Charged particle acceleration in dense plasma channels

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Reduced nonlinear equations are derived from the oscillation amplitude and the energy of a charged particle accelerated in a plasma channel. The maximum energy gain, as limited by dissipation, is described by three different scalings depending on the channel parameters. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2988772]

I. INTRODUCTION

A. Motivation

As opposed to conventional linacs, which employ radio-frequency fields limited to roughly 100 MV/m, plasma-based accelerators can operate at much higher magnitudes as they utilize electrostatic fields due to charge separation in dense media.[1] With typical electron densities of 10^{18} cm^{-3}, the accelerating gradients in contemporary experiments reach 1 GV/cm, allowing for inexpensive generation of high-quality electron beams in the GeV energy range.[2–5] However, with solid-state densities, even higher, TV/cm fields are, in principle, possibly offering access to the yet unattainable TeV domain.

To propel particles inside a dense plasma, an x-ray laser will be necessary,[6] whereas efficient acceleration might require relativistic intensities, exceeding those conceivable for x rays as of today.[7] However, given the prospective applications, particularly in high-energy physics,[8] there are benefits to utilizing novel acceleration techniques,[7] and they are attracting heightened attention due to the recent promises in laser technology.[9]

The new effects at higher densities are due to increased scattering rates, which result in greater collisional energy losses and faster pitch-angle diffusion. The latter leads to particles escaping from the driving field; thus channeling is required for efficient acceleration and, in turn, brings in additional radiative dissipation.

In what follows, we study the maximum energy, which a channeled particle can attain, by exploring the possible acceleration regimes depending on the plasma parameters. Having the emphasis on electrostatic confinement at solid-state densities, our analysis also applies when channeling is due to magnetic fields (e.g., in muon cooling experiments[10]) or a nonlinear action of optical[11,12] or particle beams in dilute plasmas.[13–16]

B. Historical background

The influence of a preformed channel on charged particle acceleration in plasma was discussed in a number of contexts, including beam evolution due to multiple scattering in beat-wave[17,18] and wakefield accelerators in the blowout regime,[16] radiative losses and particle dynamics in the wake field at laser-plasma interactions in the so-called bubble regime,[12] and others.[1] Channeling with nanotubes was also discussed in Refs. 19 and 20, and a related problem of acceleration in capillaries where high near-wall densities could produce large longitudinal fields was addressed in Refs. 7 and 21–23.

Another proposal was offered in Refs. 24–27, most comprehensively reviewed in Ref. 28. It was suggested that particles are accelerated in solids along major crystallographic directions, which provide a channeling effect[29–33] in combination with low emittance determined by an Ångström-scale aperture of the atomic "tubes." Positively charged particles are channeled more robustly, as they are repelled from ions and thus experience weaker scattering. Assuming that radiation emission due to betatron oscillations[34] is the major source of dissipation, the maximum energies are limited to, roughly, 300 GeV for positrons and 10^6 TeV for protons. The corresponding accelerating gradients are estimated at 100 GeV/cm level feasible at electron densities of the order of 10^{22} cm^{-3} and could possibly be produced in an x-ray wake or via resonant excitation of an acoustic wave.[28]

Similar ideas were discussed in Ref. 35, where it was suggested that crystal channels can act as waveguides for x-ray pulses driving the acceleration. The natural rippling of these waveguides on the atomic scale is beneficial, too, because it supplies the pulses with a longitudinal component, which can accelerate particles directly.

Accelerated beam dynamics in dense plasma channels was also addressed in Refs. 36–38, and a comprehensive treatment was attempted in Ref. 39. As in Ref. 28, linear energy gain was assumed; on the other hand, the beam kinetics was explored for the first time, simultaneously taking into account longitudinal forces, transverse cooling, and multiple scattering.

The scalings yielded by Ref. 39 are at variance with those from Ref. 28; thus the correct particle distribution and the relative impact of competing effects (transverse cooling, multiple scattering, bremsstrahlung, radiative dissipation) are yet to be calculated. The present paper is intended, in part, to resolve these issues and also to offer, for the first time, a nonlinear treatment of the channeled beam acceleration in dense plasmas.

C. Outline

The purpose of the paper is twofold. First, we derive systematically and resolve discrepancies in the previously obtained results pertaining to charged particle acceleration in plasma channels. Second, we identify the parameters that determine the theoretical maximum for the particle energy gain, as limited by dissipation,[40] and find distinct regimes differing in scalings for the maximum energy. Through that, we answer the open questions addressed in Refs. 28 and 39 and provide a uniform treatment of the acceleration problem in general; hence our analysis applies to arbitrary plasma channels and, as a spin-off, yields known results for lower-density plasmas within a unified theoretical framework.

The paper is organized as follows: In Sec. II, we derive general motion equations for a particle traveling in a plasma channel, accounting for an accelerating force, nonlinear radiative dissipation, linear transverse cooling, longitudinal friction, and multiple scattering. In Sec. III, we assess the major processes affecting the particle dynamics and specify the plasma parameters entering the motion equations. In Sec. IV, we normalize these equations accordingly. In Sec. V, we solve for the particle motion under the approximations of linear acceleration and negligible dissipation used in Refs. 16–18, 28, and 39. We derive the bivariate distribution of the particle transverse coordinate and velocity and show that our results support those of Refs. 16–18 and 28 but are in variance with Ref. 39, which is found to have omitted important effects. In Sec. VI, we obtain reduced nonlinear equations for the particle oscillation amplitude and energy and prove them to yield those in Refs. 10, 12, 16–18, 28, and 41–43 as particular cases, including the equation for the normalized beam emittance.

In Sec. VII, we identify the parameters that determine the maximum energy gain $\mathcal{E}_{\max}$. We find that, in principle, three acceleration regimes are possible depending on these parameters. In each regime, particles can be channeled indefinitely given a sufficiently strong accelerating force, and larger maximum energies are expected in wider channels. For practical applications, the most realistic regime is where $\mathcal{E}_{\max}$ is limited by the nonlinear radiative dissipation, whereas the transverse cooling and bremsstrahlung remain insignificant. This supports the estimate of Ref. 28 for particles channeled in crystals, yielding $\mathcal{E}_{\max}\sim 300$ GeV for positrons, assuming a feasible 100 GeV/cm accelerating gradient.

In Sec. VIII, we summarize our main ideas. Supplementary material pertaining to the definition of the beam emittance is given in the Appendix.

II. MOTION EQUATIONS

Consider the particle motion equation,

$$\frac{dp_i}{d\tau}=\mathcal{F}_i+\mathcal{G}_i, \tag{1}$$

where $p_i=(\mathbf{p},\mathcal{E}/c)$ is the particle four-momentum, $\mathbf{p}=m\mathbf{v}\gamma$ is the kinetic momentum, $\mathcal{E}=mc\gamma$ is the energy, $\mathbf{v}$ is the velocity, $\gamma=(1-v^2/c^2)^{-1/2}$, c is the speed of light, τ is the proper time, $\mathcal{F}_i$ is the external four-force,

$$\mathcal{G}_i=\frac{2e^2}{3mc^3}\left[\frac{d^2p_i}{d\tau^2}-\frac{p_i}{m^2c^2}\left(\frac{dp_j}{d\tau}\frac{dp_j}{d\tau}\right)\right] \tag{2}$$

is the radiation reaction force (see Ref. 44, Sec. 17), and m and e are the particle mass and charge. (For brevity, we assume $-|e|$ equal to the electron charge.) Rewrite Eq. (1) as

$$\frac{d\mathbf{p}}{dt}=\mathbf{F}+\mathbf{G}_{\mathbf{p}},\quad \frac{d\gamma}{dt}=\frac{\mathbf{F}\cdot\mathbf{v}}{mc^2}+G_\gamma, \tag{3}$$

where t is time $(dt=\gamma d\tau)$, $\mathbf{F}$ is the three-vector component of $\mathcal{F}_i$, and

$$\mathbf{G}_{\mathbf{p}}=\frac{2e^2}{3mc^3}\left[\frac{d}{dt}\left(\gamma\frac{d\mathbf{p}}{dt}\right)-\frac{\gamma^2\mathbf{v}}{mc^2}Q\right], \tag{4}$$

$$G_\gamma=\frac{2e^2}{3mc^3}\left[\frac{d}{dt}\left(\gamma\frac{d\gamma}{dt}\right)-\frac{\gamma^2}{m^2c^2}Q\right], \tag{5}$$

$$Q=\left(\frac{d\mathbf{p}}{dt}\right)^2-\left(\frac{1}{c}\frac{d\mathcal{E}}{dt}\right)^2. \tag{6}$$

Consider the radiation reaction as a perturbation, so $\mathbf{G}_{\mathbf{p}}$ and G_γ are evaluated assuming $\dot{\mathbf{p}}=\mathbf{F}$ (Ref. 45, Sec. 76); then

$$m\gamma\frac{d\mathbf{v}}{dt}=\mathbf{F}-\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{F})+\frac{2e^2}{3mc^3}\left\{\frac{1}{mc}\mathbf{F}\times(\mathbf{F}\times\boldsymbol{\beta})+\gamma[\dot{\mathbf{F}}-\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\dot{\mathbf{F}})]\right\}, \tag{7}$$

$$\frac{d\gamma}{dt}=\frac{\mathbf{F}\cdot\boldsymbol{\beta}}{mc}+\frac{2e^2}{3m^3c^5}\{\mathbf{p}\cdot\dot{\mathbf{F}}+\gamma^2[(\mathbf{F}\cdot\boldsymbol{\beta})^2-F^2\beta^2)]\}, \tag{8}$$

where $\beta=\mathbf{v}/c$. Suppose one-dimensional (1D) transverse motion in x for a particle channeled along the z axis.[46] Approximate $\mathbf{F}$ with $\mathfrak{F}\equiv-\dot{\mathbf{x}}U'(x)+\dot{\mathbf{z}}F_z$ in the curly brackets [Eqs. (7) and (8)], where U is the channeling potential, and F_z is a slowly varying longitudinal force,[47] so $\dot{\mathbf{F}}\approx-\dot{\mathbf{x}}\dot{x}U''(x)$. Thus,

$$m\gamma\ddot{x}+\dot{x}\left[\frac{\beta_zF_z}{c}\left(1+\frac{2F_zr_c}{3mc^2\beta_z}\right)+\frac{2e^2\gamma}{3mc^3}(1-\beta_x^2)U''(x)\right]-F_x\left(1+\frac{2F_zr_c\beta_z}{3mc^2}-\beta_x^2\right)=0, \tag{9}$$

$$\dot{\gamma}=\frac{F_x\beta_x+F_z\beta_z}{mc}-\frac{2e^2}{3m^3c^5}\{\gamma mv_x^2U''(x)+\gamma^2[U'(x)\beta_z+F_z\beta_x]^2\}, \tag{10}$$

where $r_c=e^2/mc^2\approx 2.82(m_e/m)\times 10^{-13}$ cm is the particle classical radius, m_e is the electron mass.

Suppose $\beta_z\approx 1$ (and thus $\gamma\gg 1$) and assume the channel height $U_{\max}$ is small compared to $\mathcal{E}$, so $\beta_x\sim\sqrt{U_{\max}/\mathcal{E}}\ll 1$. (For a particle channeled in a crystal, $U_{\max}/\mathcal{E}\sim 10^{-5}m_e/\gamma m$.) Then one can further assume that $\beta_x\ll F_x/F_z$ (which is true in crystals) because, normally, F_x and F_z are both due to charge separation, so their ratio should not be *too* small. Also,

$$\frac{\gamma mv_x^2U''}{\gamma^2U'^2}\sim\frac{1}{\gamma^2}\ll 1, \tag{11}$$

and $F_zr_c/mc^2\ll 1$ for conceivable F_z. Hence, Eqs. (9) and (10) are simplified as follows:

$$m\gamma\ddot{x}+\dot{x}\left[\frac{F_z}{c}+\frac{2e^2}{3mc^3}\gamma U''(x)\right]-F_x=0, \tag{12}$$

$$\dot{\gamma}=\frac{F_z}{mc}+\frac{F_x\dot{x}}{mc^2}-\frac{2e^2}{3m^3c^5}[\gamma U'(x)]^2. \tag{13}$$

To model multiple collisions, introduce an average friction superimposed on a Langevin force causing pitch-angle scattering. At small $\theta\equiv\angle(\mathbf{p},\hat{\mathbf{z}})\approx\beta_x$, the latter is in the x direction; hence $F_x=\mathfrak{F}_x+F_s$, where $F_s=-\nu_x p_x+mc\gamma\dot{\theta}_s$, and θ_s stands for the stochastic variation of θ; also, $F_z=F_a-\nu_z p_z$, where F_a is the accelerating force, and $p_z\approx mc\gamma$. For the effective friction coefficients, assume $\nu_x\lesssim\nu_z$, so the contribution of the x-friction to Eq. (13) is at least β_x^2 smaller than due to that in the z direction and hence will be neglected. Also, the rms energy variation due to pitch-angle scattering after time t is $\delta\gamma\sim\gamma\beta_x\sqrt{\nu_\theta t}$, where ν_θ is the scattering rate. Thus, on the time scale of interest,

$$\delta\gamma/\gamma\lesssim\beta_x\sqrt{\nu_\theta/\nu_z}\ll 1 \tag{14}$$

(Sec. III), so the whole contribution of F_s is negligible in Eq. (13). With a parabolic approximation for the channeling potential, $U=\frac{1}{2}\kappa x^2$,[8,29] one gets then

$$\ddot{x}+\dot{x}\left[\frac{F_a}{mc\gamma}-\nu_z+(\eta_r+\nu_x)\right]+\frac{\kappa}{m\gamma}x=c\dot{\theta}_s, \tag{15}$$

$$\dot{\gamma}=\frac{F_a}{mc}-\nu_z\gamma-\frac{\kappa}{mc^2}x\dot{x}-\eta_r\gamma^2\frac{\kappa x^2}{mc^2}, \tag{16}$$

where $\eta_r=2\kappa r_c/3mc$ is the radiative cooling rate.[48]

Equations (15) and (16) generalize those obtained in Refs. 16, 43, and 49–52 for particular channeling regimes. The third term in Eq. (16), omitted in the cited papers, is of minor importance for it does not affect the particle energy gain (Sec. VI). On the other hand, the first two terms in the square brackets of Eq. (15) are also missing in Ref. 39; hence the latter will not predict the acceleration correctly.

III. PARAMETERS OF MOTION

In this section, we specify the heuristically introduced quantities F_a, κ, $\dot{\theta}_s$, ν_z, ν_x entering Eqs. (15) and (16).

A. Accelerating force

For simplicity, assume a constant accelerating force $F_a=eE_a$ due to charge separation, so the electrostatic field E_a can be estimated from $\nabla\cdot\mathbf{E}_a=-4\pi e\delta n_e$, where δn_e is the perturbation of the electron density n_e. The characteristic gradient scales like $\nabla\sim\omega_p/c$, $\omega_p=\sqrt{4\pi n_e e^2/m_e}$ being the plasma frequency. Then the largest E_a, corresponding to $\delta n_e\sim n_e$, equals $E_{\max}\sim\sqrt{4\pi m_e c^2 n_e}$, or

$$E_{\mathrm{V/cm}}\approx 0.96 n_{\mathrm{cm}^{-3}}^{1/2} \tag{17}$$

(here the subindexes denotes the measuring units), yielding $F_a\sim 100$ GeV/cm for $n_e=10^{22}$ cm^{-3}.

B. Focusing strength

Like F_a, the channeling potential U would normally be due to charge separation as well. In this case, $U\propto x^2$ corresponds to a uniform charge density ρ; hence $\kappa\sim\bar{\kappa}$, where $\bar{\kappa}=4\pi n_e e^2$ is the focusing strength of a planar channel with $\rho=en_e$.[28]

For channeling in a crystal, the characteristic value is $\bar{\kappa}\sim 20$ eV/Å^2;[28] thus, $\Omega\sim 10^{16}$ s$^{-1}\times\sqrt{m_e/\gamma m}$. Assuming that the transverse energy $\frac{1}{2}\gamma m v_x^2$ is of the order of $U_{\max}=\frac{1}{2}\kappa R^2$, where $R\sim 1$ Å is the channel width, particles will occupy highly excited states, because

$$U_{\max}/\hbar\Omega\sim\sqrt{\gamma m/m_e}\gg 1. \tag{18}$$

Hence classical treatment of the transverse motion is justified (Ref. 33, Sec. 9.3; Ref. 32, Sec. I C).

We now check if the radiation reaction force can be treated classically, too. The latter requires that the energy of radiated photons $\hbar\omega$ is small compared to that of a particle. To calculate the characteristic photon frequency ω, consider the reference frame K' (further denoted by prime) where the particle average velocity is zero, so ω is determined by the energy variation $\delta\gamma'$. Since p_z is not affected by the transverse motion, the Lorenz transformation yields $\delta\gamma'=\gamma_V\delta\gamma$, where $\gamma_V=(1-V^2/c^2)^{-1/2}$, V is the velocity of K' with respect to the laboratory frame K, and $\delta\gamma\sim U/mc^2$; thus,

$$\delta\gamma'\sim\frac{\gamma}{\hat{\gamma}},\quad\hat{\gamma}=\frac{mc^2}{\kappa R^2}. \tag{19}$$

In the so-called undulator regime, when $\gamma\ll\hat{\gamma}$, the oscillations in K' are nonrelativistic; thus the particle radiates at the fundamental harmonic only, $\omega'=\Omega'$. The largest frequency in K then equals $\omega\approx 2\gamma^2\Omega$, yielding[51]

$$\frac{\hbar\omega}{\mathcal{E}}\sim\frac{\lambda\!\!\!^{-}}{R}\left(\frac{m_e}{m}\right)\sqrt{\frac{\gamma}{\hat{\gamma}}}<\frac{\lambda\!\!\!^{-}}{R}\ll 1, \tag{20}$$

so the classical approximation holds. (Here $\lambda\!\!\!^{-}=\hbar/m_e c\approx 3.86\times 10^{-11}$ cm is the Compton wavelength.)

In the "wiggler" regime, when $\gamma\gg\hat{\gamma}$, the motion in K' is ultrarelativistic, so a particle radiates a continuous spectrum with the cutoff frequency $\omega_c\sim\gamma^3 c/\mathcal{R}$, $\mathcal{R}$ being the trajectory curvature radius (Ref. 44, p. 485). Use $x\approx r\sin(\int^z d\tilde{z}/\mathfrak{B})$, where r is a constant amplitude, and $\mathfrak{B}\approx c/\Omega$ is the betatron function; hence $\mathcal{R}=(c/\Omega)^2/r$,[18] and $\omega_c\sim\gamma^3 r\Omega^2/c$ (cf. Refs. 12 and 53). Then, to treat the radiation recoil classically, one must have

$$\gamma\ll\hat{\gamma}^2\frac{R}{\lambda\!\!\!^{-}}\left(\frac{m}{m_e}\right), \tag{21}$$

where we used $r\sim R$. For crystal channels, Eq. (21) reads $\mathcal{E}\ll 3$ TeV for electrons and positrons, and $\mathcal{E}\ll 2\times 10^{10}$ TeV for protons. Both limits exceed the energies of interest; thus the quantum effects are negligible.

C. Pitch-angle scattering

Assume a Langevin force $\dot{\theta}_s$ due to the particle multiple scattering inside the channel, such that

$$\langle \dot{\theta}_s(t_1)\,\dot{\theta}_s(t_2)\rangle = 2\nu_\theta \delta(t_1 - t_2), \quad (22)$$

where the angle brackets denote ensemble averaging, and

$$\nu_\theta = \frac{1}{2}\frac{d\langle \theta_s^2\rangle}{dt} \quad (23)$$

is the pitch-angle scattering rate. The latter is a combination of $\nu_{\theta 1}$ due to the momentum transfer to scattering centers and $\nu_{\theta 2}$ due to photon recoil. Yet $\nu_{\theta 2} \ll \nu_{\theta 1}$ (Sec. III E); thus $\nu_\theta \approx \nu_{\theta 1}$, which is calculated as follows.

Use $\nu_{\theta 1} = n_s v \sigma_\theta$, where n_s is the scattering center density, and σ_θ is the collision cross section, $\sigma_\theta \sim \pi b^2$. The effective interaction scale b equals the distance r at which the deflection angle $\Delta\theta \approx \Delta p/p$ becomes of the order of unity. Use $\Delta p \sim F_C \Delta t$, where $F_C \sim Z_s e^2/r^2$ is the Coulomb force, $Z_s e$ is the charge of the scattering center, and $\Delta t \sim r/v$ is the characteristic interaction time. Hence $\Delta\theta \sim Z_s e^2/rpv$, and $b \sim Z_s e^2/pv$; thus, $\nu_{\theta 1} \sim \pi n_s Z_s^2 e^4/vp^2$.[54] A detailed derivation shows the presence of an additional logarithmic factor due to the Coulomb scattering being a long-range interaction,

$$\nu_{\theta 1} = \frac{4\pi n_s Z_s^2 e^4 \Lambda_\theta}{vp^2}, \quad (24)$$

where $\Lambda_\theta \sim 10$ both for amorphous plasmas[17,18,55,56] and crystal channels (Ref. 32, Sec. 1.4, Refs. 49, 50, 52, and 57).

At $\gamma \gg 1$, Eq. (24) yields $\nu_{\theta 1} = \bar{\nu}_\theta/\gamma^2$, where $\bar{\nu}_\theta \approx \text{const}$ is determined by scattering centers. Since ions have higher Z_s as compared to electrons, scattering on ions usually dominates; yet sometimes the electron contribution can prevail as well. For example, a positively charged particle channeled in a crystal avoid ion collisions;[8,29] hence scattering is determined by electrons only, and

$$n_s Z_s^2 = n_e. \quad (25)$$

Therefore, $\bar{\nu}_\theta$ is decreased as compared to the amorphous medium[57] (Ref. 32, Sec. 1.4 and Ref. 33, Sec. 10).

We now check if the effective pitch-angle scattering rate is small compared to the transverse oscillation frequency, as required for the derivation of Eqs. (15) and (16). The effective rate ν_{eff} is defined as the inverse time after which the particle gets scattered on an angle comparable to $p_x/p_z \approx \beta_x$. Thus, $\nu_{\text{eff}} \sim \nu_{\theta 1}/\beta_x^2$, or, roughly,

$$\left(\frac{\nu_{\text{eff}}}{\Omega}\right)^2 \sim \frac{1}{\gamma}\left(\frac{Z_s}{n_e R^3}\right)\left(\frac{Z_s r_c}{R}\right), \quad (26)$$

where we used $\beta_x^2 \sim U_{\max}/\mathcal{E}$. Since $R^3 \gtrsim n_i^{-1}$, and $n_e \geqslant Z_s n_i$ (where n_i is the ion density), one has $\nu_{\text{eff}}/\Omega \lesssim \sqrt{Z_s r_c/\gamma R} \ll 1$. Hence treating scattering as perturbation is justified.

D. Friction force

The friction rates ν_x and ν_z are determined by the energy transfer to cold particles and bremsstrahlung, with the corresponding partial rates estimated as follows.

In the former case, the friction is isotropic, i.e., $\nu_{x1} = \nu_{1z}$. A particle transfers momentum Δp each time it collides with a scattering center; hence the associated energy loss reads $\Delta\mathcal{E} = (\Delta p)^2/2m_s$. From Eq. (23), one has $(\Delta p)^2/\Delta t \sim \nu_{\theta 1} p^2$; therefore, $\langle d\mathcal{E}/dt\rangle \sim -\nu_{\theta 1} p^2/m_s$, or (Ref. 32, Sec. 1.4 and Ref. 44, Sec. 13.1),[30,31,43,58]

$$\nu_{x1} = \nu_{z1} \sim (\gamma m/m_s)\nu_{\theta 1}. \quad (27)$$

This shows that the energy dissipation due to scattering on electrons is, in any case, faster than that on ions; hence the corresponding ion contribution will be neglected.[59]

In contrast, bremsstrahlung is anisotropic and can be determined by either species s. The momentum decays due to being carried away by photons, and those are emitted within a typical opening angle $\theta_{\text{ph}} \sim 1/\gamma$ (Ref. 44, Sec. 14.3); thus, $\nu_{x2} \sim \nu_{z2}/\gamma$, and ν_{z2} depends on the channel type. In an amorphous channel, the major effect is due to ions because of their higher charge.[8,56,60–63] In a crystal channel though, collisions with ions can be suppressed (Sec. III C); hence the effect becomes determined by electrons.[35,49,50,52,58,64] Assuming Eq. (25), the dissipation rate can be expressed in a general form, $\nu_{z2} = c/X_0$, where X_0 is an approximately constant[65] radiation length given by

$$X_0^{-1} = 4\alpha n_s Z_s^2 r_c^2 \Lambda_b, \quad (28)$$

where $\alpha = e^2/\hbar c \approx \frac{1}{137}$ is the fine-structure constant and Λ_b is a logarithmic factor of the order of 5.

For the ratio of the dissipation rates, one has

$$\frac{\nu_{x1}}{\nu_{x2}} \gtrsim \frac{1}{\alpha} \gg 1, \quad \frac{\nu_{z1}}{\nu_{z2}} \sim \frac{\pi\Lambda_\theta}{Z_s\Lambda_b}\left(\frac{m}{m_e}\right)\frac{1}{\alpha\gamma}. \quad (29)$$

Thus bremsstrahlung is insignificant as a transverse friction (so $\nu_x \approx \nu_{x1}$); yet, as compared to the energy transfer to cold particles, it has a larger longitudinal effect for electrons and positrons at energies above tens of MeV,[61] and for protons at energies above tens of TeV. Also,

$$\frac{\nu_{x1}}{\eta_r} \sim \left(\frac{m}{m_e}\right)\frac{\Lambda_\theta}{\gamma}; \quad (30)$$

hence, at $\mathcal{E}$ exceeding about the same threshold, ν_{x1} can be neglected as compared to the radiative cooling η_r.

E. Photon recoil

Now we can also estimate the bremsstrahlung contribution to pitch-angle scattering. At each collision, a particle radiates energy $\Delta\mathcal{E} \sim (\Delta z/X_0)\mathcal{E}$, where Δz is about the Wigner–Seitz radius, or the characteristic distance between scattering centers. The associated change of the transverse momentum is then $\Delta p_\perp \sim \theta_{\text{ph}}\Delta\mathcal{E}/c$. With $\theta_{\text{ph}} \sim 1/\gamma$ and $\Delta\theta \approx \Delta p_\perp/p$, this allows writing the scattering rate $\nu_{\theta 2} \sim c\Delta\theta^2/\Delta z$ as

$$\nu_{\theta 2} \sim \frac{c n_s^{-1/3}}{X_0^2 \gamma^2}. \quad (31)$$

Comparing $\nu_{\theta 2}$ with $\nu_{\theta 1}$, we get

$$\frac{\nu_{\theta 2}}{\nu_{\theta 1}} \sim (\alpha Z_s)^2 (n_s r_c^3)^{2/3}, \quad (32)$$

which is several orders of magnitude less than unity. Hence the contribution of photon recoil to pitch-angle scattering is negligible.[26,51]

IV. NORMALIZED MOTION EQUATIONS

We now can put the motion equations (15) and (16) in a dimensionless form as follows. Introduce new variables

$$\xi = x/\bar{x}, \quad \zeta = t/\bar{t}, \quad \Gamma = \gamma/\bar{\gamma}, \quad (33)$$

where the normalization constants are chosen such that

$$\bar{\gamma}(\eta_r \bar{t})\left(\frac{\kappa \bar{x}^2}{mc^2}\right) = \frac{\kappa \bar{t}^2}{m\bar{\gamma}} = \frac{F_a X_0}{mc^2 \bar{\gamma}} = 1, \quad (34)$$

or, explicitly,

$$\bar{x} = r_c \sqrt{\frac{3}{2}} \left(\frac{Z_s \Lambda_b}{\pi \kappa_* \mathfrak{E}}\right)^{3/4}, \quad (35)$$

$$\bar{t} = \frac{r_c}{4c\mathcal{N}} \sqrt{\frac{\mathfrak{E}}{\pi Z_s \kappa_* \Lambda_b}}, \quad (36)$$

$$\bar{\gamma} = \frac{\mathfrak{E}}{4\mathcal{N} Z_s \Lambda_b}, \quad (37)$$

where $\kappa_* \equiv \kappa/\bar{\kappa}$, $\mathcal{N} \equiv n_e r_c^3$, $\mathfrak{E} \equiv F_a/eE_S$, and $E_S \equiv m^2c^3/e\hbar$, which, for electrons, equals the Schwinger field, $E_S \approx 1.32 \times 10^{16}$ V/cm.[8] Hence the new equations read

$$\xi'' + \left[\frac{\nu}{\Gamma} - (\nu + \lambda) + 2\eta\right]\xi' + \frac{\xi}{\Gamma} = \chi, \quad (38)$$

$$\Gamma' = \nu - (\nu + \lambda)\Gamma - \varsigma \xi \xi' - \xi^2 \Gamma^2, \quad (39)$$

where the prime denotes derivative with respect to ζ. The new dimensionless parameters can be written as

$$\nu = \nu_{z2}\bar{t}, \quad \lambda = \nu_{z1}\bar{t}, \quad \eta = \frac{1}{2}(\eta_r + \nu_x)\bar{t}, \quad \varsigma = \frac{\kappa \bar{x}^2}{mc^2 \bar{\gamma}},$$

and the stochastic force χ is normalized such that

$$\langle \chi(\zeta_1)\chi(\zeta_2)\rangle = \frac{2D}{\Gamma^2}\delta(\zeta_1 - \zeta_2), \quad D = \frac{\bar{\nu}_\theta \bar{t}}{\bar{\gamma}^2}\left(\frac{c\bar{t}}{\bar{x}}\right)^2. \quad (40)$$

V. LINEAR ENERGY GAIN

At the initial acceleration stage, the energy gain is approximately linear,

$$\Gamma = \nu\zeta + \Gamma_0. \quad (41)$$

Hence, assuming that the transverse cooling is negligible, one can obtain an exact solution for $\xi(\zeta)$, which is done as follows. Equation (38) is simplified so as to read

$$\xi'' + \frac{\nu \xi'}{\Gamma(\zeta)} + \frac{\xi}{\Gamma(\zeta)} = \chi, \quad (42)$$

yielding

$$\xi = h_1 \psi_1(\zeta) + h_2 \psi_2(\zeta) + \xi_s(\zeta). \quad (43)$$

Here ψ_i are the fundamental set of solutions,

$$\psi_1(\zeta) = J_0[u(\zeta)], \quad \psi_2(\zeta) = Y_0[u(\zeta)], \quad (44)$$

J_0 and Y_0 are the zeroth-order Bessel functions of the first and second kind, $u = 2\sqrt{\Gamma(\zeta)}/\nu$, h_i are given by

$$h_1 = \frac{\xi_0 \psi_{20}' - \xi_0' \psi_{20}}{W(\zeta)}, \quad h_2 = -\frac{\xi_0 \psi_{10}' - \xi_0' \psi_{10}}{W(\zeta)}, \quad (45)$$

the index 0 refers to $\zeta = 0$, $W(\zeta) = \nu/\pi\Gamma(\zeta)$ is the Wronskian, and the forced term equals

$$\xi_s = \int_0^\zeta \frac{\chi(\tilde{\zeta})}{W(\tilde{\zeta})}[\psi_1(\tilde{\zeta})\psi_2(\zeta) - \psi_1(\zeta)\psi_2(\tilde{\zeta})]d\tilde{\zeta}. \quad (46)$$

The bivariate probability distribution of ξ_s and ξ_s' is Gaussian (see Ref. 66, p. 26).[67] Equation (46) yields

$$\langle \xi_s^2 \rangle = \frac{2D}{\nu}\left\{1 - \frac{\pi^2 u_0^2}{4}[w^2 + (\partial_{u_0} w)^2]\right\}, \quad (47)$$

$$\langle \xi_s \xi_s' \rangle = -\frac{D}{4\Gamma}\pi^2 u_0^2 u[w\partial_u w + \partial_{u_0} w \partial_{uu_0}^2 w], \quad (48)$$

$$\langle \xi_s'^2 \rangle = \frac{2D}{\nu\Gamma}\left\{1 - \frac{\pi^2 u_0^2}{4}[(\partial_u w)^2 + (\partial_{uu_0}^2 w)^2]\right\}, \quad (49)$$

where $w(u, u_0) = J_0(u)Y_0(u_0) - Y_0(u)J_0(u_0)$. With Eq. (43), this allows us to find the distribution of (ξ, ξ'),

$$f(\xi, \xi'; \zeta | \xi_0, \xi_0') = \frac{1}{2\pi\sqrt{\sigma}} \times \exp\left[-\frac{1}{2\sigma}(\sigma_{22}\tilde{\xi}_s^2 - 2\sigma_{12}\tilde{\xi}_s\tilde{\xi}_s' + \sigma_{11}\tilde{\xi}_s'^2)\right], \quad (50)$$

where, on the right-hand side, we substitute

$$\tilde{\xi}_s(\xi, \xi'; \zeta | \xi_0, \xi_0') = \xi - [h_1(\xi_0, \xi_0')\psi_1(\zeta) + h_2(\xi_0, \xi_0')\psi_2(\zeta)], \quad (51)$$

and σ_{ij} are obtained from

$$\sigma_{11} = \langle \xi_s^2 \rangle, \quad \sigma_{12} = \langle \xi_s \xi_s' \rangle, \quad \sigma_{22} = \langle \xi_s'^2 \rangle, \quad (52)$$

$$\sigma = \sigma_{11}\sigma_{22} - \sigma_{12}^2 \quad (53)$$

using Eqs. (47)–(49).

Since

$$u \geqslant u_0 = \frac{2\sqrt{\mathcal{E}_0 \kappa}}{F_a} \sim \sqrt{\frac{\mathcal{E}_0}{12.5\ \mathrm{keV}}} \gg 1 \quad (54)$$

(assuming $\kappa \sim 20$ eV/Å^2 and $F_a \sim 100$ GeV/cm), one can expand the moments of $f(\xi, \xi'; \zeta | \xi_0, \xi_0')$ to obtain

$$\langle \xi_s \xi_s' \rangle = \frac{D}{2\Gamma}\left[\sqrt{\frac{\Gamma_0}{\Gamma}} - \cos 2(u - u_0) \right] \to 0. \quad (55)$$

One also gets

$$\langle \xi_s^2 \rangle \approx \Gamma \langle \xi_s'^2 \rangle \approx \frac{2D}{\nu}\left[1 - \sqrt{\frac{\Gamma_0}{\Gamma}} \right] \to \text{const}, \quad (56)$$

in agreement with Refs. 16–18 and 28, yet in contrast with Ref. 39, where the second term in Eq. (42) is missed. From Eq. (56), it follows that the oscillation rms amplitude $x_{\text{rms}} = \bar{x}\sqrt{\langle \xi_s^2 \rangle}$ saturates for $\mathcal{E} \gg \mathcal{E}_0$ at

$$x_{\text{rms}} \approx \sqrt{\frac{2 Z_s \Lambda_\theta e^2}{F_a \kappa_*}}. \quad (57)$$

Thus a sufficiently strong accelerating field, which provides $x_{\text{rms}} \ll R$, allows one to channel particles indefinitely (although the energy gain is limited, as discussed in Sec. VII). For $F_a = 100$ GeV/cm, with $\Lambda_\theta = 10$, $Z_s = 1$, and $\kappa_* = 1$, Eq. (57) yields $x_{\text{rms}} \approx 0.54$ Å; hence perpetual channeling in crystals with $R \gtrsim 1$ Å is possible.

VI. REDUCED NONLINEAR EQUATIONS

A. Basic equations

Without assuming Eq. (41), the particle dynamics is described by approximate reduced equations, which are derived as follows.[68] Consider a complex variable

$$\mathcal{Z} = \left(\xi' + \frac{i\xi}{\sqrt{\Gamma}} \right) \Gamma^{3/4}, \quad (58)$$

so as to write the motion equation as

$$\mathcal{Z}' = i\varpi \mathcal{Z} - \eta \mathcal{Z} - \tfrac{3}{16}\sqrt{\Gamma}|\mathcal{Z}|^2 \mathcal{Z} + \chi \Gamma^{3/4} + q, \quad (59)$$

$$\varpi = \frac{1}{\sqrt{\Gamma}} + \frac{\varsigma |\mathcal{Z}|^2}{16\Gamma^2}, \quad (60)$$

$$q = \frac{i\varsigma}{8\Gamma^2}\left(\mathcal{Z}^3 - |\mathcal{Z}|^2 \mathcal{Z}^* - \frac{1}{2}\mathcal{Z}^{*3} \right) + \frac{\sqrt{\Gamma}}{8}\left(\mathcal{Z}^3 + \frac{1}{2}\mathcal{Z}^{*3} \right) - \left[\frac{\nu}{\Gamma} - (\nu + \lambda) + 4\eta \right] \frac{\mathcal{Z}^*}{4}. \quad (61)$$

Further introduce

$$b = \frac{1}{\sqrt{2}} \mathcal{Z} e^{i\phi}, \quad \phi = \int_0^\zeta \varpi(\tilde{\zeta}) d\tilde{\zeta}, \quad (62)$$

so as to rewrite Eq. (59) as

$$b' = -\eta b - \frac{3}{8}\sqrt{\Gamma}|b|^2 b + \frac{e^{-i\phi}}{\sqrt{2}}(\chi \Gamma^{3/4} + q). \quad (63)$$

Except for the jitter due to the Langevin force, $q e^{-i\phi}$ represents a nonresonant oscillatory drive and thus does not affect the long-term dynamics of b. Yet the jitter is small, so the stochastic part of $q e^{-i\phi}$ is negligible, too (as compared to $\chi \Gamma^{3/4} e^{-i\phi}$), and the same applies to the term proportional to ς in Eq. (39).

Hence reduced equations are obtained,

$$b' = -\eta b - \tfrac{3}{8}\sqrt{\Gamma}|b|^2 b + \chi_n, \quad (64)$$

$$\Gamma' = \nu - (\nu + \lambda)\Gamma - |b|^2 \Gamma^{3/2}, \quad (65)$$

where the normalized force χ_n is given by

$$\chi_n = \frac{e^{-i\phi}}{\sqrt{2}} \chi \Gamma^{3/4}, \quad \langle \chi_n(\zeta_1) \chi_n^*(\zeta_2) \rangle = \frac{D}{\sqrt{\Gamma}} \delta(\zeta_1 - \zeta_2).$$

As shown above, Eqs. (64) and (65) put together and generalize those derived in Refs. 10, 12, 16–18, 28, and 41–43.

B. Radiative deceleration

In the absence of the longitudinal force ($\nu = 0$) and multiple scattering ($\lambda = 0$, $\chi_n = 0$), Eqs. (64) and (65) read

$$b' = -\eta b - \tfrac{3}{8}\sqrt{\Gamma}|b|^2 b, \quad \Gamma' = -|b|^2 \Gamma^{3/2}. \quad (66)$$

Introduce $\mu = |b|^2 e^{2\varphi}$, where $\varphi(\zeta) = \int_0^\zeta \eta(\tilde{\zeta}) d\tilde{\zeta}$, so as to rewrite those as follows:

$$\mu' = -\tfrac{3}{4}\sqrt{\Gamma}\mu^2 e^{-2\varphi}, \quad \Gamma' = -\mu \Gamma^{3/2} e^{-2\varphi}. \quad (67)$$

Hence $\mu \Gamma^{-3/4} = \text{const}$, so the solutions for Γ and the oscillation amplitude ξ_m are obtained,

$$\Gamma = \Gamma_0 \left[1 + \frac{5}{8} \xi_{m0}^2 \Gamma_0 \int_0^\zeta e^{-2\varphi(\tilde{\zeta})} d\tilde{\zeta} \right]^{-4/5}, \quad (68)$$

$$\xi_m = \xi_{m0} \left(\frac{\Gamma}{\Gamma_0} \right)^{1/8} e^{-\varphi}, \quad (69)$$

in agreement with Refs. 12, 43, and 42 (in the case of Ref. 12, see erratum).

C. Linear equation for the beam emittance

Assume now that the nonlinear term in Eq. (64) is negligible (Sec. VII). Then one can derive an equation for the ensemble-averaged $|b|^2$, which coincides with the normalized beam emittance ϵ_n (see the Appendix),

$$\langle |b|^2 \rangle = \langle \xi^2 \rangle \sqrt{\Gamma} = \epsilon_n. \quad (70)$$

Using

$$b = b_0 + \int_0^\zeta \chi_n(\tilde{\zeta}) \exp[\varphi(\tilde{\zeta}) - \varphi(\zeta)] d\tilde{\zeta} \quad (71)$$

together with Eqs. (40) and (70), one obtains the equation

$$\epsilon_n' = -2\eta \epsilon_n + \frac{D}{\sqrt{\Gamma}}, \quad (72)$$

also known from the theory of muon cooling.[10]

At negligible transverse cooling ($\eta = 0$) and multiple scattering ($D = 0$), the normalized emittance is an invariant; otherwise ϵ_n can be obtained by integrating Eq. (72) in quadratures for a given $\Gamma(\zeta)$. For the linear energy gain [Eq. (41)] at $\nu_x \gg \eta_r$, one has $\eta \approx h/\Gamma$, where $h = \text{const}$; hence

TABLE I. Possible acceleration regimes depending on the saturation mechanisms for the normalized energy $\Gamma=\mathcal{E}/mc^2\bar{\gamma}$ and the normalized beam emittance ϵ_n. Here ξ_m is the oscillation amplitude; $\nu_*=\nu/\sqrt{D}$ and $\eta_*=\eta/\sqrt{D}$ are dimensionless parameters.

	ϵ_n is saturated		
	Linear saturation of ϵ_n	Nonlinear saturation of ϵ_n	ϵ_n is not saturated
Γ is saturated by radiative dissipation	**Regime 1** $\nu_*\eta_*\ll 1,\ \eta_*\gg 1$ $\Gamma_{\max}\sim\eta\nu/D,\ \xi_{\mathrm{m}}\sim D/(\eta\sqrt{\nu})$	**Regime 2** $\nu_*\ll 1,\ \eta_*\ll 1$ $\Gamma_{\max}\sim\nu/\sqrt{D},\ \xi_m\sim\sqrt{D/\nu}$	
Γ is saturated by bremsstrahlung	**Regime 4 (inaccessible)** $\nu_*/\eta_*\ll 1,\ \nu_*\eta_*\gg 1$ $\Gamma_{\max}\sim 1,\ \xi_m\sim\sqrt{D/\eta}$	Mutually exclusive conditions	**Regime 3** $\nu_*\gg 1,\ \nu_*/\eta_*\gg 1$ $\Gamma_{\max}\sim 1,\ \xi_m\sim\sqrt{D/\nu}$
	ξ_m is limited by η, as in Eq. (76)	ξ_m is not limited by η, as in Eq. (74)	

$$\epsilon_n=\left(\epsilon_{n0}-\frac{2D\sqrt{\Gamma_0}}{2h+\nu}\right)\left(\frac{\Gamma_0}{\Gamma}\right)^{h/\nu}+\frac{2D\sqrt{\Gamma}}{2h+\nu},\tag{73}$$

in agreement with Ref. 28. Thus, at $h/\nu\ll 1$,

$$\epsilon_n=\epsilon_{n0}+\frac{2D}{\nu}(\sqrt{\Gamma}-\sqrt{\Gamma_0})\approx\frac{2D}{\nu}\sqrt{\Gamma}\tag{74}$$

(here the approximation corresponds to negligible ϵ_{n0} and Γ_0), so Eq. (56) is recovered.

For the linear energy gain [Eq. (41)] and negligible collisional deceleration ($\nu_x\ll\eta_r$, so $\eta\approx$const), the solution reads

$$\epsilon_n=\epsilon_{n0}e^{\ell_0-\ell}+D\sqrt{\frac{\pi}{2\eta\nu}}(\operatorname{erfi}\sqrt{\ell}-\operatorname{erfi}\sqrt{\ell_0})e^{-\ell},\tag{75}$$

where $\ell=2\eta\Gamma/\nu$, and $\operatorname{erfi}(x)=(2/\sqrt{\pi})\int_0^x e^{y^2}dy$ is the imaginary error function. At $\ell_0,\ell\gg 1$, one finds

$$\epsilon_n=\epsilon_{n0}e^{\ell_0-\ell}+\frac{De^{-\ell}}{\sqrt{2\nu\eta}}\left(\frac{e^{\ell}}{\sqrt{\ell}}-\frac{e^{\ell_0}}{\sqrt{\ell_0}}\right)\approx\frac{D}{2\eta\sqrt{\Gamma}}.\tag{76}$$

At $\ell_0,\ell\ll 1$, one again obtains Eq. (74) yielding Eq. (56), in agreement with Refs. 16–18, 28, and 41.

D. Nonlinear equation for the beam emittance

With the nonlinearity kept in Eq. (64), the emittance equation is derived as follows. Consider a time interval $\Delta\zeta$ on which b and Γ change little; thus, Eq. (64) yields

$$\Delta b=\left(-\eta b-\frac{3}{8}\sqrt{\Gamma}|b|^2b\right)\Delta\zeta+\int_{\Delta\zeta}\chi_n(\tilde{\zeta})d\tilde{\zeta}.\tag{77}$$

Using $\Delta\langle|b|^2\rangle=\langle|\Delta b|^2\rangle$, one then gets, with $\Delta\zeta\to 0$,

$$\epsilon_n'=-2\eta\epsilon_n-\frac{3}{4}\sqrt{\Gamma}\epsilon_n^2+\frac{D}{\sqrt{\Gamma}},\tag{78}$$

$$\Gamma'=\nu-(\nu+\lambda)\Gamma-\epsilon_n\Gamma^{3/2},\tag{79}$$

in agreement with Ref. 43.

Strictly speaking, Eqs. (78) and (79) hold only for a monoenergetic beam. However, if Γ is limited by the nonlinear radiative dissipation, then so is ϵ_n (Sec. VII). In this case, transverse oscillations randomize on the time scale Γ/Γ' [Eqs. (12) and (56)]. The latter is exactly the acceleration time scale; thus, different particles will gain different energies. This fact limits the applicability of Eqs. (78) and (79) and also makes the results of Ref. 43 approximate. Yet, Eqs. (78) and (79) are sufficient to *estimate* the maximum acceleration gain, as discussed in Sec. VII.

VII. ACCELERATION REGIMES

Consider limitations on the maximum energy gain $\Gamma_{\max}$ allowed by Eq. (79). At the extreme energies of interest, bremsstrahlung results in larger dissipation than that by the energy transfer to cold particles [$\lambda\ll\nu$; Eq. (29)], so $\Gamma_{\max}$ is limited by either bremsstrahlung or radiative dissipation. Simultaneously, ϵ_n may or may not saturate, the limiting magnitude being determined by the linear or the nonlinear term in Eq. (78), whereas the transverse cooling is primarily due to η_r [$\nu_x\ll\eta_r$; Eq. (30)], so $\eta\approx$const. Hence up to six different regimes are possible, as listed in Table I. Yet one of these regimes implies mutually exclusive conditions so it cannot be realized, and the other two yield equivalent scalings; thus, four distinct regimes remain.

The latter are characterized by the parameters

$$\nu_*=\nu/\sqrt{D},\quad \eta_*=\eta/\sqrt{D},\tag{80}$$

which determine the bremsstrahlung impact on $\Gamma_{\max}$ and the radiative cooling influence on the transverse oscillations, respectively (Table I). Equation (80) also rewrites as

$$\nu_*=\frac{\alpha\Lambda_b}{2\pi}\sqrt{\frac{6Z_s}{\kappa_*\Lambda_\theta}},\quad \eta_*=\sqrt{\frac{\kappa_*}{6Z_s\Lambda_\theta}}.\tag{81}$$

Effectively, the parameter space (of the normalized system) is one-dimensional then, because

$$\nu_*\eta_*=\frac{\alpha}{2\pi}\left(\frac{\Lambda_b}{\Lambda_\theta}\right)\equiv\varepsilon\ll 1,\tag{82}$$

where ε varies little among different channels.

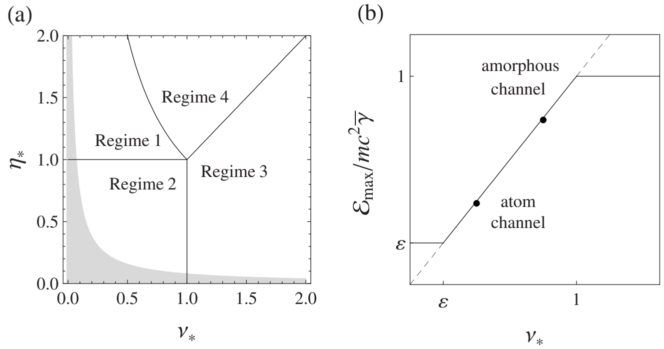

FIG. 1. (a) Acceleration regimes (Table I) in the parameter space (ν_*, η_*) [Eq. (80)]. Shaded is the realistic operation domain [Eq. (82), not to scale]. (b) Schematic dependence of $\mathcal{E}_{\max}=mc^2\bar{\gamma}\Gamma_{\max}$ [Eq. (85), not to scale] vs $\nu_* \propto \kappa_*^{-1/2}$ at fixed ε [Eq. (82)], where $\kappa_*=\kappa/\bar{\kappa}$ is the normalized focusing strength. The dashed line corresponds to $\Gamma_{\max}=\nu_*$; the circles denote the operating points for (i) a positive particle channeled in a crystal and (ii) a particle in an amorphous channel.

Since $\varepsilon \sim \alpha/8\pi \approx 3\times 10^{-4}$ (where we substituted $\Lambda_\theta/\Lambda_b \approx 4$, as for an amorphous channel[56,65,69]), one of the four regimes is inaccessible [Fig. 1(a)]. For the three other regimes, the oscillation amplitude x_m reads

$$x_m \sim \bar{x}\sqrt{\frac{D}{\nu}} \times \begin{cases} \eta_*^{-1}, & \text{if } \eta_* > 1 \quad \text{(regime 1)}, \\ 1, & \text{if } \eta_* < 1 \quad \text{(regimes 2,3)} \end{cases} \tag{83}$$

[cf. Eq. (56)], so

$$x_m^{(1)} < x_m^{(2)} \sim x_m^{(3)}. \tag{84}$$

Thus x_m is limited from above by that flowing from Eq. (57); hence, given a large accelerating force, particles can be channeled indefinitely, as described in Sec. V.

The maximum energy $\mathcal{E}_{\max}=mc^2\bar{\gamma}\Gamma_{\max}$ is given by

$$\mathcal{E}_{\max} \sim mc^2\bar{\gamma} \times \begin{cases} \varepsilon, & \text{if } \nu_* < \varepsilon \quad \text{(regime 1)}, \\ \nu_*, & \text{if } \varepsilon < \nu_* < 1 \quad \text{(regime 2)}, \\ 1, & \text{if } \nu_* > 1 \quad \text{(regime 3)} \end{cases} \tag{85}$$

[Fig. 1(b)]; therefore, $\mathcal{E}_{\max}$ is larger in wider channels corresponding to smaller $\kappa_* \propto \nu_*^{-2}$,

$$\mathcal{E}_{\max}^{(1)} < \mathcal{E}_{\max}^{(2)} < \mathcal{E}_{\max}^{(3)}. \tag{86}$$

For positively charged particles channeled in a crystal, one has $\nu_* \sim 0.003$, $\eta_* \sim 0.1$, assuming $Z_s=1$ (i.e., no scattering on ions), $\Lambda_\theta=4\Lambda_b$, $\Lambda_b=4$, $\kappa_*=1$; hence regime 2 is realized. In this case, $\mathcal{E}_{\max}$ is limited by the nonlinear radiative dissipation ($\nu_* \ll 1$), yet the transverse oscillations are not affected by the radiative cooling ($\eta_* \ll 1$). For positrons and protons, the maximum energy can then be estimated as

$$\mathcal{E}_{e^+} \sim 300 \text{ GeV}, \quad \mathcal{E}_{p^+} \sim 10^6 \text{ TeV}, \tag{87}$$

respectively, in agreement with Ref. 28. In fully ionized plasmas with the same parameters and $Z_s \sim 50$, one has $\nu_* \sim 0.02$, $\eta_* \sim 0.014$; hence regime 2 is implemented again, yielding about seven times smaller energies. To access regimes 1 or 3 requires increasing η_* or ν_* up to unity, meaning much higher or lower κ_*; thus, for practical applications, regime 2 remains most probable.

We now consider $\mathcal{E}_{\max}$ as a function of density. At fixed κ_*, both ν_* and η_* are constant, and $F_a \propto \sqrt{n_e}$. Hence $\mathcal{E}_{\max} \propto n_e^{-1/2}$, means that higher densities correspond to a lower energy limit (and larger x_m). For example, $n_e=10^{19}$ cm^{-3} allows tens of TeV for positrons (or electrons), at $F_a \sim 3$ GeV/cm, even in an amorphous channel, in agreement with Ref. 12. However, the required acceleration distance would be of the order of 100 m, whereas in crystals the theoretical energy maximum is attained on a feasible 3 cm interval. Assuming $n_e \sim 10^{22}$ cm^{-3}, which yields the plasma wavelength $\lambda_p \sim 50$ nm, the corresponding accelerated charge is estimated as $en_e\lambda_p^3\mathcal{S} \sim 0.2\mathcal{S}$ pC, where $\mathcal{S}$ is the beam transverse area normalized to λ_p^2. This is about $10^{-3}\mathcal{S}$ of that available via the existing plasma-based schemes.[2–5] On the other hand, in crystals, $\mathcal{S}$ can be made large compared to unity due to the waveguide effect;[35] also sustaining plasma homogeneity is easier, particle channeling is naturally provided by the lattice, and extremely low emittance of accelerated beams can be attained. Thus solid-density plasmas are promising as active media for particle acceleration and, as such, merit further assessment.

VIII. CONCLUSIONS

In summary, we systematically treated charged particle acceleration in dense plasma channels, resolving discrepancies in the previously obtained results. We also identified the parameters that determine the theoretical maximum for the particle energy gain, as limited by dissipation, and found distinct regimes differing in scalings for the maximum energy. Through that, we answer the open questions addressed in Refs. 28 and 39 and provide a uniform treatment of the acceleration problem in general; hence our analysis applies to arbitrary plasma channels and, as a spin-off, yields known results for lower-density plasmas within a unified theoretical framework.

Our main results are summarized as follows:

- The equations of the particle motion in a plasma channel are derived [Eqs. (15) and (16)] accounting for an accelerating force, nonlinear radiative dissipation, linear transverse cooling, longitudinal friction, and multiple scattering. At energies beyond a few tens of MeV for electrons and positrons and tens of TeV for protons, the transverse cooling is dominated by radiative losses, and the longitudinal friction is mainly due to bremsstrahlung.
- Under the approximations of linear acceleration and negligible dissipation adopted in Refs. 16–18, 28, and 39, the bivariate distribution of the particle transverse coordinate and velocity is derived. The result supports those of Refs. 16–18 and 28 but is at variance with Ref. 39, which is found to have omitted important effects.
- Reduced nonlinear equations are obtained for the particle oscillation amplitude and energy [Eqs. (64) and (65)] and yield those in Refs. 10, 12, 16–18, 28, and 41–43 as particular cases, including the equation for the normalized beam emittance ϵ_n [Eqs. (72) and (78)].

- From the reduced equations, the parameters ν_* and η_* are identified [Eq. (80)] that determine the maximum energy gain $\mathcal{E}_{\max}$; hence four acceleration regimes [Fig. 1(a)]. However, since the product $\nu_* \eta_* \equiv \varepsilon \ll 1$ varies little among different channels [Eq. (82)], $\mathcal{E}_{\max}$ is effectively parameterized with a single dimensionless quantity, and only three regimes remain.
- Given a sufficiently strong accelerating force, particles can be channeled indefinitely, with the maximum energy [Eq. (85)] being higher in wider channels corresponding to larger ν_* [Fig. 1(b)]. For practical applications, most probable is the regime where $\mathcal{E}_{\max} \propto \nu_*$ is limited by the nonlinear radiative dissipation, whereas transverse cooling and bremsstrahlung are insignificant. This supports the estimate of Ref. 28 for particles channeled in crystals, yielding $\mathcal{E}_{\max} \sim 300$ GeV for positrons at a feasible 100 GeV/cm accelerating gradient.
- Greater energies are, in principle, possible in lower-density plasmas yet the required acceleration distances are unrealistically large, as opposed to those in crystals. On the other hand, in crystals, sustaining plasma homogeneity is easier, channeling is naturally provided by the lattice, and extremely low emittance of accelerated beams can be attained. Thus solid-density plasmas are promising as active media for particle acceleration and, as such, merit further assessment.

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APPENDIX: BEAM EMITTANCE

The normalized emittance is the area $\mathfrak{G}$ occupied by the beam in the transverse phase space (x,p_x),[70]

$$\epsilon_n = \mathfrak{G}/\pi mc. \tag{A1}$$

Equation (A1) rewrites as $\epsilon_n = \gamma\beta_z\epsilon$, where $\epsilon = A/\pi$ is the geometric emittance, A is the area in the space (x,x'), and the prime denotes (in this section) the derivative with respect to z. Suppose that the beam distribution is characterized by a quadratic function of the vector $\mathbf{X} = (x,x')^{\mathrm{T}}$,

$$f(x,x') = f(\mathbf{X}\cdot\hat{\sigma}^{-1}\mathbf{X}), \tag{A2}$$

where $\hat{\sigma}$ is a symmetric matrix

$$\hat{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}. \tag{A3}$$

Then $dA = \sqrt{\sigma}\, d\mathcal{A}$, where $\sigma = \det\hat{\sigma}$, and $d\mathcal{A}$ is the area element in the normalized variables $\boldsymbol{\mathcal{X}}$. The latter are defined such that $\boldsymbol{\mathcal{X}}\cdot\boldsymbol{\mathcal{X}} = \mathbf{X}\cdot\hat{\sigma}^{-1}\mathbf{X}$, so the distribution has a unit width in the $\boldsymbol{\mathcal{X}}$ space; hence $\epsilon \sim A \sim \sqrt{\sigma}$. For a Gaussian distribution

$$f(x,x') = \frac{1}{2\pi\sqrt{\sigma}} \exp\left[-\frac{1}{2\sigma}(\sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2)\right], \tag{A4}$$

one has

$$\sigma_{11} = \langle x^2\rangle, \quad \sigma_{12} = \langle xx'\rangle, \quad \sigma_{22} = \langle x'^2\rangle \tag{A5}$$

[cf. Eqs. (50) and (52)]. Hence ϵ is defined as

$$\epsilon = \sqrt{\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2}. \tag{A6}$$

In our case, $\langle xx'\rangle^2 \ll \langle x^2\rangle\langle x'^2\rangle$, and $\langle x'^2\rangle \approx (\Omega/c)^2\langle x^2\rangle$ [Eqs. (55) and (56); Sec. VI A]; thus $\epsilon \approx (\Omega/c)\langle x^2\rangle$. Using also that $\beta_z \approx 1$, one gets

$$\epsilon_n \approx \frac{\gamma\Omega}{c}\langle x^2\rangle = \bar{\epsilon}_n\langle\xi^2\rangle\sqrt{\Gamma}, \quad \bar{\epsilon}_n = \bar{x}\bar{\gamma}\sqrt{\varsigma}, \tag{A7}$$

and we assume measuring ϵ_n in units $\bar{\epsilon}_n$ in Secs. VI and VII.

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