Physics Letters A 372 (2008) 6094–6096

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Manley–Rowe relations for an arbitrary discrete system

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ARTICLE INFO

Article history:
Received 20 June 2008
Received in revised form 1 August 2008
Accepted 6 August 2008
Available online 12 August 2008
Communicated by F. Porcelli

PACS:
45.05.+x
02.10.Ud
45.20.Jj
05.45.-a

ABSTRACT

Manley–Rowe relations are formulated for a discrete Hamiltonian system with an arbitrary number of resonances. Assuming that the resonances are defined as $\hat{R}|\omega\rangle = 0$, where $\hat{R}$ is an $n \times n$ integer matrix of rank $r < n$, and $|\omega\rangle \equiv (\omega_1, \ldots, \omega_n)^{\mathrm{T}}$ is the frequency vector, the projection of the action vector $|J\rangle$ on $\ker \hat{R}$ is an adiabatic invariant. Hence $n - r$ independent integrals, from where the conventional Manley–Rowe relations for a single resonance follow as a particular case.

Manley–Rowe relations are conservation laws, which constrain the energies exchanged by oscillatory degrees of freedom under nonlinear resonant interactions [1–3]. Originally obtained for electrical circuits [4–24], these relations yield a quantum analogy [25–27] and occur at classical wave interactions [28–38], sharing a common mathematical description [39–44]. However, their general form for multiple resonances has not been understood[1]; hence the problem is yet to be solved.

The purpose of this Letter is to point out that the general Manley–Rowe relations can be derived deductively via the formalism of Ref. [45], offering a compact vector form of the integrals for any dynamical system with an arbitrary number of resonances. We restate the results of Ref. [45], suggest their quantum mechanical interpretation, and show how the conventional Manley–Rowe relations follow for a single resonance as a particular case. We also consider a sample system with multiple resonances to illustrate how the general formalism can yield conservation laws concisely as compared with *ad hoc* techniques.

Consider a classical dynamical system described by n actions $|J\rangle$ and conjugate phases $|\varphi\rangle$ exhibiting oscillations at frequencies $|\dot{\varphi}\rangle = |\omega\rangle$, where we assume the notation $|x\rangle \equiv (x_1, \ldots, x_n)^{\mathrm{T}}$. Suppose that some of these oscillations are in resonance,[2] the condition reading

$$\hat{R}|\omega\rangle = 0, \tag{1}$$

where $\hat{R}$ is an $n \times n$ matrix of rank $r < n$,

$$\hat{R} = \sum_{i=1}^{n} |e^{(i)}\rangle\langle r^{(i)}|, \tag{2}$$

$e_j^{(i)} = \delta_{ij}$, and $|r^{(i)}\rangle$ are integer vectors. Eq. (1) yields r independent equations:

$$\langle r^{(i)}|\omega\rangle = 0, \quad i = 1, \ldots, r, \tag{3}$$

where we assume, for brevity, that those are the first r of $|r^{(i)}\rangle$ vectors that are linearly independent; then Eqs. (3) are equivalent to

$$\sum_{i=1}^{r} |e^{(i)}\rangle\langle r^{(i)}|\omega\rangle = 0. \tag{4}$$

Introduce

$$|q^{(i)}\rangle = \begin{cases} |r^{(i)}\rangle, & i = 1, \ldots, r, \\ |g^{(i-r)}\rangle, & i = r+1, \ldots, n, \end{cases} \tag{5}$$

where $\{|g^{(i)}\rangle\}$ is any integer basis in $\ker \hat{R}$ found from

$$\hat{R}|g^{(i)}\rangle = 0. \tag{6}$$

Then Eq. (4) rewrites as

$$\hat{P}\hat{Q}|\omega\rangle = 0, \tag{7}$$

where $\hat{P}$ is a projection operator of rank r:

$$\hat{P} = \sum_{i=1}^{r} |e^{(i)}\rangle\langle e^{(i)}|, \tag{8}$$

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[1] See Ref. [3, p. 5] for a discussion on the insufficiency of the analysis offered in Ref. [6].

[2] Any of the frequencies being commensurate is also considered a resonance.

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doi:10.1016/j.physleta.2008.08.011

and $\hat{Q}$ is an integer matrix of rank n:

$$\hat{Q} = \sum_{i=1}^{n} |e^{(i)}\rangle\langle q^{(i)}|. \tag{9}$$

Since $\hat{Q}$ is invertible, there exists a canonical transformation $(J, \varphi) \to (I, \theta)$, such that $|\theta\rangle = \hat{U}|\varphi\rangle$, with $\hat{U} = \hat{Q}\,\det\hat{Q}^{-1}$, so Eq. (7) yields

$$\hat{P}|\Omega\rangle = 0, \tag{10}$$

where $|\Omega\rangle \equiv |\dot{\theta}\rangle = \hat{U}|\omega\rangle$. The corresponding generating function $F(\varphi, I)$ is obtained from $\theta_i = \partial_{I_i} F$ and reads

$$F(\varphi, I) = \langle I|\hat{U}|\varphi\rangle, \tag{11}$$

therefore $|J\rangle = \hat{U}^\dagger|I\rangle$, where we used $J_i = \partial_{\varphi_i} F$.

For $|\theta\rangle$ increased by $2\pi|k\rangle$, where $|k\rangle$ is an arbitrary integer vector, $|\varphi\rangle = \hat{U}^{-1}|\theta\rangle$ is increased by $2\pi\hat{U}^{-1}|k\rangle$, where $\hat{U}^{-1}|k\rangle$ is also integer. Thus the system is periodic in $|\theta\rangle$, meaning that $|\theta\rangle$ can be considered phase variables, and the corresponding frequencies read

$$|\Omega\rangle = \det\hat{Q}^{-1} \sum_{i=r+1}^{n} |e^{(i)}\rangle\langle g^{(i-r)}|\omega\rangle. \tag{12}$$

Here $\Omega_i = \langle e^{(i)}|\Omega\rangle$ is zero for $i = 1, \dots, r$ but nonzero for $i = (r+1), \dots, n$, because $|g^{(i)}\rangle$ are linearly independent from all integer vectors orthogonal to $|\omega\rangle$ [Eqs. (3)]. By definition, Ω_i are also mutually incommensurate; hence, assuming that they (and the beat frequencies) are large compared to the rest inverse time scales in the system, the corresponding $n - r$ actions I_i are adiabatic invariants [46]. Then, for each $i = (r+1), \dots, n$,

$$\langle \dot{J}|g^{(i)}\rangle = \det\hat{Q}^{-1}\left\{\sum_{j=1}^{r}\langle\dot{I}|e^{(j)}\rangle\underbrace{\langle r^{(j)}|g^{(i-r)}\rangle}_{0} + \sum_{j=r+1}^{n}\underbrace{\langle\dot{I}|e^{(j)}\rangle}_{0}\langle g^{(j-r)}|g^{(i-r)}\rangle\right\} = 0, \tag{13}$$

which yields the desired integrals (cf. Refs. [45,47–50]):

$$\langle J|g^{(i)}\rangle = \text{const}, \quad i = 1, \dots, (n-r). \tag{14}$$

Eqs. (14) show that the projection of $|J\rangle$ on the $\hat{R}$ null space is conserved, in a vector form reading

$$\hat{P}_{\text{ker}}|J\rangle = \text{const}, \tag{15}$$

where $\hat{P}_{\text{ker}}$ is the operator projecting on $\ker\hat{R}$. Eq. (15) generalizes the conventional Manley–Rowe relations [4,39] as it applies to any discrete system with an arbitrary number of resonances.

Since $|\omega\rangle$ belongs to $\ker\hat{R}$, one also obtains a corollary

$$\langle\dot{J}|\omega\rangle = 0, \tag{16}$$

known as the energy conservation law; hence a quantum interpretation of the above results: *Given* Eq. (16), the change of the number of quanta $|N\rangle = \hbar^{-1}|J\rangle$ satisfies

$$\langle\Delta N|\omega\rangle = 0. \tag{17}$$

Since $|\Delta N\rangle$ is integer, Eq. (17) cannot be independent from Eq. (1), which, by definition, lists all resonant combinations of frequencies. Thus, $\langle\Delta N| = \langle\psi|\hat{R}$, where $\langle\psi|$ is some vector, meaning that $\langle\Delta N|g^{(i)}\rangle = 0$; hence Eqs. (13)–(15) are recovered.[3]

[3] For quantum mechanical derivations of the conventional Manley–Rowe relations, see Refs. [17,25–27,51–53].

Using the above results, the conventional Manley–Rowe relations [4] can be derived as a particular case corresponding to a single resonance of the form

$$\sum_{i=1}^{n} \nu_i\omega_i = 0, \tag{18}$$

with integer ν_i. The matrix $\hat{R}$, of rank 1, is written as

$$\hat{R} = \begin{pmatrix} \nu_1 & \nu_2 & \dots & \nu_n \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}, \tag{19}$$

hence Eq. (6) yields $n-1$ independent null space vectors:

$$|g^{(i)}\rangle = \sum_{j=1}^{n}(\nu_{i+1}\delta_{i,j} - \nu_i\delta_{i+1,j})|e^{(j)}\rangle. \tag{20}$$

Thus Eq. (16) gives

$$\nu_{i+1}J_i - \nu_i J_{i+1} = \text{const}, \tag{21}$$

and the latter rewrites as

$$dJ_1/\nu_1 = dJ_2/\nu_2 = \dots = dJ_n/\nu_n, \tag{22}$$

which is a form equivalent [39] to the original result by Manley and Rowe [4].

Unlike *ad hoc* techniques, the above derivation yields the integrals deductively and without specifying the interaction details. Hence the analysis remains concise also for multiple resonances, as seen in the following example. Consider a system of n oscillators with

$$\omega_1 = \omega_2 = \dots = \omega_n, \tag{23}$$

so $\hat{R}$ is upper bidiagonal:

$$\hat{R} = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & -1 \\ 0 & \cdots & \cdots & 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

As $\operatorname{rank}\hat{R} = n-1$, there is a single independent vector in $\ker\hat{R}$, particularly

$$|g^{(1)}\rangle = (1, 1, \dots, 1)^{\mathrm{T}}. \tag{25}$$

Therefore one integral is obtained from Eqs. (14), reading

$$\sum_{i=1}^{n} J_i = \text{const}, \tag{26}$$

that is, conserved is only the total number of quanta. A comparison of this analysis with an *ad hoc* derivation such as that in Ref. [44] illustrates the power of Eqs. (14), (15) in application to multi-resonance classical systems ranging, in general, from electrical circuits [3,6] to particle traps, rf-heated mirror plasmas, or Rydberg atoms and molecules in laser fields [44].

In summary, we point out that Manley–Rowe relations are derived from the first principles of Hamiltonian mechanics via the formalism of Ref. [45], offering a compact vector form of the integrals for any dynamical system with an arbitrary number of resonances. Assuming that the resonances are defined as $\hat{R}|\omega\rangle = 0$, where $\hat{R}$ is an $n \times n$ integer matrix of rank $r < n$, and $|\omega\rangle \equiv (\omega_1, \dots, \omega_n)^{\mathrm{T}}$ is the frequency vector, the projection of the action vector $|J\rangle$ on $\ker\hat{R}$ is an adiabatic invariant; hence $n-r$ independent integrals. We suggest a quantum mechanical interpretation of this result and show how conventional Manley–Rowe relations are yielded for a single resonance. We also consider a sample system with multiple resonances to illustrate how the general formalism can yield conservation laws concisely as compared with *ad hoc* techniques.

Acknowledgements

This work was supported by DOE Contract No. DEFG02-05ER54838 and DE-AC02-76-CH03073, and by the NNSA under the SSAA Program through DOE Research Grant No. DE-FG52-04NA00139.

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