Discrete particle noise in a nonlinearly saturated plasma

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Understanding discrete particle noise in an equilibrium plasma has been an important topic since the early days of particle-in-cell simulation [1]. In this paper, particle noise in a nonlinearly saturated system is investigated. We explore the usefulness of the fluctuation-dissipation theorem (FDT) in a regime where drift instabilities are nonlinearly saturated. We obtain excellent agreement between the simulation results and our theoretical predictions of the noise properties. It is found that discrete particle noise is independent of the saturation level and transport properties associated with long-wavelength drift waves when mode coupling is ignored.

[1] C. K. Birdsall and A. B. Langdon, *Plasma Physics via Computer Simulation*, McGraw-Hill, New York (1985).

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Gyrokinetic Vlasov equation, $(k_\perp \rho_i \ll 1)$:

$$\frac{\partial F_\alpha}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla F_\alpha - \nabla\phi \times \mathbf{b} \cdot \nabla F_\alpha - \frac{q_\alpha m_i}{q_i m_\alpha} \mathbf{b} \cdot \nabla\phi \frac{\partial F_\alpha}{\partial v_\parallel} = 0$$

Gyrokinetic Poisson equation:

$$\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} k_\perp^2(l,m)\phi_{l,m}(t) e^{ik_x l x} e^{ik_y m y} = \sum_\alpha \frac{q_\alpha}{q_i} \int_{-\infty}^{\infty} F_\alpha dv_\parallel$$

- $F_\alpha = F_\alpha(x, y, v_\parallel, t)$ is the distribution function of species α (two species, ions and electrons).
- $\phi = \phi(x, y, t)$ is the electrostatic potential, normalized to T/q_i; T is the temperature of both species.
- $\mathbf{b}= \theta\hat{y} + \hat{z}$, $\theta \ll 1$; $v_\parallel$ is the velocity parallel to the magnetic field $B\mathbf{b}$.
- Lengths normalized to $\rho_i = \sqrt{\frac{T}{m_i}}\frac{1}{\Omega_i}$, times normalized to $\Omega_i^{-1} = (q_i B/m_i)^{-1}$.
- $k_\perp^2(l,m) = k_x^2 l^2 + k_y^2 m^2 (1-\theta^2)$ is the (square of the) wavevector component perpendicular to the magnetic field; $k_\parallel = \theta k_y$ is the component parallel to the field.
- Periodic boundary conditions in a slab of size L_x, L_y; $k_x = 2\pi/L_x$, $k_y = 2\pi/L_y$.
- m_α, q_α are the mass and charge of species α. With normalization, thermal velocity $v_{t\alpha}^2 = m_i/m_\alpha$.

Equilibrium

$$F_{\alpha 0}(v_\parallel) = \frac{1}{\sqrt{2}\sqrt{\pi} v_{t\alpha}} \exp\left(-\frac{v_\parallel^2}{2v_{t\alpha}^2}\right) \quad ; \quad \phi(x,y,t) = 0$$

Drive drift waves in the system by letting

$$F_{\alpha 0}(x, v_\parallel) = F_{\alpha 0}(v_\parallel) \left[\frac{\epsilon \exp\left(-\frac{\epsilon x}{L_x}\right)}{1 - \exp(-\epsilon)} \right]$$

- $\epsilon = \kappa_N L_x \ll 1$
- To lowest order $F_{\alpha 0}(x, v_\parallel) = F_{\alpha 0}(v_\parallel)$ (independent of x)
- To lowest order $\nabla F_{\alpha 0}(x, v_\parallel) = -\kappa_N F_{\alpha 0}(v_\parallel)\hat{x}$ (independent of x). Define $\omega_N^* = k_y \kappa_N$.

Linear dielectric for this system is

$$\mathcal{D}_{l,m}(\omega) = 1 + \left(1 + \frac{m\omega_N^*}{\omega}\right) \frac{X_i}{k_\perp^2(l,m)} + \left(1 - \frac{m\omega_N^*}{\omega}\right) \frac{X_e}{k_\perp^2(l,m)}$$

$$X_\alpha = 1 + \xi_\alpha Z(\xi_\alpha) \quad ; \quad \xi_\alpha = \frac{\omega}{\sqrt{2} k_\parallel m v_{t\alpha}}$$

where $Z(\xi_\alpha)$ is the plasma dispersion function.

For undriven system ($\omega_N^* = 0$):

$$\lim_{\omega \to \infty} \mathcal{D}_{l,m}(\omega) = 1$$

- No dielectric response if the frequency is too high

$$\lim_{\omega \to 0} \mathcal{D}_{l,m}(\omega) = 1 + \frac{2}{k_\perp^2(l,m)}$$

- The gyrokinetic equivalent of Debye shielding; will use later

Undriven system ($\omega_N^* = 0$) - normal modes

• Set $\mathcal{D}_{l,m}(\omega) = 0$:

$$1 + \frac{X_i}{k_\perp^2(l,m)} + \frac{X_e}{k_\perp^2(l,m)} = 0$$

• Assume both species are fluidlike:

$\xi_\alpha \gg 1 \quad \Rightarrow X_\alpha \sim -1/2\xi_\alpha^2 - 3/4\xi_\alpha^4 - \ldots$

• Neglect ion contribution; $v_{te}^2 \gg v_{ti}^2$ (ion distribution is a narrow Maxwellian, electron distribution is a wide Maxwellian, so at high velocities the electron contribution is more significant)

• Obtain ω_H-modes:

$$0 = 1 - \frac{m^2 k_\parallel^2 v_{te}^2}{\omega^2 k_\perp^2(l,m)} = 1 - \frac{\omega_H^2}{\omega^2} \quad ; \quad \omega_H = \pm \frac{m k_\parallel v_{te}}{k_\perp(l,m)}$$

• These modes can be shown to be weakly damped.

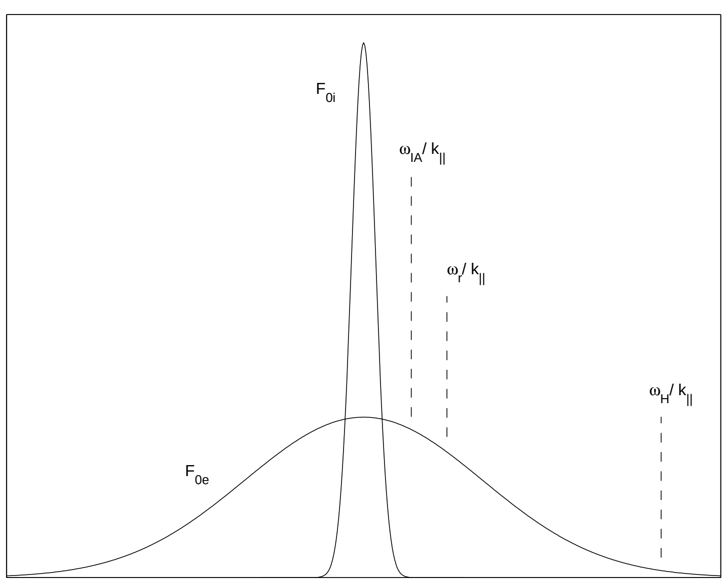
• Assume fluidlike ions, kinetic electrons:

$\xi_i \gg 1, \xi_e \ll 1 \quad \Rightarrow X_i \sim -1/2\xi_i^2, X_e \sim 1$

• Ion acoustic modes:

$$1 - \frac{m^2 k_\parallel^2}{k_\perp^2(l,m)\omega^2} + \frac{1}{k_\perp^2(l,m)} = 0 \quad ; \quad \omega_{IA} = \pm \frac{k_\parallel m}{\sqrt{1 + k_\perp^2(l,m)}}$$

• These modes can be shown to be strongly Landau damped when $T_i = T_e$.



• Fluctuation-dissipation theorem tells us that (in thermal equilibrium)

$$\langle \delta\phi\delta\phi\rangle_{l,m}(\omega) = \frac{2}{N\omega k_\perp^2(l,m)}\text{Im }\left(1 - \frac{1}{\mathcal{D}_{l,m}(\omega)}\right)$$

• Integrate over all frequencies:

$$\langle \delta\phi\delta\phi\rangle_{l,m} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{N\omega k_\perp^2(l,m)}\text{Im }\left(1 - \frac{1}{\mathcal{D}_{l,m}(\omega)}\right) d\omega$$

• If the dielectric has no poles in the upper half-plane, take principal value and perform integration:

$$\langle \delta\phi\delta\phi\rangle_{l,m} = \text{Im P }\frac{1}{2\pi}\frac{2}{Nk_\perp^2(l,m)}\int_{-\infty}^{\infty}\left(\frac{\mathcal{D}_{l,m}(\omega)-1}{\omega\mathcal{D}_{l,m}(\omega)}\right) d\omega$$

$$\langle \delta\phi\delta\phi\rangle_{l,m} = \frac{1}{Nk_\perp^2(l,m)}\left(\frac{\mathcal{D}_{l,m}(\omega=0)-1}{\mathcal{D}_{l,m}(\omega=0)}\right)$$

• For undriven system, we obtain

$$\langle \delta\phi\delta\phi\rangle_{l,m} = \frac{1}{Nk_\perp^2(l,m)}\left(\frac{2}{k_\perp^2(l,m)+2}\right)$$

• Doesn't work for driven system; the drift wave grows, implying a pole in the upper ω-half-plane; also, original formula is only strictly valid in thermal equilibrium.

Drift waves

- Fluidlike ions, kinetic electrons -

$\xi_e \ll 1, \quad \xi_i \gg 1 \quad \Rightarrow \quad X_i \sim 0,\ X_e \sim 1 + i\sqrt{\pi}\xi_e.$

- Mode frequency and growth rate - set $\mathcal{D}_{l,m}(\omega) = 0$:

$$\mathcal{D}_{l,m}(\omega) = 0 = \frac{\left[k_\perp^2(l,m) + \left(1 - \frac{m\omega_N^*}{\omega}\right)(1 + i\sqrt{\pi}\xi_e)\right]}{k_\perp^2(l,m)}$$

- Assume $\omega = \omega_r + i\gamma_r, \gamma_r \ll \omega_r$; to lowest order

$$\omega_r = \frac{m\omega_N^*}{1 + k_\perp^2(l,m)} \quad ; \quad \gamma_r = \frac{\sqrt{\pi}\omega_r^2 k_\perp^2(l,m)}{(1 + k_\perp^2(l,m))\sqrt{2}k_\parallel v_{te}}$$

- Drift wave frequency much smaller than ω_H.

• For normal modes, we can approximate the function

$$\langle \delta\phi\delta\phi \rangle_{l,m}(\omega) = \frac{2}{N\omega k_\perp^2(l,m)} \mathrm{Im} \left(1 - \frac{1}{\mathcal{D}_{l,m}(\omega)}\right)$$

• $\mathcal{D}_{l,m}(\omega) = \mathcal{D}' + i\mathcal{D}''$

$$\langle \delta\phi\delta\phi \rangle_{l,m}(\omega) = \frac{2}{N\omega k_\perp^2(l,m)} \left[\frac{\mathcal{D}''}{\mathcal{D}'^2 + \mathcal{D}''^2}\right]$$

• For small $\mathcal{D}''$ (normal modes that are weakly damped), the Lorentzian function approaches a delta function;

$$\langle \delta\phi\delta\phi \rangle_{l,m}(\omega) \approx \frac{2}{N\omega k_\perp^2(l,m)} \pi\delta(\mathcal{D}') = \frac{2\pi}{N k_\perp^2(l,m)} \sum_{p=1}^{p_0} \frac{\delta(\omega - \omega_p)}{\omega \left(\frac{\partial \mathcal{D}'}{\partial \omega}\Big|_{\omega - \omega_p}\right.}$$

for the p_0 normal modes with real frequency ω_p.

• Apply to ion acoustic and ω_H-modes:

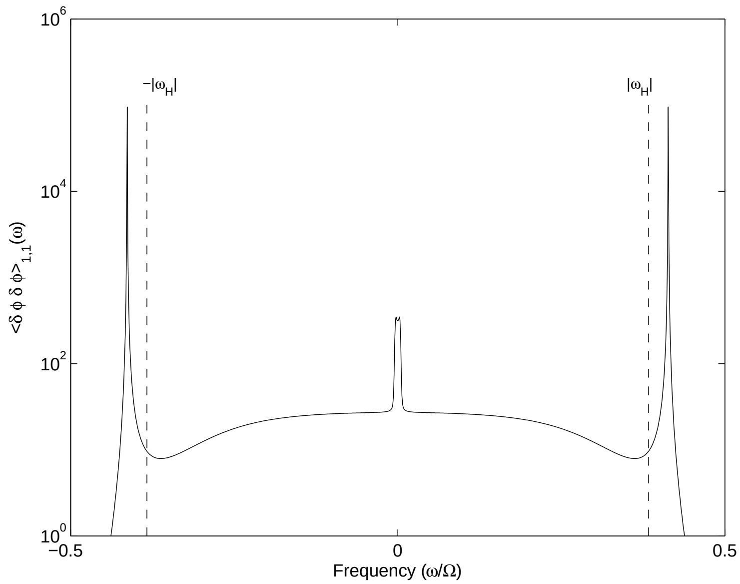
$$\left(\frac{1 + k_\perp^2(l,m)}{k_\perp^2(l,m)}\right)\left(1 - \frac{\omega_{IA}^2}{\omega^2}\right) = 0 \Rightarrow \omega\left(\frac{\partial \mathcal{D}'}{\partial \omega}\right) = \frac{2(1 + k_\perp^2(l,m))}{k_\perp^2(l,m)}$$

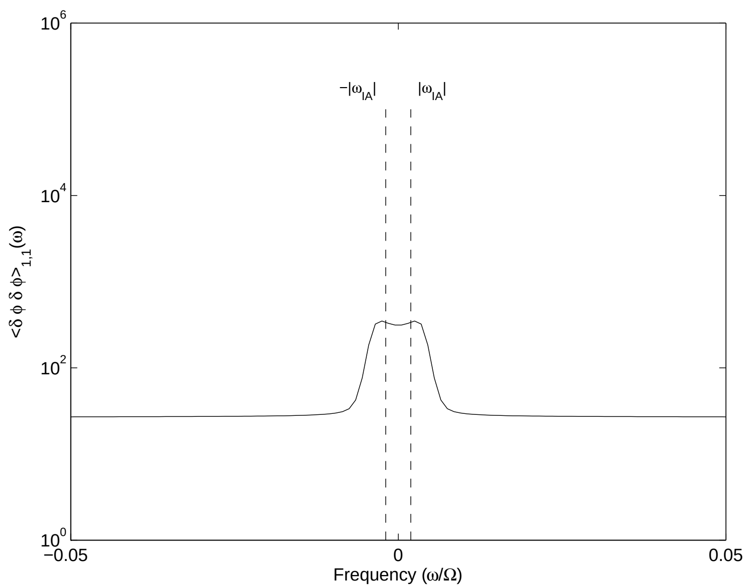
$$\left(1 - \frac{\omega_H^2}{\omega^2}\right) = 0 \Rightarrow \omega\left(\frac{\partial \mathcal{D}'}{\partial \omega}\right) = 2$$

$$\langle \delta\phi\delta\phi \rangle_{l,m}(\omega) = \frac{2\pi}{N}\left[\frac{\delta(\omega - |\omega_H|)}{2k_\perp^2(l,m)} + \frac{\delta(\omega + |\omega_H|)}{2k_\perp^2(l,m)} + \right.$$

$$\left. \frac{\delta(\omega - |\omega_{IA}|)}{2(1 + k_\perp^2(l,m))} + \frac{\delta(\omega + |\omega_{IA}|)}{2(1 + k_\perp^2(l,m))}\right]$$

• Since $k_\perp^2(l,m)$ is small, the ω_H-terms are dominant; spectral density function is composed of localized peaks at normal modes.





• Numerical representation of $\langle \delta\phi\delta\phi \rangle_{l,m}(\omega)$; $\theta = 0.01$, $L_x = 64$, $L_y = 32$, $(l,m) = (1,1)$, $v_{te} = \sqrt{1837.0}$. Qualitatively correct; large-amplitude sharp peaks near ω_H modes; smaller-amplitude, less well-defined peaks near ion acoustic modes.

• Compare ω-integrated spectral density approximation with exact answer:

$$\langle \delta\phi\delta\phi \rangle_{l,m(approximate)} = \frac{1}{N}\left[\frac{1}{k_\perp^2(l,m)} + \frac{1}{(1+k_\perp^2(l,m))}\right]$$

$$\langle \delta\phi\delta\phi \rangle_{l,m(exact)} = \frac{1}{Nk_\perp^2(l,m)[1+k_\perp^2(l,m)/2]}$$

• Slight overestimate, but corrections are of order $k_\perp^2(l,m)$, which is small.

• Keeping more terms in the expansion of $Z(\xi_\alpha)$ improves both this estimate and the prediction for the location of peaks in the spectral density.

• Spectral density $\sim 1/N$, where N is the number of particles in the system.

• Include drift waves - what changes? Assume ω_N^* is small and find its effect on normal modes

• For ω_H-modes:

$$0 = 1 - \frac{\omega_H^2}{\omega^2} + \frac{m\omega_N^*\omega_H^2}{\omega^3} \Rightarrow \omega = \pm\omega_H - \frac{m\omega_N^*}{2}$$

• ω_H-modes slightly downshifted.

• For ion acoustic modes:

$$0 = \left(\frac{1 + k_\perp^2(l,m)}{k_\perp^2(l,m)}\right)\left[1 - \frac{\omega_{IA}^2}{\omega^2} - \frac{m\omega_N^*\omega_{IA}^2}{\omega^3} - \frac{m\omega_N^*}{\omega(1 + k_\perp^2(l,m))}\right]$$

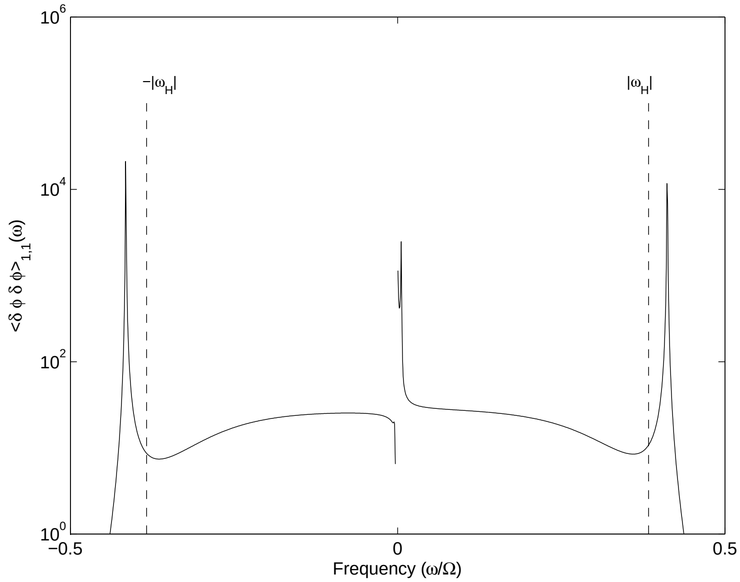
$$\Rightarrow \omega = \pm\omega_{IA} + \frac{m\omega_N^*}{2}\left[\frac{2 + k_\perp^2(l,m)}{1 + k_\perp^2(l,m)}\right]$$

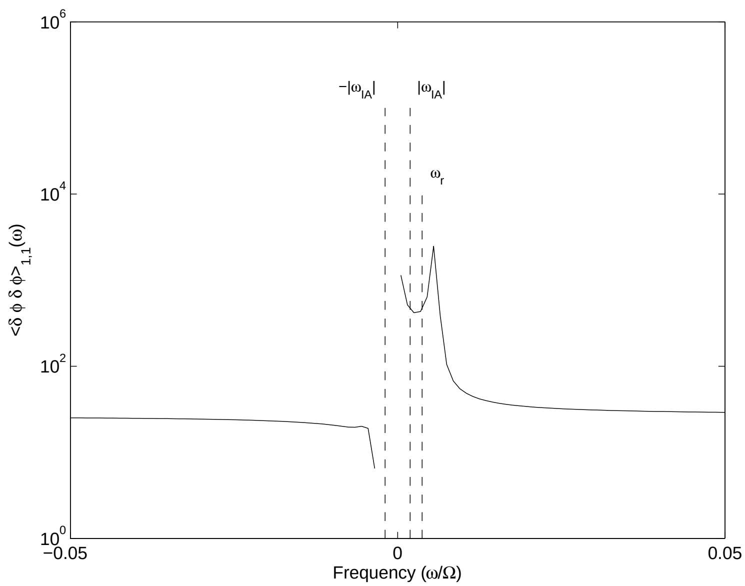
• ω_{IA}-modes slightly upshifted.

• Estimate for ω-integrated $\langle\delta\phi\delta\phi\rangle_{l,m}$ remains the same, to lowest order in ω_N^*.

• Can't perform the ω-integral analytically when growing modes are present; numerical representation has problems capturing transition from damped to growing modes

• The (positive definite) spectral density function is still composed of localized peaks at normal modes, but its numerical representation is not correct for all ω.





• Numerical representation of $\langle \delta\phi\delta\phi \rangle_{l,m}(\omega)$; same parameters with $\kappa_N = 0.02$. Modes shift in the proper direction; transition from damped to growing modes causes spurious numerical results.

Drift waves - nonlinear saturation

• As the drift wave nonlinearly saturates, its growth rate decreases to zero (marginally stable state); relevant frequency is now no longer in upper ω-half-plane.

• Keep only a single mode and its conjugates ($l = \pm 1, m = \pm 1$), estimate properties of nonlinearly saturated state using quasilinear approach (no mode coupling).

• Electron physics drives the drift wave, so neglect ion nonlinearities and perturbed ion current - we obtain (from Vlasov and Poisson equations)

$$\frac{\partial \delta f_e}{\partial t} + v_\parallel \theta \frac{\partial \delta f_e}{\partial y} - \nabla\phi \times \mathbf{b} \cdot \nabla \delta f_e + v_{te}^2 \theta \frac{\partial \phi}{\partial y} \frac{\partial \delta f_e}{\partial v_\parallel} +$$

$$\frac{\partial \phi}{\partial y} F_{0e} \kappa_N - \theta v_\parallel \frac{\partial \phi}{\partial y} F_{0e} = 0$$

$$\frac{\partial \delta n_i}{\partial t} + \kappa_N \frac{\partial \phi}{\partial y} = 0$$

$$\sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} k_\perp^2(l,m) \phi_{l,m}(t) e^{ik_x l x} e^{ik_y m y} = \delta n_i - \int_{-\infty}^{\infty} \delta f_e dv_\parallel$$

• Neglect parallel velocity nonlinearity, Fourier transform in time and space:

$$-i\omega_{l,m} \delta f_{elm}(\omega) + imk_\parallel v_\parallel \delta f_{elm}(\omega) + i\phi_{l,m}(\omega)(m\omega_N^* - mk_\parallel v_\parallel) F_{0e} +$$

$$\sum_{l'=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} (m'l - l'm) k_x k_y \phi_{l',m'}(\omega) \delta f_{l-l',m-m'}(\omega)$$

$$e^{-i(\omega_{l',m'} + \omega_{l-l',m-m'} - \omega_{l,m})t} = 0$$

$$-i\omega_{l,m} \delta n_{ilm}(\omega) + im\omega_N^* \phi_{l,m}(\omega) = 0$$

$$k_\perp^2(l,m) \phi_{l,m}(\omega) = \delta n_{ilm}(\omega) - \int_{-\infty}^{\infty} \delta f_{elm}(\omega) dv_\parallel$$

$$\omega_{-l,-m}^* = \omega_{l,m} \quad ; \quad \phi_{-l,-m}^*(\omega) = \phi_{l,m}(\omega) \quad ; \quad \delta f_{e-l,-m}^*(\omega) = \delta f_{el,m}(\omega)$$

• From linear dispersion relation, $(l=\pm 1, m=\pm 1)$:

$$\omega^*_{1,-1} = -\omega_{1,1} \quad ; \quad \phi^*_{1,-1}(\omega) = \phi_{1,1}(\omega) \quad ; \quad \delta f^*_{e1,-1}(\omega) = \delta f_{e1,1}(\omega)$$

$$\delta f_{e1,1}(\omega) = \phi_{1,1}(\omega) F_{0e} \frac{(\omega^*_N - k_\parallel v_\parallel)}{(\omega_{1,1} - k_\parallel v_\parallel)}$$

• Substitute these into $l=2, m=0$ equation:

$$\delta f_{e2,0}(\omega) = -\frac{2ik_x k_y F_{0e} |\phi_{1,1}(\omega)|^2 (\omega^*_N - k_\parallel v_\parallel)}{|\omega_{1,1} - k_\parallel v_\parallel|^2} \quad ; \quad \omega_{2,0} = 2i \ \mathrm{Im}\ \omega_{1,1}$$

• Nonlinear $l=1, m=1$ equation; $\omega_{1,1} = \omega_{1,1r} + i\gamma_{1,1}$:

$$\delta f_{e1,1}(\omega) = \phi_{1,1}(\omega) F_{0e} \frac{(\omega^*_N - k_\parallel v_\parallel)}{(\omega_{1,1} - k_\parallel v_\parallel)} - \frac{2ik_x k_y \phi_{1,1}(\omega) \delta f_{e2,0}(\omega) e^{2\gamma_{1,1} t}}{(\omega_{1,1} - k_\parallel v_\parallel)}$$

• Nonlinear dispersion relation (only retaining growth rate corrections);

$$k^2_\perp(l,m) + 1 - \frac{\omega^*_N}{\omega_{1,1}} + \frac{i\sqrt{\pi}(\omega_{1,1r} - \omega^*_N)}{\sqrt{2} k_\parallel v_{te}} \left[1 - \frac{4k_x^2 k_y^2 |\phi_{1,1}(t)|^2}{\gamma^2_{1,1}} \right]$$

• Nonlinear growth rate:

$$\gamma_{1,1NL} = \gamma_{1,1} \left[1 - \frac{4k_x^2 k_y^2 |\phi_{1,1}(t)|^2}{\gamma^2_{1,1}} \right] \quad ; \quad \text{Saturation } \phi_{1,1}(t) \sim \frac{\gamma_{1,1}}{2k_x k_y}$$

• Drift wave grows up, saturates, and is (nonlinearly) marginally stable: at saturation,

$$\mathcal{D}_{l,m}(\omega) = \left[\frac{1 + k^2_\perp(l,m)}{k^2_\perp(l,m)} \right] \left[1 - \frac{\omega_{1,1r}}{\omega} \right]$$

• Now, can find approximate contribution of drift wave to ω-integrated spectral density:

$$\langle\delta\phi\delta\phi\rangle_{l,m}(\omega) \approx \frac{2\pi}{Nk_\perp^2(l,m)} \sum_{p=1}^{p_0} \frac{\delta(\omega-\omega_p)}{\omega\left(\frac{\partial\mathcal{D}'}{\partial\omega}\Big|_{\omega-\omega_p}\right.}$$

• This yields the expression

$$\langle\delta\phi\delta\phi\rangle_{l,m}(\omega) \approx \frac{2\pi}{N}\left[\frac{\delta(\omega-\omega_{1,1r})}{1+k_\perp^2(l,m)} \quad + \right.$$

(nonlinearly saturated drift waves)

$$\frac{\delta\left[\omega-|\omega_H|+\frac{\omega_N^*}{2}\right]}{k_\perp^2(l,m)\left(2-\frac{\omega_N^*}{|\omega_H|}\right)} \quad + \quad \frac{\delta\left[\omega+\ |\omega_H|+\frac{\omega_N^*}{2}\right]}{k_\perp^2(l,m)\left(2-\frac{\omega_N^*}{|\omega_H|}\right)} \quad +$$

(ω_H modes)

$$\left.\frac{\delta\left[\omega-|\omega_{IA}|-\omega_{1,1r}(1+\frac{k_\perp^2(l,m)}{2})\right]}{[1+k_\perp^2(l,m)]\left[2+\frac{\omega_{1,1r}k_\perp^2(l,m)}{|\omega_{IA}|}\right]} + \frac{\delta\left[\omega+|\omega_{IA}|-\omega_{1,1r}(1+\frac{k_\perp^2(l,m)}{2})\right]}{[1+k_\perp^2(l,m)]\left[2-\frac{\omega_{1,1r}k_\perp^2(l,m)}{|\omega_{IA}|}\right]}\right]$$

(Ion acoustic modes)

• Now, integrate this formula over high frequencies only; nonzero but small ω_N^* in the dispersion relation only leads to slight shifts in the normal mode frequencies:

$$\int_{-\infty}^{\infty} \frac{\langle\delta\phi\delta\phi\rangle_{l,m}(\omega)}{2\pi} d\omega\Bigg|_{large\ |\omega|} \approx \frac{1}{Nk_\perp^2(l,m)}\left[\frac{1}{1+\frac{\omega_N^{*2}}{4\omega_H^2}}\right]$$

• For the undriven system,

$$\int_{-\infty}^{\infty} \frac{\langle\delta\phi\delta\phi\rangle_{l,m}(\omega)}{2\pi} d\omega\Bigg|_{large\ |\omega|} \approx \frac{1}{Nk_\perp^2(l,m)}$$

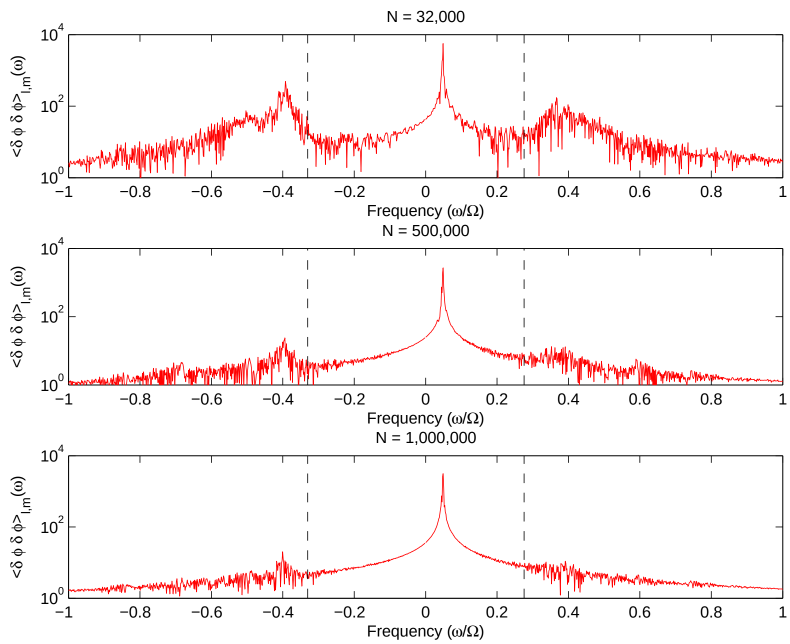
• The spectral wavenumber density, at large $|\omega|$, is nearly unaltered by the presence of the drift wave. It continues to scale as N^{-1}. For drift waves, however, the fluctuations amplify, and the mode grows until the nonlinear saturation effects (independent of N) halt its growth. Thus, we expect this function to have a peak at the drift frequency $\omega_{1,1r}$ with an amplitude independent of N; as well, we expect peaks at the frequencies $\pm\omega_H$, with amplitudes which decrease with increasing N.

Properties of discrete particle noise

• In Fourier space, spectral density function for $\omega_N^* = 0$ system indicates that discrete particle noise resides prominently at frequencies $\omega = \pm|\omega_H|$; its amplitude scales as N^{-1}.

• For small ω_N^*, the high-frequency (discreteness-related) components of the spectral density do not change appreciably. Drift waves occur, but at a much lower frequency for a given wavenumber.

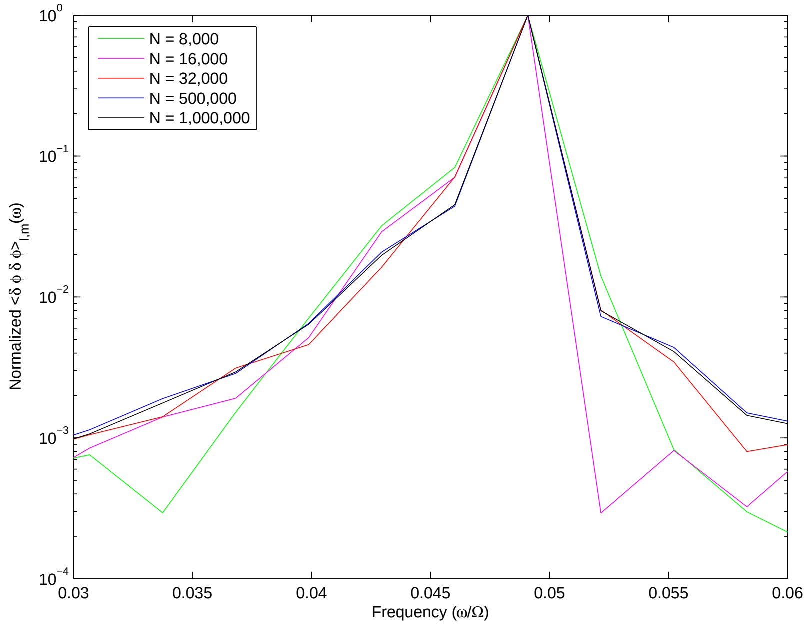
• The drift waves saturate at a level independent of N (assuming large enough N for accurate results).



• Simulations confirm these predictions. Here, we simulate drift instabilities ($\kappa_N = 0.2$) on a 128×128 grid, with $\theta = 0.01$, $(l, m) = (\pm 1, \pm 1)$, $L_x = L_y = 23$, and $\Delta t = 0.125$.

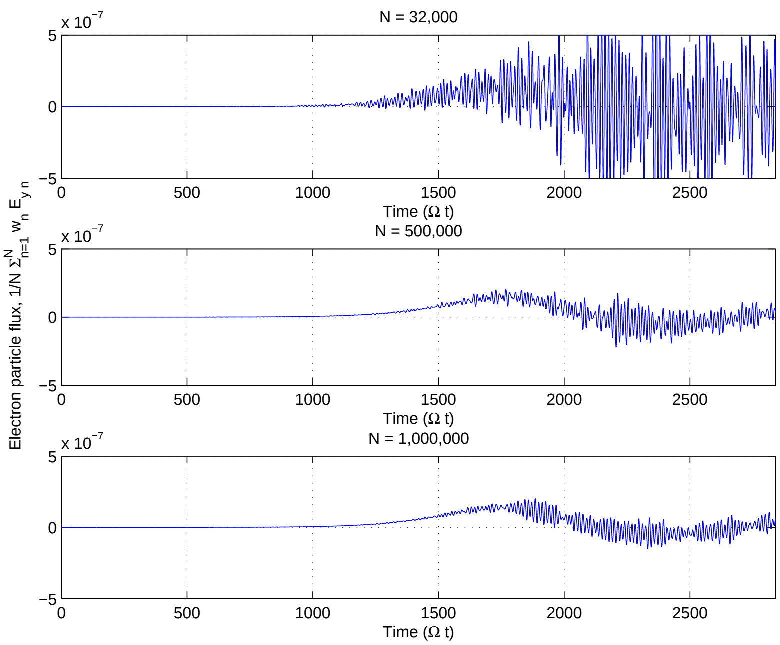
• To ensure that the ω_H-noise is not affecting the drift wave, see if drift-wave peak in the spectral density broadens as a function of N.

• No broadening seen; suggests that noise is not substantially affecting physics of drift wave.



• Discreteness-induced noise can also influence measurements of particle flux; can decrease average flux levels spuriously if too few particles are used. Average flux converges quickly with N once enough particles are used to obtain good statistics.

• Signal-to-noise ratio improves as particle number is increased, even after average flux level has stabilized.



• Same parameters as on previous page; the noise on the particle flux measurement decreases as N is increased.

Particle simulations

• Evolve particles according to characteristics of Vlasov equation, value of F_α is conserved along these orbits.

$$\frac{\partial F_\alpha}{\partial t} + v_\parallel \mathbf{b}\cdot\nabla F_\alpha - \nabla\phi\times\mathbf{b}\cdot\nabla F_\alpha - \frac{q_\alpha m_i}{q_i m_\alpha}\mathbf{b}\cdot\nabla\phi\frac{\partial F_\alpha}{\partial v_\parallel} = 0$$

• Characteristics:

$$\frac{dx_{j\alpha}(t)}{dt} = -\left(\frac{\partial\phi(x,y,t)}{\partial y}\right|_{x=x_{j\alpha}(t),y=y_{j\alpha}(t)}$$

$$\frac{dy_{j\alpha}(t)}{dt} = \theta v_{\parallel j\alpha}(t) + \left(\frac{\partial\phi(x,y,t)}{\partial x}\right|_{x=x_{j\alpha}(t),y=y_{j\alpha}(t)}$$

$$\frac{dv_{\parallel j\alpha}(t)}{dt} = -\frac{q_\alpha m_i}{q_i m_\alpha}\theta\left(\frac{\partial\phi(x,y,t)}{\partial y}\right|_{x=x_{j\alpha}(t),y=y_{j\alpha}(t)}$$

• Define weight function $w_\alpha \equiv \delta f_\alpha / F_\alpha$, where $F_\alpha = F_{0\alpha} + \delta f_\alpha$.

• From Vlasov equation,

$$\frac{dw(x,y,v_\parallel,t)_\alpha}{dt} = [1 - w(x,y,v_\parallel,t)_\alpha]\left(\frac{q_\alpha}{q_i}\theta v_\parallel - \kappa_N\right)\frac{\partial\phi(x,y,t)}{\partial y}$$

• Monte Carlo sample:

$$\frac{dw_{j\alpha}(t)}{dt} = [1 - w_{j\alpha}(t)]\left(\frac{q_\alpha}{q_i}\theta v_{\parallel j\alpha}(t) - \kappa_N\right)\left(\frac{\partial\phi(x,y,t)}{\partial y}\right|_{x=x_{j\alpha}(t),y=y_{j\alpha}(t)}$$

• Representation:

$$F_\alpha = F_{0\alpha} + \sum_{j=1}^{N} w_{j\alpha}(t)\frac{L_x L_y}{N}\delta[x - x_{j\alpha}(t)]\delta[y - y_{j\alpha}(t)]\delta[v_\parallel - v_{\parallel j\alpha}(t)]$$

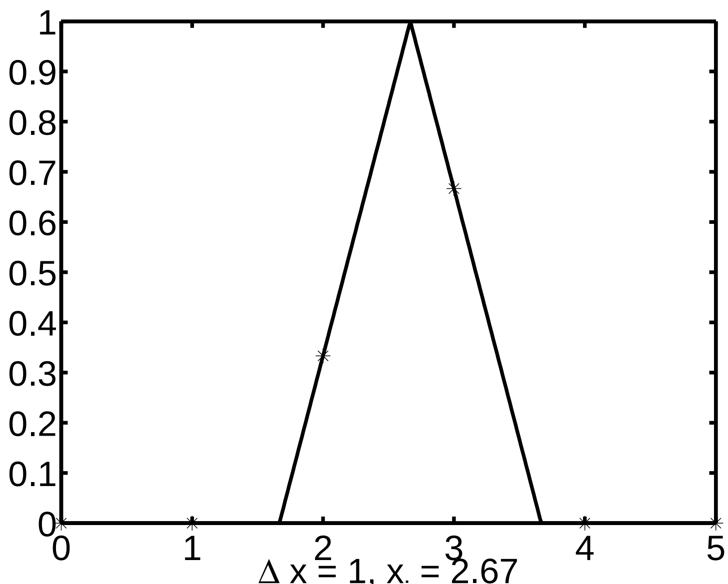
• Potential only represented at gridpoints - interpolate particle positions from continuous space to gridpoints using finite-size particles.

$$\delta[x - x_j(t)] \rightarrow \frac{1}{2\Delta x^2}\left[\Delta x - |x - x_j(t)| + |\Delta x - |x - x_j(t)||\right]$$

where Δx is the grid spacing. If $x = x_n = n\Delta x$, and $x_j(t) = (m+\epsilon)\Delta x$, for integer m, n and $0 \leq \epsilon < 1$, we obtain

$$S(x_n) = \frac{1}{2\Delta x}\left[1 - |n - m - \epsilon| + |1 - |n - m - \epsilon||\right]$$

$$S(x_n) = \frac{1}{\Delta x}(1-\epsilon)\delta_{n,m} + \frac{1}{\Delta x}\epsilon\delta_{n,m+1}$$



which is just bilinear interpolation.

• In Fourier space, with $\Delta x = L_x/N_x$ (number of x gridpoints),

$$S[x - x_j(t)] = \sum_{l=-\infty}^{\infty} \frac{\sin^2\left(\frac{\pi l}{N_x}\right) e^{-2\pi i l x_j(t)/L_x}}{\left(\frac{\pi l}{N_x}\right)^2} e^{2\pi i l x/L_x}(1 - \delta_{l,0})$$

• For $x = n\Delta x$,

$$S[x_n - x_j(t)] = \sum_{l=-\infty}^{\infty} \frac{\sin^2\left(\frac{\pi l}{N_x}\right) e^{-2\pi i l x_j(t)/L_x}}{\left(\frac{\pi l}{N_x}\right)^2} e^{2\pi i l n/N_x}(1 - \delta_{l,0})$$

• With aliasing,

$$S[x_n - x_j(t)] = \sum_{l=-Nx/2,\neq 0}^{Nx/2-1} \sin^2\left(\frac{\pi l}{N_x}\right) e^{-2\pi i l x_j(t)/L_x} \left[\sum_{p=-\infty}^{\infty} \frac{e^{-2\pi i p x_j(t)/\Delta x}}{\left(\frac{\pi l}{N_x} + p\pi\right)^2}\right] e^{2\pi i l n/N_x}$$

• Simplify notation:

$$\sum_{p=-\infty}^{\infty} \frac{e^{-2\pi i p x_{j\alpha}(t)/\Delta x}}{\left(\frac{\pi l}{N_x} + p\pi\right)^2} \equiv A_{jl\alpha} \quad ; \quad \sum_{q=-\infty}^{\infty} \frac{e^{-2\pi i q y_{j\alpha}(t)/\Delta y}}{\left(\frac{\pi m}{N_y} + q\pi\right)^2} \equiv B_{jm\alpha}$$

• Poisson equation:

$$\phi_{l,m}(t) = \frac{\sum_\alpha \frac{q_\alpha}{q_i} \sum_{j=1}^{N} \frac{w_{j\alpha}(t)}{N} \sin^2\left(\frac{\pi l}{N_x}\right) e^{-2\pi i l x_{j\alpha}(t)/L_x} \sin^2\left(\frac{\pi m}{N_y}\right) e^{-2\pi i m y_{j\alpha}(t)/L_y} A_{jl\alpha} B_{jm\alpha}}{\left(\frac{4\pi^2 l^2}{N_x^2} + \frac{4\pi^2 m^2(1-\theta^2)}{N_y^2}\right)}$$

if $(l, m) \neq (0, 0)$.

• Can add (l, m)-dependent filter $S_{l,m}$ to restrict to specific values of (l, m) (as in simulations).

• From fluctuation-dissipation theorem (assuming undriven system),

$$\langle \delta\phi\delta\phi \rangle_{l,m} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \langle |\delta\phi_{l,m}(t)|^2 \rangle dt$$

• Can determine an upper bound on discrete particle noise at a given wavenumber by calculating this quantity for totally uncorrelated particles; in a discrete time representation [W. M. Nevins *et al.*, Phys. Plasmas **12**, 122305 (2005)],

$$\langle \delta\phi\delta\phi \rangle_{l,m} = \frac{1}{N_t} \sum_{t=0}^{N_t-1} \sum_{\alpha} \sum_{\alpha'} \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{q_\alpha q_{\alpha'}}{q_i^2 N^2} \sin^4\left(\frac{\pi l}{N_x}\right) \sin^4\left(\frac{\pi m}{N_y}\right) |S_{l,m}|^2$$

$$\left\langle \frac{w_{j\alpha t} w_{k\alpha' t} e^{-2\pi i l [x_{j\alpha t} - x_{k\alpha' t}]/L_x} e^{-2\pi i m [y_{j\alpha t} - y_{k\alpha' t}]/L_y} A_{jl\alpha} B_{jm\alpha} A^*_{kl\alpha'} B^*_{km\alpha'}}{\left(\frac{4\pi^2 l^2}{N_x^2} + \frac{4\pi^2 m^2 (1-\theta^2)}{N_y^2}\right)^2} \right\rangle$$

• Grid effects make this cumbersome; the different wavenumbers of an indidividual particle couple together.

• Ignoring all grid effects (assuming ϕ can be represented throughout the domain, not just at gridpoints) yields (for the uncorrelated result)

$$\langle \delta\phi\delta\phi \rangle_{l,m} \leq \frac{1}{N_t} \sum_{t=0}^{N_t-1} \frac{(\bar{w_{et}}^2 + \bar{w_{it}}^2)}{N k_\perp(l,m)^4}$$

• Compare with exact answer (for undriven system):

$$\langle \delta\phi\delta\phi \rangle_{l,m} = \frac{1}{N k_\perp^2(l,m)} \left(\frac{2}{k_\perp^2(l,m) + 2}\right) \leq \frac{1}{N_t} \sum_{t=0}^{N_t-1} \frac{(\bar{w_{et}}^2 + \bar{w_{it}}^2)}{N k_\perp(l,m)^4}$$

• Rearrange:

$$1 \leq W^2 \frac{1 + k_\perp^2(l,m)/2}{k_\perp^2(l,m)}$$

where W^2 is the time average of the sum of the ion and electron mean square weights. This quantity can be small, but $k_\perp^2(l,m)$ is also small; in addition, the weight representation requires us to normalize ϕ to a typical weight $\bar{W}$, since differing values of $w_{j\alpha}(t=0)$ yield different fluctuation levels for undriven systems [G. Hu and J. A. Krommes, Phys. Plasmas **1**, 863 (1994)]. Roughly, then, we get

$$1 \leq \frac{W^2}{\bar{W}^2} \frac{1 + k_\perp^2(l,m)/2}{k_\perp^2(l,m)}$$

which is obviously true for $k_\perp^2(l,m) \ll 1$.

• When the inequality is suspect, the bulk of the fluctuations at a given wavenumber are caused by noise.

• However, this method gives no information about the frequency dependence of the noise; high-frequency discreteness-induced noise may not substantially affect low-frequency turbulent phenomena.

Conclusions

• Linear growth, saturation, and transport associated with long-wavelength drift modes (ignoring mode coupling effects) appear to be independent of the high-frequency discrete particle noise contained in the normal (ω_H) modes.

• This high-frequency noise scales inversely as the number of particles N in the simulation; higher N also improves signal-to-noise ratio associated with transport quantities (flux).