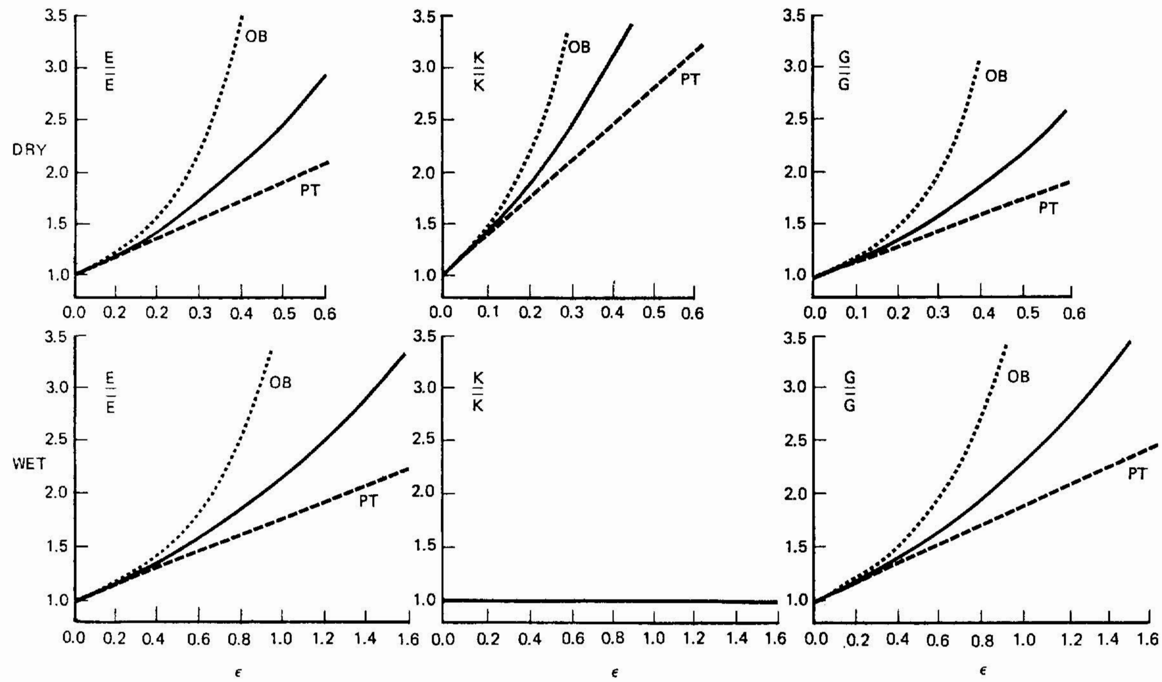

Fig. 1. Coefficients $1/\bar{E}$, $1/\bar{K}$, $1/\bar{G}$, of energy in terms of stress, normalized by their unperturbed values (without bars). Unperturbed Poisson's ratio $\sigma = 1/4$. Our values, solid curves, intermediate between O'Connell and Budiansky (OB) and linear extrapolation of perturbation theory (PT).

elastic moduli $\bar{C}$ of the rock with all cracks present: $d\Phi/d\varepsilon|_{OB} = F(\bar{C})$. The expression which incorporates the one-by-one feature of the correct energy estimates $d\Phi$ by using the effective elastic moduli $C(\varepsilon)$ when only a fraction of the cracks are present: $d\Phi/d\varepsilon = F\big(C(\varepsilon)\big)$. The left hand side of each of these expressions can be written in terms of the elastic moduli

$$\left.\frac{dC(\varepsilon)}{d\varepsilon}\right|_{PT} = f(C)$$

$$\left.\frac{dC(\varepsilon)}{d\varepsilon}\right|_{OB} = f(\bar{C}) \qquad (1)$$

$$\frac{dC(\varepsilon)}{d\varepsilon} = f\big(C(\varepsilon)\big)$$

Each one of these has the properties $C(0) = C$, $C(\varepsilon) = \bar{C}$ for the actual ε of the rock. The first two can be integrated immediately since their right sides are independent of ε:

$$\bar{C}_{PT} - C = \varepsilon\, f(C)$$

$$\bar{C}_{OB} - C = \varepsilon\, f(\bar{C}) \qquad (2)$$

The solutions of these two equations depend on whether the energy is expressed in terms of the strain or whether it is expressed in terms of the stress. In the former case the C's are the Lamé constants λ and μ (isotropic solid) while in the latter case, chosen by OB, any two of $1/E$, $1/K$, $1/G$ are the C's. The corrected equation (1c), on the other hand, is a set of differential equations whose solutions are independent of the choice of the C's.

For an isotropic cracked solid there is no need for us to calculate the functions f which appear in Eqs. (1) and (2), since they have already been calculated by Garbin and Knopoff and by O'Connell and Budiansky. The differential equations (1c) are

$$\frac{d\lambda}{d\varepsilon} = \begin{cases} \dfrac{-16}{45}\,\lambda\,\dfrac{(1-\sigma)(\sigma^2-16\sigma+19)}{(2-\sigma)(1-2\sigma)} & \text{DRY} \\[2ex] \dfrac{+32}{45}\,\lambda\,\dfrac{(1-\sigma)(1-2\sigma)}{(2-\sigma)\sigma}\,, & \text{WET} \end{cases}$$

$$\frac{d\mu}{d\varepsilon} = \begin{cases} \dfrac{-32}{45}\,\mu\,\dfrac{(1-\sigma)(5-\sigma)}{(2-\sigma)} & \text{DRY} \\[2ex] \dfrac{-32}{15}\,\mu\,\dfrac{(1-\sigma)}{(2-\sigma)}\,. & \text{WET} \end{cases} \qquad (3)$$

Here λ and μ are understood to be functions of ε, and $\sigma = \lambda/2(\lambda+\mu)$ is Poisson's ratio.

For the case of dry cracks, the self-consistent differential equations (3) are most easily solved by computer. For wet cracks, however, a simple analytic solution for $\bar{\lambda}$, $\bar{\mu}$, and derived quantities is available.

$$\bar{\sigma} = \frac{2x-\sqrt{3x+1}}{4x+1}\,, \text{ where } x = \frac{(1-\sigma^2)}{(1-2\sigma)^2}\, e^{\frac{64}{45}\varepsilon}\,;$$

$$\frac{\bar{\lambda}}{\lambda} = \left(\frac{1+\sigma}{1+\bar{\sigma}}\right)\,,$$

$$\frac{\bar{\mu}}{\mu} = \left(\frac{1+\sigma}{1+\bar{\sigma}}\right)\left(\frac{1-2\bar{\sigma}}{1-2\sigma}\right)\,, \qquad (4)$$

$$\frac{\bar{\lambda}+2\bar{\mu}}{\lambda+2\mu} = \left(\frac{1+\sigma}{1+\bar{\sigma}}\right)\left(1-2\left(\frac{\bar{\sigma}-\sigma}{1-\sigma}\right)\right)\,,$$

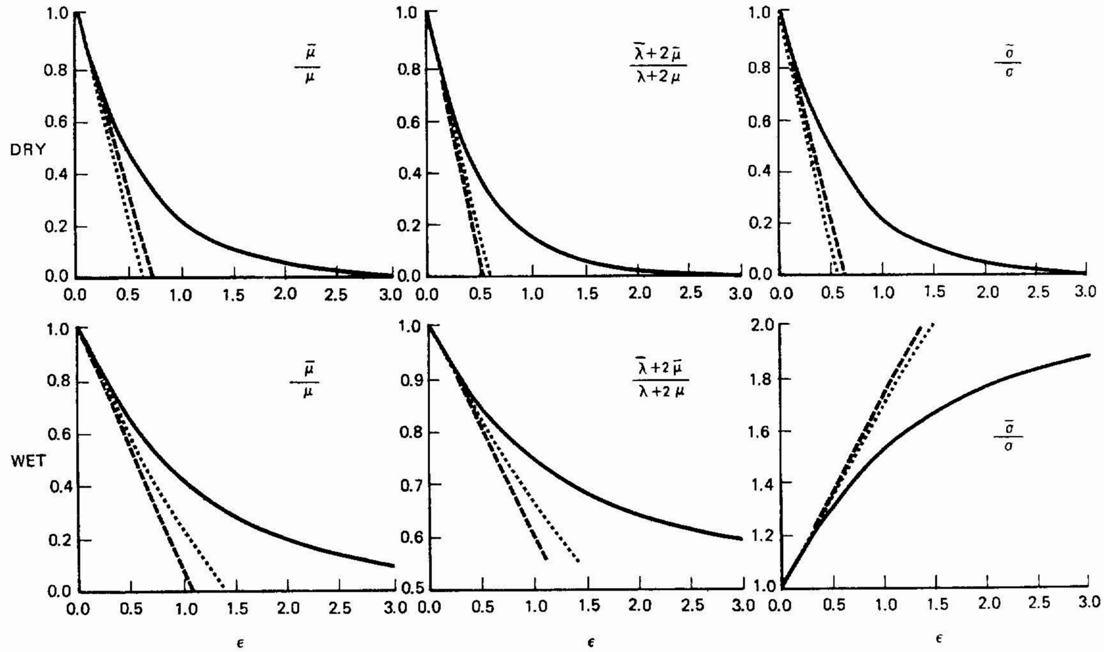

Fig. 2. Seismic velocities and Poisson's ratio normalized to their unperturbed values. Linear extrapolations of PT for different quantities are mutually inconsistent, and in this figure lie close to the results of OB. Our result for $\bar{\sigma}$, unlike that of OB, remains within the physical region for all values of crack density ε.

$$\frac{\bar{K}}{K} = 1$$

$$\frac{\bar{E}}{E} = \left(\frac{1-2\bar{\sigma}}{1-2\sigma}\right)$$

Results and Discussion

Figure 1 shows the values of Young's modulus $E/\bar{E}$, bulk modulus $K/\bar{K}$ and shear modulus $G/\bar{G}$ as functions of crack density. Results are displayed for both dry and wet cracks, comparing the results of our new self-consistent calculation with those of O'Connell and Budiansky and perturbation theory. The new results are intermediate between those of OB and perturbation theory. This illustrates the fact that OB have counted the interaction energy between a pair of cracks twice, attributing the entire interaction energy to each crack separately. For interactions among more than two cracks the overcounting is even more serious; it turns out that the N-crack interaction energy has been overcounted by a factor N factorial. Beyond $\varepsilon = 9/16$ for dry cracks and $\varepsilon = 45/32$ for wet cracks the OB theory does not give physical results; their energies become infinite at these critical values.

Figure 2 shows similar plots for the S and P wave velocities, $\bar{\mu}/\mu$ and $(\bar{\lambda}+2\bar{\mu})/(\lambda+2\mu)$ respectively, and for Poisson's ratio $\bar{\sigma}/\sigma$ (note that $\bar{\mu}/\mu$ and $(G/\bar{G})^{-1}$ are not equal in perturbation theory). There is seen to be little difference between the OB results and perturbation theory. The reason is simply explained: The first of Eqs. (2) can be written as

$$\bar{C}_{PT} = C + \varepsilon C f(\sigma) \quad .$$

Now consider perturbation theory for the reciprocal elastic modulus. This gives

$$(\bar{C}^{-1})_{PT} = C^{-1} - \varepsilon C^{-1} f(\sigma) . \qquad (5)$$

The second of Eqs. (2) is

$$\bar{C}_{OB} = C + \varepsilon \bar{C}_{OB}\, f(\bar{\sigma}) \quad ,$$

so that

$$\bar{C}_{OB} = C/\big(1 - \varepsilon f(\bar{\sigma})\big) \quad . \qquad (6)$$

If the function f were independent of σ, Eqs. (5) and (6) would imply that $(\bar{C}_{OB})^{-1} = (\bar{C}^{-1})_{PT}$. The true f is actually a weak function of σ so that $(\bar{C}_{OB})^{-1} \approx (\bar{C}^{-1})_{PT}$. Hence the small difference between the O'Connell and Budiansky results for the Lamé constants (and derived quantities) and perturbation theory. When compared with the new self-consistent results, the previous theories are seen to severely overestimate the influence of cracks. In particular, the new Poisson's ratio never leaves the physical range $0 \le \sigma \le 1/2$, yet sensibly approach the boundaries of this range as the cracks become infinitely dense. See, also, Bruner [1976] .

The remote sensing of rock fractures by seismic techniques requires knowledge of the influence of crack parameters on the elastic properties of solid media. The self-consistent theory presented in this paper is one step in this direction. The necessity of applying our considerations to other cases, such as polycrystalline materials, is an open question.

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