Calculation of the Neoclassical Radial Electric Field using the Global Toroidal Code

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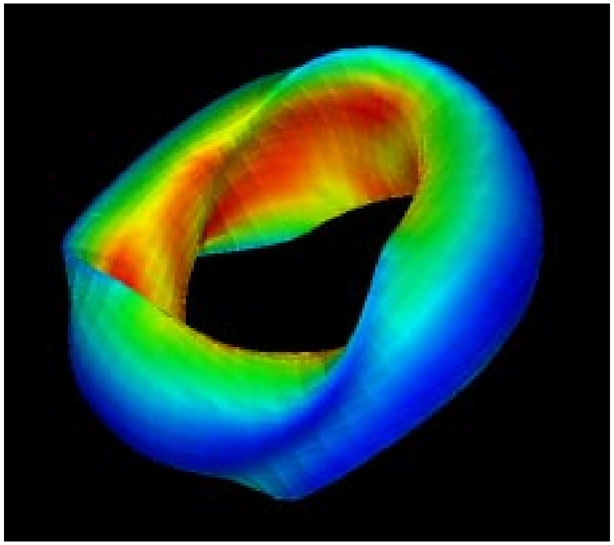

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Motivation

- Neoclassical radial electric field is important for stellarator transport (e.g. collisionless particle dynamics; zonal flow physics; transport barriers)
- Departure of axi-symmetry can be weak (i.e. QA concept): standard calculation of E_r can be difficult
- Well-established gyro-kinetic particle simulation techniques offer alternative possibility to determine E_r
- Method can be generalized to viscous flow damping in fully three-dimensional, non-axisymmetric geometries

The Method

Write confining $\mathbf{B}$ field in Boozer coordinates (ψ, θ, ζ) as

$$\begin{aligned} \mathbf{B} &= \iota(\psi)\, \boldsymbol{\nabla}\zeta \times \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\theta \\ \mathbf{B} &= g(\psi)\, \boldsymbol{\nabla}\zeta + I(\psi)\, \boldsymbol{\nabla}\theta + \beta_\star \boldsymbol{\nabla}\psi \end{aligned} \tag{1}$$

with $g(\psi) \propto$ poloidal current; $I(\psi) \propto$ toroidal current.
Jacobian of transformation $\mathcal{J} \equiv \left[\boldsymbol{\nabla}\psi \cdot (\boldsymbol{\nabla}\theta \times \boldsymbol{\nabla}\zeta)\right]^{-1}$ satisfies

$$\mathcal{J}B^2 = g(\psi) + \iota(\psi)\, I(\psi) \equiv f(\psi) \quad \text{(flux surface quantity)} \tag{2}$$

Ion Momentum Balance

$$\rho \frac{d\mathbf{V}}{dt} = -\boldsymbol{\nabla} \cdot \mathbf{P} + en \left(\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} \right) + \mathbf{F} + \mathbf{R} \tag{3}$$

where $\mathbf{V}$: fluid velocity; ρ : mass density, $\mathbf{R}$: collisional drag; $\mathbf{F}$: (external) applied force and $\mathbf{P} = P_{||} \widehat{\mathbf{b}}\widehat{\mathbf{b}} + \left(\mathbf{I} - \widehat{\mathbf{b}}\widehat{\mathbf{b}}\right) P_\perp$: the pressure tensor; also $P_{||}\,(P_\perp)$: parallel (perpendicular) pressure.

Take scalar product of Eq.(2) with $\mathbf{e}_\zeta \equiv \partial \mathbf{r}/\partial\zeta$ (with $\mathbf{r}$ is the position vector) and operate with $\langle ... \rangle = \int\int ... \mathcal{J}(\psi, \theta, \zeta)\, d\theta d\zeta$ one obtains

$$\frac{\iota(\psi)}{c} \frac{dQ}{dt} - \left\langle \frac{dL_\zeta}{dt} \right\rangle = \left\langle \frac{\partial \widehat{P}}{\partial \zeta} \right\rangle - T_\zeta \tag{4}$$

where $\widehat{P} \equiv \left(P_{||} + P_{\perp}\right)/2$, $T_\zeta = \langle (\mathbf{R} + \mathbf{F}) \cdot \mathbf{e}_\zeta \rangle$ is the torque due to applied forces and collisional drag; L_ζ is the toroidal component of the canonical momentum $\mathbf{L} = \rho \mathbf{V} + e\mathbf{A}/c$ where $\mathbf{A} = \psi \boldsymbol{\nabla} \theta - \chi \boldsymbol{\nabla} \zeta$ is the vector potential and $2\pi\chi$ is the poloidal flux.

To derive Eq.(4), note that

$$\left\langle en\mathbf{e}_\zeta \cdot \left(\frac{\mathbf{V} \times \mathbf{B}}{c}\right)\right\rangle = \frac{e}{c} \int\int n\mathbf{V} \cdot \left(\mathbf{B} \times \mathbf{e}_\zeta\right) \mathcal{J} d\theta d\zeta = \frac{e}{c} \iota\left(\psi\right) \int \boldsymbol{\Gamma} \cdot d\boldsymbol{\sigma}_n = \frac{\iota\left(\psi\right)}{c} \frac{dQ}{dt}$$

Here Q is the total charge, $\boldsymbol{\Gamma} = n\mathbf{V}$ is the particle flux, and $d\boldsymbol{\sigma}_n \equiv \mathcal{J} \boldsymbol{\nabla} \psi d\theta d\zeta$ is an area element normal to the magnetic surface $\psi = \mathrm{const}$ and pointing outwards.

For zero applied force and after a few ion-ion collision times, toroidal balance equation reads

$$\frac{\iota\left(\psi\right)}{c} \frac{dQ}{dt} = S \equiv \left\langle \frac{\partial \widehat{P}}{\partial \zeta} \right\rangle \tag{5}$$

Knowing the parallel and perpendicular pressures on the magnetic surface (velocity moments of δf), one obtains a measure of the radial particle flux on that surface through Eq.(5)

Calculation of Parallel & Perpendicular Pressures *PPPL*

Write perpendicular pressure as

$$P_\perp = \sum_{m,n} (P_\perp)_{m,n} \exp\left[i\left(m\theta + nN_p\zeta\right)\right] \tag{6}$$

where N_p is the number of field periods of the configuration and the Fourier coefficients are calculated according to

$$(P_\perp)_{m,n} = \frac{\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \left(mv_\perp^2/2\right) \delta f \exp\left[-i\left(m\theta + nN_p\zeta\right)\right] d^3v}{\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta} \tag{7}$$

Guiding center motion and collisions will spread the particles toward equal density in pitch and over the magnetic surface

Then, in a small layer $\delta\psi \ll \psi_b$ (boundary), one notes that

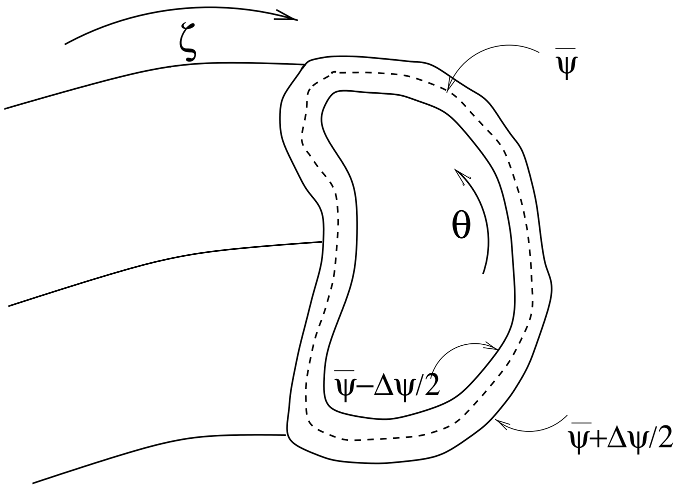
$$\int\int d\theta d\zeta \implies \int \mathcal{J}^{-1} (\delta\psi)^{-1} d^3x \implies \left[F\left(\overline{\psi}\right)\delta\psi\right]^{-1} \int B^2 d^3x$$

and the $(P_\perp)_{m,n}$ Fourier components become

$$(P_\perp)_{m,n} = \int \frac{\int d^3x \left(mv_\perp^2/2\right) \delta f B^2 \exp\left[-i\left(m\theta + nN_p\zeta\right)\right]}{\int d^3x B^2} d^3v \tag{8}$$

Same method applies for $P_{||}$.

Numerical Method



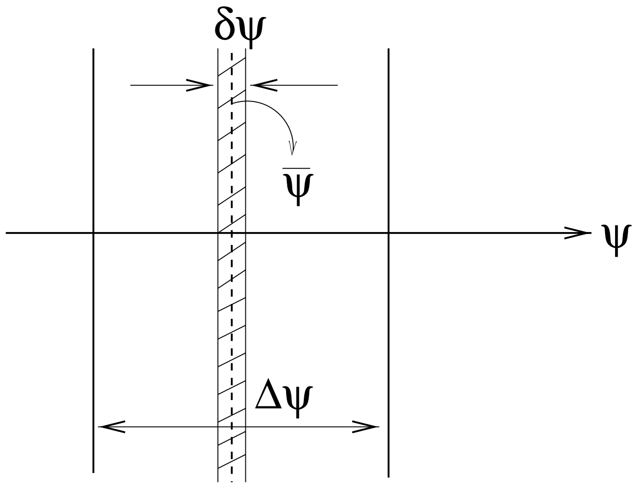
Particles are initially randomized in θ and ζ, and between $\overline{\psi} - \Delta\psi/2$ and $\overline{\psi} + \Delta\psi/2$.

Introduce radial coordinate $r = \sqrt{\psi/B_0}$ so that $\delta r \approx \Delta\psi/\left(2\sqrt{B_0\psi}\right)$; typical drift time is $\tau_d \approx \Delta r/V_d$ where $V_d \approx V_{th}\left(\rho_{th}/R_c\right)$ is the typical radial curvature drift velocity.

We must have $\tau_d \gg \tau_r$, where τ_r is the relaxation time (typically a few ion-ion collision times).

Calculation of $P_{||}$ and $P_\perp$ are carried out within an annulus $\delta\psi \ll \Delta\psi \ll \psi_b$ centered around $\overline{\psi}$.

Parallel and perpendicular pressures calculated on different processor element (PE) are collected on a single PE (PE=0), on which the Fourier components $\left(P_\perp\right)_{m,n}$ and $\left(P_{||}\right)_{m,n}$ are evaluated.



Divergence of Pressure Tensor

Write Pressure Tensor $\mathbf{P}$ as

$$\mathbf{P} = \widetilde{P}\mathbf{BB} + P_{\perp}\mathbf{I} + P_{\perp}\mathbf{I} \tag{9}$$

where $\widetilde{P} \equiv \left(P_{||} - P_{\perp}\right)/B^2$. Noting that

$$\boldsymbol{\nabla} B^2/2 = \mathbf{B}\times\left(\boldsymbol{\nabla}\times\mathbf{B}\right) + \left(\mathbf{B}\cdot\boldsymbol{\nabla}\right)\mathbf{B} \tag{10}$$

and

$$\boldsymbol{\nabla}\cdot\mathbf{P} = \mathbf{B}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\widetilde{P}\right) + \widetilde{P}\boldsymbol{\nabla}\cdot\left(\mathbf{BB}\right) + \boldsymbol{\nabla} P_{\perp} \tag{11}$$

and using Ampere's law and the radial force balance equation, we obtain

$$\boldsymbol{\nabla}\cdot\mathbf{P} = \mathbf{B}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\widetilde{P}\right) + \widetilde{P}\left(\frac{1}{2}\boldsymbol{\nabla} B^2 + 4\pi\boldsymbol{\nabla} P_0\right) + \boldsymbol{\nabla} P_{\perp} \tag{12}$$

where $P_0 = P_0(\psi)$ is the equilibrium pressure. Taking the scalar product of Eq.(12) with $\mathbf{e}_{\varphi} \equiv \partial\mathbf{r}/\partial\varphi$ where $\mathbf{r}$ is the position vector and $\varphi = \{\theta, \zeta\}$ one gets

$$\mathbf{e}_{\varphi}\cdot\left(\boldsymbol{\nabla}\cdot\mathbf{P}\right) = B_{\varphi}\left(\mathbf{B}\cdot\boldsymbol{\nabla}\widetilde{P}\right) + \frac{\widetilde{P}}{2}\frac{\partial}{\partial\varphi}B^2 + \frac{\partial P_{\perp}}{\partial\varphi} \tag{13}$$

Taking the flux-surface average $\langle \bullet \rangle \equiv \int\int \mathcal{J} \left(\bullet \right) d\theta d\zeta$ of Eq.(13) yields

$$\langle \mathbf{e}_\varphi \boldsymbol{\cdot} \left(\boldsymbol{\nabla} \boldsymbol{\cdot} \mathbf{P} \right) \rangle = \frac{1}{2} \left\langle \frac{\partial}{\partial \varphi} \left(P_{||} + P_\perp \right) \right\rangle \ , \tag{14}$$

since $\langle \mathbf{B} \boldsymbol{\cdot} \boldsymbol{\nabla} F \rangle = 0$ for any function $F = F\left(\psi, \theta, \zeta\right)$ and

$$\begin{aligned} \left\langle \widetilde{P} \frac{\partial B^2}{\partial \varphi} \right\rangle &= \left\langle \frac{P_{||} - P_\perp}{B^2} \frac{\partial B^2}{\partial \varphi} \right\rangle \\ &= \left\langle B^2 \left(P_\perp - P_{||} \right) \frac{\partial}{\partial \varphi} \left(\frac{1}{B^2} \right) \right\rangle \\ &= \left\langle \frac{f\left(\psi\right)}{\mathcal{J}} \left(P_\perp - P_{||} \right) \frac{\partial}{\partial \varphi} \left(\frac{1}{B^2} \right) \right\rangle \\ &= f\left(\psi\right) \int\int \left(P_\perp - P_{||} \right) \frac{\partial}{\partial \varphi} \left(\frac{1}{B^2} \right) d\theta d\zeta \\ &= f\left(\psi\right) \int\int \frac{1}{B^2} \frac{\partial}{\partial \varphi} \left(P_{||} - P_\perp \right) d\theta d\zeta \\ &= \int\int \mathcal{J} \frac{\partial}{\partial \varphi} \left(P_{||} - P_\perp \right) d\theta d\zeta \\ &= \left\langle \frac{\partial}{\partial \varphi} \left(P_{||} - P_\perp \right) \right\rangle \ . \end{aligned}$$

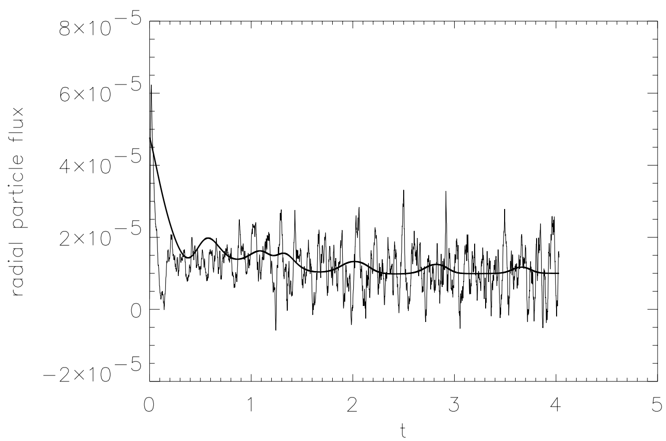
Results & Conclusions

- Run for NCSX plasma (C82 configuration); with central ion temperature $T_i(0) = 2.76$ KeV; central electron temperature $T_e(0) = 2.14$ KeV; central plasma density $n_0 = 6.73 \times 10^{13}$ cm^{-3}. Magnetic surface of reference $\psi/\psi_b = 0.7$.

- Equilibrium B field is specified using 30 Fourier harmonics

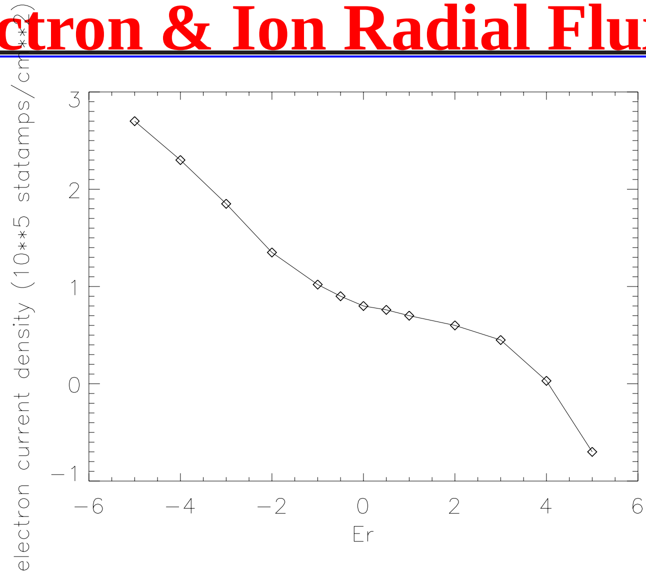
- Trajectories of 2×10^5 Lagragian markers are integrated; time step $\Delta t/\tau_{ii} = 4 \times 10^{-4}$; collisional effects are calculated every 10 time steps.

- Background distribution function f_0 loaded as a Maxwellian with $\langle V_{||}\rangle = 0$.

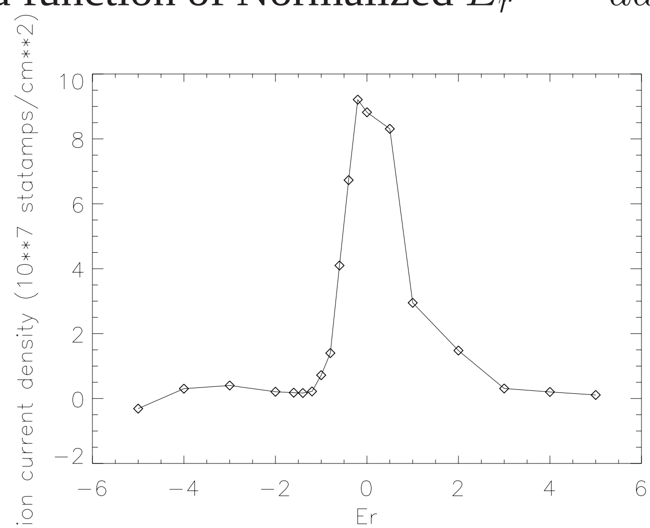


Direct measurement (broken line) and gyro-kinetic calculation (smooth curve) of Γ_r (a.u.)

Electron & Ion Radial Fluxes



Electron Current Density as a function of Normalized $E_r = -ad/dr\,(e\Phi/T_i(0))$



Ion Current Density as a function of Normalized $E_r = -ad/dr\,(e\Phi/T_i(0))$

Stable root found at $E_r \simeq -26.2\ \mathrm{kV/m}$