Langmuir wave linear evolution in inhomogeneous nonstationary anisotropic plasma

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A hydrodynamic equation describing the linear evolution of a nondissipative Langmuir wave in inhomogeneous nonstationary anisotropic plasma without magnetic field is derived in the geometrical optics approximation. The continuity equation for the wave action density, anticipated from general principles, is then confirmed *ab initio*, and the conditions for the action conservation are formulated. Given those, the wave field $\widetilde{E}$ universally scales with the electron density N as $\widetilde{E} \propto N^{3/4}$ in homogeneous plasma, whereas the wavevector evolution varies depending on the wave geometry. © *2009 American Institute of Physics*. [doi:10.1063/1.3250983]

I. INTRODUCTION

The energy of a wave propagating in nonstationary medium can be manipulated by controlling how the parameters of the medium evolve;[1–7] for example, the energy can be pumped up, transported, focused, and (or) deposited where necessary. In the case of Langmuir, or plasma waves,[8] there may be important high energy density applications connected with compressing plasma targets, because these targets may advertently or inadvertently contain wave packets that would be amplified along with the densification. Hence, understanding of the Langmuir wave evolution in nonstationary plasma is needed. To develop such understanding is the purpose of this paper.

Specifically, we assume the geometrical optics (GO) limit,[9–13] when the plasma parameters vary sufficiently slowly in time and space. We also assume that a wave is linear,[14] and no collisions, ionization, or recombination take place.[15] In this case, the plasma dynamics should allow a Lagrangian formulation;[16,17] thus, it is anticipated to comply with the general theorem of GO which states that the wave action is conserved in inhomogeneous nonstationary medium.[18–24] Previously, the theorem was independently rederived *ab initio* for a variety of oscillations,[18,19,25–28] confirming the general treatment; particularly, space-charge waves in cold electron beams were considered, similar to Langmuir waves in cold plasmas.[29–31] However, for thermal plasmas there has been less agreement, and some of the models proposed in literature do not comply with the action conservation.

The Langmuir wave action, or "plasmon" conservation theorem (PCT) was reported in Refs. 32–35, accounting for nonlinear effects; however, the inhomogeneity of the background plasma was neglected there (see also Ref. 36). The density inhomogeneity was included in a linear treatment in Ref. 37, yet within a model assuming constant temperature. More precise models of Langmuir waves in inhomogeneous plasma (see, e.g., Ref. 38 and references therein) did not specifically address PCT and assumed stationary medium; also, the wave equation derived in Ref. 39 is not entirely correct (see, e.g., Refs. 40 and 41 and Sec. IV A) and hence

is at variance with the theorem. Similarly, the kinetic models offered in Refs. 42 and 43 are erroneous, as explained in Refs. 44 and 45, and so is the corresponding part of Ref. 46, as argued in Appendix B. Thus, the accuracy of PCT with respect to the temperature corrections was not fully assessed.

An accurate kinetic treatment of the thermal effects was eventually proposed in Refs. 44 and 45. Particularly, it was shown that the Langmuir wave action in isotropic nonstationary collisionless plasma is conserved in the GO limit, assuming that Landau damping is insignificant. However, the solution in Refs. 44 and 45 is incomplete, because collisionless plasma may not remain isotropic in the presence of inhomogeneous average flow.[47] Thus, it yet remains to derive an explicit equation for a Langmuir wave in nonstationary inhomogeneous anisotropic plasma and show how the wave parameters evolve.

The present paper offers such a calculation, completing the results of Refs. 29, 37, 44, and 45 and reconciling the Langmuir wave *ab initio* theory with the action conservation theorem anticipated from general principles of Lagrangian dynamics. Specifically, we derive the wave equation in the GO limit and confirm that it yields a continuity equation for the wave action density; hence, it is demonstrated that, under certain conditions, also found explicitly, the wave action is conserved. As a corollary, it is then shown that, in homogeneous plasma carrying Langmuir oscillations, the wave field amplitude universally scales with the electron density N as $\widetilde{E} \propto N^{3/4}$, whereas the wavevector evolution varies depending on the wave geometry.

The paper is organized as follows. In Sec. II, we introduce our basic equations. In Sec. III, we find the Langmuir wave dispersion relation in homogeneous anisotropic plasma. In Sec. IV, we derive the equations for GO rays and the amplitude of Langmuir oscillations in inhomogeneous nonstationary plasma. In Sec. V, we use those to obtain a continuity equation for the wave action density; we also derive the scalings for the oscillation field amplitude and wavenumber. In Sec. VI, we summarize our main results. Supplementary calculations are given in appendices.

II. BASIC EQUATIONS

Consider a Langmuir wave in unmagnetized nonrelativistic plasma with given flow velocity $\mathbf{V}(\mathbf{r},t)$. Neglect ion oscillations and assume collisions and Landau damping to be insignificant on time scales of interest. For electrons, adopt the low-temperature approximation requiring $k\lambda_D \ll 1$, where k is the wavenumber, and λ_D is the Debye length [cf. Eq. (11) in Ref. 48]. Hence, an asymptotic closure of the hydrodynamic model is possible, via omitting the heat flux, and one obtains, by taking the first three velocity moments of the electron Vlasov equation,[48–50]

$$\partial_t N_e + \nabla \cdot (N_e \mathbf{V}_e) = 0, \tag{1}$$

$$\partial_t \mathbf{V}_e + (\mathbf{V}_e \cdot \nabla)\mathbf{V}_e = \frac{e}{m_e}\mathbf{E} - \frac{e}{m_e}\nabla\varphi - \frac{\nabla \cdot \hat{\mathbf{P}}_e}{N_e m_e}, \tag{2}$$

$$\partial_t \hat{\mathbf{P}}_e + (\mathbf{V}_e \cdot \nabla)\hat{\mathbf{P}}_e + \hat{\mathbf{P}}_e(\nabla \cdot \mathbf{V}_e)$$
$$+ [(\hat{\mathbf{P}}_e\nabla)\mathbf{V}_e] + [(\hat{\mathbf{P}}_e\nabla)\mathbf{V}_e]^{\mathrm{T}} = 0. \tag{3}$$

Here N_e is the electron density, $\mathbf{V}_e$ is the electron flow velocity, $e < 0$ and m_e are the electron charge and mass, $\hat{\mathbf{P}}_e$ is the electron pressure tensor (which is symmetric by definition), $\mathbf{E}$ is the average electric field (if any), φ is the wave electrostatic potential,[51] and the index T denotes transposition. Separate the slow and the quiver variables, correspondingly, as

$$N_e = N + \tilde{N}, \quad \mathbf{V}_e = \mathbf{V} + \tilde{\mathbf{V}}, \quad \hat{\mathbf{P}}_e = \hat{\mathbf{P}} + \tilde{\hat{\mathbf{P}}}, \tag{4}$$

and assume that the oscillations are weak, i.e.,

$$n \equiv \tilde{N}/N \ll 1, \tag{5}$$

and similarly for the pressure. (On the other hand, the absolute value of $\mathbf{V}$ is unimportant.) Then the following linear equations are obtained:

$$\partial'_t n + \mathbf{h} \cdot \tilde{\mathbf{V}} + \nabla \cdot \tilde{\mathbf{V}} = 0, \tag{6}$$

$$\partial'_t \tilde{\mathbf{V}} + (\tilde{\mathbf{V}} \cdot \nabla)\mathbf{V} - \mathbf{\Pi} + (e/m_e)\nabla\varphi = 0, \tag{7}$$

$$\partial'_t \tilde{\hat{\mathbf{P}}} + (\tilde{\mathbf{V}} \cdot \nabla)\hat{\mathbf{P}} + \tilde{\hat{\mathbf{P}}}(\mathrm{Tr}\hat{\mathbf{W}}) + (\hat{\mathbf{W}}\tilde{\hat{\mathbf{P}}}) + (\hat{\mathbf{W}}\tilde{\hat{\mathbf{P}}})^{\mathrm{T}}$$
$$+ \hat{\mathbf{P}}(\nabla \cdot \tilde{\mathbf{V}}) + [(\hat{\mathbf{P}}\nabla)\tilde{\mathbf{V}}] + [(\hat{\mathbf{P}}\nabla)\tilde{\mathbf{V}}]^{\mathrm{T}} = 0, \tag{8}$$

$$\mathbf{\Pi} = (n\nabla \cdot \hat{\mathbf{P}} - \nabla \cdot \tilde{\hat{\mathbf{P}}})/(m_e N), \tag{9}$$

$$\nabla^2\varphi = -(m_e/e)\omega_p^2 n. \tag{10}$$

Here we introduced the partial time derivative in the frame of reference K' (further denoted by prime) moving with velocity $\mathbf{V}$ with respect to the laboratory frame K:

$$\partial'_t = \partial_t + (\mathbf{V} \cdot \nabla). \tag{11}$$

In addition, we introduced $\mathbf{h} = \nabla \ln N$, $\hat{\mathbf{W}} = \nabla\mathbf{V}$ (which is a tensor with elements $W_{j\ell} = \partial V_j/\partial x_\ell$), the electron plasma

frequency $\omega_p = \sqrt{4\pi e^2 N/m_e}$, and Tr for the tensor trace. The complex notation is also henceforth assumed for the quiver variables.

III. HOMOGENEOUS STATIONARY PLASMA

In the case of homogeneous plasma with $\mathbf{V} = $ const, the exact eigenmodes of Eqs. (6)–(10) are found as follows. Assume

$$n, \tilde{\mathbf{V}}, \tilde{\hat{\mathbf{P}}} \sim \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t). \tag{12}$$

Then one gets

$$-i\omega' n + i\mathbf{k} \cdot \tilde{\mathbf{V}} = 0, \tag{13}$$

$$-i\omega'\tilde{\mathbf{V}} + i\mathbf{k}(e/m_e)\varphi - \mathbf{\Pi} = 0, \tag{14}$$

$$-i\omega'\tilde{\hat{\mathbf{P}}} + i\hat{\mathbf{P}}(\mathbf{k} \cdot \tilde{\mathbf{V}}) + i[(\hat{\mathbf{P}}\mathbf{k})\tilde{\mathbf{V}}] + i[(\hat{\mathbf{P}}\mathbf{k})\tilde{\mathbf{V}}]^{\mathrm{T}} = 0, \tag{15}$$

$$\mathbf{\Pi} = -i\mathbf{k}\tilde{\hat{\mathbf{P}}}/(m_e N), \tag{16}$$

$$-k^2\varphi = -(m_e/e)\omega_p^2 n, \tag{17}$$

with $\omega' \equiv \omega - \mathbf{k} \cdot \mathbf{V}$ being the wave frequency in the frame where the plasma average flow rests. Hence,

$$\hat{\mathbf{H}}\tilde{\mathbf{V}} = \omega'^2\tilde{\mathbf{V}}, \tag{18}$$

i.e., $\tilde{\mathbf{V}}$ must be an eigenvector of the tensor $\hat{\mathbf{H}}$ given by

$$\hat{\mathbf{H}} = \frac{\mathbf{k}\mathbf{k}}{k^2}\omega_p^2 + \frac{1}{m_e}[2(\hat{\mathbf{T}}\mathbf{k})\mathbf{k} + (\mathbf{k} \cdot \hat{\mathbf{T}}\mathbf{k})\hat{\mathbf{1}}], \tag{19}$$

where $\mathbf{k}\mathbf{k}$ is a dyad, $\hat{\mathbf{1}}$ is a unit tensor, and $\hat{\mathbf{T}} = \hat{\mathbf{P}}/N$ is the electron unperturbed temperature tensor.

For a longitudinal wave, one has from Eq. (13) that

$$\tilde{\mathbf{V}} = n\omega'\mathbf{k}/k^2. \tag{20}$$

Substituting this into Eq. (18) yields

$$\hat{\mathbf{T}}\mathbf{k} = \frac{m_e}{2k^2}\left[\omega'^2 - \omega_p^2 - \frac{(\mathbf{k} \cdot \hat{\mathbf{T}}\mathbf{k})}{m_e}\right]\mathbf{k}. \tag{21}$$

Since the coefficient on the right-hand side is a scalar, $\mathbf{k}$ must be an eigenvector of $\hat{\mathbf{T}}$:

$$\hat{\mathbf{T}}\mathbf{k} = m_e v_{Te}^2\mathbf{k}, \tag{22}$$

where the eigenvalues satisfy $v_{Te}^2 > 0$, because $\hat{\mathbf{T}}$ is positive defined; thus, v_{Te} can be understood as the electron thermal speed along $\mathbf{k}$. Then,

$$\omega = \Omega(\mathbf{k}) + \mathbf{k} \cdot \mathbf{V}, \tag{23}$$

where the function $\Omega(\mathbf{k})$ (numerically equal to ω') is given by $\Omega^2 = \omega_p^2 + 3k^2 v_{Te}^2$ (cf., e.g., Ref. 52, Chap. 8 or Refs. 53–55 for isotropic plasma), or

$$\Omega^2 = \omega_p^2 + \mathbf{k} \cdot \hat{\mathbf{C}}\mathbf{k}, \quad \hat{\mathbf{C}} = 3\hat{\mathbf{T}}/m_e. \tag{24}$$

Therefore, similarly to elastic media,[56–60] a longitudinal wave in plasma must have $\mathbf{k}$ along a principal axis of the

temperature tensor.[61] Since $\hat{\mathbf{T}}$ is symmetric, there always exist at least three such directions. On the other hand, in isotropic plasma, any axis can be considered a principle axis of $\hat{\mathbf{T}}$; then, Langmuir waves can propagate in arbitrary direction.

IV. GEOMETRICAL OPTICS EQUATIONS

A. Approximate wave equation

Consider now a more general case of inhomogeneous nonstationary plasma. Assume, however, that the wave amplitude, the frequency, and the local wavevector vary on *large* temporal and spatial scales $\mathcal{T}$ and L in the plasma average flow rest frame K'. Specifically, we assume

$$\epsilon \equiv 1/\min\{\omega'\mathcal{T}, k'L\} \ll 1, \tag{25}$$

with $k'=k$ in the nonrelativistic limit used here. In this case, henceforth referred as the GO limit,[9,10] an approximate scalar equation for a Langmuir wave can be obtained which will capture effects to the leading order in the parameter ϵ.

To derive this equation, first, apply ∂_t' to Eq. (6) to get

$$\partial_t'^2 n + \partial_t'(\mathbf{h} \cdot \tilde{\mathbf{V}}) + \partial_t'(\nabla \cdot \tilde{\mathbf{V}}) = 0. \tag{26}$$

The third term on the left-hand side here is found by taking the divergence of Eq. (7):

$$\partial_t'(\nabla \cdot \tilde{\mathbf{V}}) = \omega_p^2 n + \nabla \cdot \mathbf{\Pi} - (\tilde{\mathbf{V}} \cdot \nabla)(\nabla \cdot \mathbf{V}) - \{\tilde{\mathbf{V}}, \mathbf{V}\},$$

where we substituted Eq. (10) and introduced

$$\{\tilde{\mathbf{V}}, \mathbf{V}\} \equiv \nabla \cdot [(\tilde{\mathbf{V}} \cdot \nabla)\mathbf{V} + (\mathbf{V} \cdot \nabla)\tilde{\mathbf{V}}]$$
$$- (\tilde{\mathbf{V}} \cdot \nabla)(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla)(\nabla \cdot \tilde{\mathbf{V}}), \tag{27}$$

accounting for the fact that ∂_t' and ∇ may not commute. Therefore Eq. (26) takes the form

$$\partial_t'^2 n + \omega_p^2 n + \nabla \cdot \mathbf{\Pi} + R = 0, \tag{28}$$

with R given by

$$R = -(\tilde{\mathbf{V}} \cdot \nabla)(\nabla \cdot \mathbf{V}) + \partial_t'(\mathbf{h} \cdot \tilde{\mathbf{V}}) - \{\tilde{\mathbf{V}}, \mathbf{V}\}. \tag{29}$$

The first term in Eq. (29) is of the order of ϵ^2 and can be neglected. The second term is evaluated as

$$\frac{\partial'(\mathbf{h} \cdot \tilde{\mathbf{V}})}{\partial t} \approx 2\frac{\partial' n}{\partial t}\frac{\Omega}{\omega_p}\nabla\omega_p \cdot \frac{\mathbf{k}}{k^2}, \tag{30}$$

$(\partial'/\partial t \equiv \partial_t')$, where we used that, to the zeroth order in ϵ, one can employ Eq. (20) and

$$\partial_t' \approx -i\Omega, \quad \nabla \approx i\mathbf{k}. \tag{31}$$

The third term in Eq. (29) can be put as (Appendix A)

$$\{\tilde{\mathbf{V}}, \mathbf{V}\} = 2(\nabla\tilde{V}_j) \cdot (\nabla V_j) - 2(\nabla \times \tilde{\mathbf{V}}) \cdot (\nabla \times \mathbf{V}), \tag{32}$$

where we assume summation over repeated indices. Both curls are of the order of ϵ here; thus, $\{\tilde{\mathbf{V}}, \mathbf{V}\} \approx 2(\nabla\tilde{V}_j) \cdot (\nabla V_j)$. Equations (20) and (31) further yield

$$\{\tilde{\mathbf{V}}, \mathbf{V}\} \approx -2(\partial_t'n)\mathbf{k}\hat{\mathbf{W}} \cdot \mathbf{k}/k^2. \tag{33}$$

Hence, Eq. (29) rewrites as

$$R = 2\frac{\partial' n}{\partial t}\left(\frac{\Omega}{\omega_p}\nabla\omega_p + \mathbf{k}\hat{\mathbf{W}}\right) \cdot \frac{\mathbf{k}}{k^2}. \tag{34}$$

An approximate expression for $\nabla \cdot \mathbf{\Pi}$ is obtained similarly (Appendix B). Substituting that and Eq. (34) into Eq. (28) one gets for isotropic plasma

$$\frac{\partial'^2 n}{\partial t^2} + \omega_p^2 n - 3v_{Te}^2\nabla^2 n + 2\frac{\partial' n}{\partial t}\left(\frac{\Omega}{\omega_p}\nabla\omega_p + \mathbf{k}\hat{\mathbf{W}}\right) \cdot \frac{\mathbf{k}}{k^2}$$
$$- 6\nabla n \cdot \nabla v_{Te}^2 = 0, \tag{35}$$

and in the general case of anisotropic plasma, which we study below,

$$\frac{\partial'^2 n}{\partial t^2} + \omega_p^2 n - C_{j\ell}\frac{\partial^2 n}{\partial x_j \partial x_\ell} + 2\frac{\partial' n}{\partial t}\left(\frac{\Omega}{\omega_p}\frac{\partial\omega_p}{\partial x_\ell} + k_j W_{j\ell}\right)\frac{k_\ell}{k^2}$$
$$- \left(\delta_{js} + \frac{k_j k_s}{k^2}\right)\frac{\partial C_{s\ell}}{\partial x_j}\frac{\partial n}{\partial x_\ell} = 0, \tag{36}$$

assuming Cartesian coordinates.[62]

B. Eikonal equation

Equation (36) can be solved using the GO approach,[9,63] specifically as follows. Take

$$n = \mathcal{N}e^{i\theta}, \tag{37}$$

where $\mathcal{N}$ is the slowly varying envelope. Substitute Eq. (37) into Eq. (36) and first consider the terms of order ϵ^0; hence the eikonal equation

$$[-(\partial_t\theta - \nabla\theta \cdot \mathbf{V})^2 + \omega_p^2 + \nabla\theta \cdot \hat{\mathbf{C}}\nabla\theta]\mathcal{N} = 0. \tag{38}$$

Since, by definition,

$$\partial_t\theta = -\omega, \quad \nabla\theta = \mathbf{k}. \tag{39}$$

Equation (38) is equivalent to Eq. (23), except now the plasma parameters may slowly depend on $\mathbf{r}$ and t:

$$\omega = \Omega(\mathbf{k}; \mathbf{r}, t) + \mathbf{k} \cdot \mathbf{V}(\mathbf{r}, t). \tag{40}$$

Differentiate Eq. (40) with respect to t and with respect to $\mathbf{r}$ and use $\nabla\omega = -\partial_t\mathbf{k}$, as flows from Eqs. (39). Then,

$$d_t\omega = \partial_t\omega(\mathbf{k}, \mathbf{r}, t), \quad d_t\mathbf{k} = -\nabla\omega(\mathbf{k}, \mathbf{r}, t), \tag{41}$$

where the partial derivatives are taken at fixed $\mathbf{k}$; also,

$$d_t \equiv \partial_t + (\mathbf{v}_g \cdot \nabla), \tag{42}$$

and $\mathbf{v}_g \equiv \partial_{\mathbf{k}}\omega(\mathbf{k}, \mathbf{r}, t)$ is the group velocity:

$$\mathbf{v}_g = \mathbf{U} + \mathbf{V}, \quad \mathbf{U} \equiv \partial_{\mathbf{k}}\Omega(\mathbf{k}, \mathbf{r}, t). \tag{43}$$

Since $\mathbf{v}_g$ equals the velocity at which the envelope propagates (Ref. 52, Chap. 4), one can complement these with

$$d_t\mathbf{r} = \partial_{\mathbf{k}}\omega. \tag{44}$$

Together, Eqs. (41) and (44) are known as GO ray equations (Ref. 52, Chap. 4) and can be considered as canonical equations [with the Hamiltonian $\hbar\omega(\mathbf{k}, \mathbf{r}, t)$; Eq. (40)] which

determine the dynamics of plasmons, i.e., quasiparticles with velocity $\mathbf{v}_g$, momentum $\hbar\mathbf{k}$, and energy $\hbar\omega$; see also Refs. 32 and 33.

C. Amplitude equation

The equation obtained from Eq. (36) in the first order in ϵ reads as

$$-2i\Omega\frac{\partial'\mathcal{N}}{\partial t} - i\frac{\partial'\Omega}{\partial t}\mathcal{N} - iC_{j\ell}\left(k_\ell\frac{\partial\mathcal{N}}{\partial x_j} + k_j\frac{\partial\mathcal{N}}{\partial x_\ell}\right) - iC_{j\ell}\mathcal{N}\frac{\partial k_\ell}{\partial x_j}$$
$$-2i\Omega\Gamma\mathcal{N} - ik_\ell\mathcal{N}\frac{\partial C_{s\ell}}{\partial x_j}\left(\delta_{js} + \frac{k_jk_s}{k^2}\right) = 0, \qquad (45)$$

where $\mathbf{k}$ is a function of $\mathbf{r}$ and t [unlike in Eqs. (41), where $\mathbf{k}$ is a variable], and

$$\Gamma = \left(\frac{\Omega}{\omega_p}\frac{\partial\omega_p}{\partial x_\ell} + k_jW_{j\ell}\right)\frac{k_\ell}{k^2}. \qquad (46)$$

The same expression can be written also as follows. Use Eq. (41) for $d_t\mathbf{k}$ to get

$$\frac{dk_\ell}{dt} = -\frac{\omega_p}{\Omega}\frac{\partial\omega_p}{\partial x_\ell} - \frac{k_jk_s}{2\Omega}\frac{\partial C_{js}}{\partial x_\ell} - k_jW_{j\ell}. \qquad (47)$$

Hence, Eq. (46) is put in the form

$$\Gamma = -\frac{k_\ell}{k^2}\frac{dk_\ell}{dt} + \left(\frac{\Omega}{\omega_p} - \frac{\omega_p}{\Omega}\right)\frac{k_\ell}{k^2}\frac{\partial\omega_p}{\partial x_\ell} - \frac{k_jk_\ell k_s}{2\Omega k^2}\frac{\partial C_{js}}{\partial x_\ell}. \qquad (48)$$

We now use the expression for Ω [Eq. (24)], $\nabla\omega_p/\omega_p = \nabla N/(2N)$, and

$$\mathbf{k}\cdot d_t\mathbf{k}/k = d_tk. \qquad (49)$$

Hence, Eq. (46) can be represented as

$$\Gamma = -\frac{d\ln k}{dt} + \frac{k_jk_\ell k_s}{2\Omega k^2}\left(C_{js}h_\ell - \frac{\partial C_{js}}{\partial x_\ell}\right). \qquad (50)$$

Since the wave is assumed propagating along a local principal axis of the temperature tensor,[64] one also has $C_{j\ell}k_jk_\ell/k^2 = 3v_{Te}^2$ and $3v_{Te}^2k_s = C_{s\ell}k_\ell$; thus, Eq. (50) can be further put as

$$\Gamma = -\frac{d\ln k}{dt} + \frac{U_jh_j}{2} - \frac{k_jk_\ell k_s}{2\Omega k^2}\frac{\partial C_{j\ell}}{\partial x_s}, \qquad (51)$$

where we used $U_j = C_{j\ell}k_\ell/\Omega$ [Eq. (43)]. Then Eq. (45) rewrites as

$$\partial_t'(\Omega|\mathcal{N}|^2) + \nabla\cdot(\Omega|\mathcal{N}|^2\mathbf{U}) + \Omega|\mathcal{N}|^2(\mathbf{U}\cdot\mathbf{h} - d_t\ln k^2) = 0. \qquad (52)$$

We now use

$$\partial_t'(\Omega|\mathcal{N}|^2) + \nabla\cdot(\Omega|\mathcal{N}|^2\mathbf{U}) = d_t(\Omega|\mathcal{N}|^2) + \Omega|\mathcal{N}|^2\nabla\cdot\mathbf{U}, \qquad (53)$$

$$\mathbf{U}\cdot\mathbf{h} = d_t\ln N + \nabla\cdot\mathbf{V},$$

the latter being due to

$$\partial_t'N = -N\nabla\cdot\mathbf{V}, \qquad (54)$$

which flows from Eq. (1). Hence, Eq. (53) reads as

$$d_t\ln(\Omega|\mathcal{N}|^2) + d_t\ln N - d_t\ln k^2 + \nabla\cdot\mathbf{v}_g = 0. \qquad (55)$$

In principle, this allows one to calculate the envelope amplitude $|\mathcal{N}|$ along the GO rays, as discussed in Sec. V A.

V. DISCUSSION

A. Wave action. Equation of state

Introduce the wave average energy density,

$$\mathcal{E}' = \frac{|\widetilde{E}'|^2}{16\pi}\frac{\partial(\varepsilon_l'\omega')}{\partial\omega'}, \qquad (56)$$

in the frame K' traveling with velocity $\mathbf{V}$. Here $\widetilde{\mathbf{E}}' \approx \widetilde{\mathbf{E}}$, and the longitudinal permittivity in K' equals that in the laboratory frame K: $\varepsilon_l' = \varepsilon_l$.[65] Neglecting the corrections due to finite ϵ and using Eqs. (6)–(16), ε_l is derived like for isotropic plasma (Ref. 52, Chap. 3):

$$\varepsilon_l = 1 - \frac{\omega_p^2}{\omega'^2 - 3k^2v_{Te}^2}. \qquad (57)$$

(Here $\mathbf{k}$ parallel to the principal axis of $\hat{\mathbf{T}}$ is assumed, as before.[64]) Thus, Eq. (56) rewrites as

$$\mathcal{E}' = \frac{\Omega^2}{\omega_p^2}\frac{|\widetilde{E}|^2}{8\pi} \qquad (58)$$

[where we used $\varepsilon_l(\Omega + \mathbf{k}\cdot\mathbf{V};\mathbf{k}) = 0$], so *this* energy density is always positive, unlike that in K.[66]

Further, define the wave action density J, or the number of quanta (plasmons) per unit volume, as $J = \mathcal{E}'/\omega'$,[18,67] where we take $\omega' > 0$, by analogy with discrete systems (see, e.g., Sec. III of Ref. 68). Then,

$$J \propto \Omega|\mathcal{N}|^2N/k^2, \qquad (59)$$

where we used $\widetilde{\mathbf{E}} \approx -i\mathbf{k}\varphi$ and Eq. (17) for φ. From Eq. (55), it follows that $d_t\ln J + \nabla\cdot\mathbf{v}_g = 0$, or

$$d_tJ + J\nabla\cdot\mathbf{v}_g = 0. \qquad (60)$$

The latter is also equivalent to a continuity equation:

$$\partial_tJ + \nabla\cdot(\mathbf{v}_gJ) = 0. \qquad (61)$$

Hence, the wave total action is conserved:

$$\int Jd^3\mathbf{r} = \text{inv}, \qquad (62)$$

and the dynamics is thereby called adiabatic.

Equations (60)–(62) agree with the previous results for space-charge waves in cold plasmas and electron beams,[29–31] as well as phenomenological hydrodynamical treatment of the corrections due to the electron homogeneous temperature[37] and kinetic treatment for inhomogeneous non-stationary but isotropic plasmas.[44,45] By construction,[48–50] the hydrodynamic calculation offered here is asymptotically precise at small temperatures and, apart from missing Landau damping (Sec. IV B), just as accurate as the perturbative kinetic calculation in Refs. 44 and 45. On the other hand, it also accounts for the temperature anisotropy, which is expected at collisionless compression or rarefaction[47] yet

missed in Refs. 44 and 45. Therefore, our results complete those of Refs. 44 and 45 and finally reconcile the Langmuir wave *ab initio* theory with the action conservation theorem anticipated from general principles of Lagrangian dynamics.[10,18–20,27,67]

Besides, the above results show explicitly how the Langmuir wave parameters evolve. Consider, for instance, homogeneous plasma, assuming that the envelope shape remains fixed. Then Eq. (62) rewrites as

$$J/N = \text{inv}, \tag{63}$$

where we used that the total number of electrons is conserved. In the absence of Landau damping (Sec. IV B) one has $kv_{Te} \ll \omega_p$, and therefore

$$J \approx |\widetilde{E}|^2/(8\pi\omega_p). \tag{64}$$

Together with Eq. (63), this yields (similarly to Ref. 29)

$$\widetilde{E} = \widetilde{E}_0 (N/N_0)^{3/4}, \tag{65}$$

where the index 0 denotes the initial values. Thus, the wave field increases when the plasma is compressed and decreases when the plasma is rarefied.

Finally, Eq. (65) also results in an effective adiabatic index γ for the ponderomotive pressure p_E. To see this, consider the known expression for p_E,[69] using that the field $\widetilde{\mathbf{E}}$ oscillates at the frequency $\omega = \omega_p$:

$$p_E = |\widetilde{E}|^2/(16\pi). \tag{66}$$

[In fact, Eq. (66) itself is also derivable from Eq. (63), as shown in Appendix C.] Hence, from Eq. (65), one obtains

$$p_E = \frac{|\widetilde{E}_0|^2}{16\pi} \left(\frac{N}{N_0} \right)^{3/2}. \tag{67}$$

Therefore, for the ponderomotive pressure one has $\gamma = 3/2$, which is different, say, from $\gamma = (D+2)/D$ for the kinetic pressure of D-dimensional thermal electron gas without a wave.[70]

B. Wavevector. Adiabaticity conditions

The scalings (63)–(67) hold for any wave geometry, whereas the dependence of the frequency and the wavevector on plasma parameters may vary, as governed by Eq. (41). Particularly, ω is conserved only in stationary medium, and the dynamics of $\mathbf{k}$ is discussed below.

1. Homogeneous plasma

To illustrate the evolution of $\mathbf{k}$, first consider plasma compression such that N remains homogeneous [which is possible at homogeneous yet not necessarily zero $\nabla \cdot \mathbf{V}$; see Eq. (54)]. Then the wavevector is conserved if $\mathbf{V}$ is transverse to $\mathbf{k}$, an example being radial compression of a cylindrical plasma column with $\mathbf{k}$ along the axis of symmetry. However, if $\mathbf{k}$ has a component along $\mathbf{V}$, the wavevector will evolve; specifically,

$$k = k_0 \exp\left[\int_0^t \nu(t')dt' \right], \tag{68}$$

with Eqs. (47) and (49) yielding $\nu = \nu_V$,

$$\nu_V \equiv -\mathbf{k} \cdot \hat{\mathbf{W}}\mathbf{k}/k^2 \sim V/L_V, \tag{69}$$

where L_V is the spatial scale on which the compression takes place. For instance, radial compression with $\mathbf{V} = \chi(t)\mathbf{r}$ and $\mathbf{k}$ along $\mathbf{V}$ in spherical, cylindrical, and linear geometry equally yield $\nu_V = \chi$.

2. Inhomogeneous plasma

As the next step, consider inhomogeneous plasma, for now assuming $\mathbf{V} = 0$. In this case k can increase or decrease, depending on $\mathbf{k}_0$ as well as the density and temperature gradients, so Eq. (68) holds with $\nu = \nu_\nabla$,

$$\nu_\nabla \equiv -\mathbf{k} \cdot \nabla\Omega/k^2 \sim v_{\text{ph}}/L_\Omega, \tag{70}$$

with $v_{\text{ph}} \approx \omega_p/k$ being the phase speed and $L_\Omega \equiv \Omega/|\nabla\Omega|$ being the characteristic spatial scale.

First, suppose that k increases. Then, on the time scale of the order of $\tau_e \sim L_\Omega/v_{Te}$, the wavelength becomes comparable to the Debye length $\lambda_D = v_{Te}/\omega_p$, regardless of whether the plasma inhomogeneity is due to the density or the temperature; hence, the wave decays because of Landau damping (see also Refs. 42 and 71). In other words, dissipation is negligible only at

$$t \lesssim L_\Omega/v_{Te}. \tag{71}$$

Thus, when compression is added, it will proceed adiabatically only if $\nu_V \tau_e \gtrsim 1$, or

$$V/L_V \gtrsim v_{Te}/L_\Omega. \tag{72}$$

Assuming that the plasma average flow is entirely controlled by the large ion mass $m_i \gg m_e$,[72] one can rewrite V in Eq. (72) as follows. Express $\mathbf{E}$ from Eq. (2) and substitute it into a similar equation for ions; hence,

$$\partial_t' \mathbf{V} \approx -\nabla \cdot \hat{\mathbf{P}}_\Sigma/(m_i N), \tag{73}$$

where we neglected the electron inertia and introduced the total kinetic pressure $\hat{\mathbf{P}}_\Sigma \sim NT$.[73] Use $\partial_t' \mathbf{V} \sim V^2/L_V$, yielding

$$V \sim c_s \sqrt{L_V/L_\Omega}, \tag{74}$$

where $c_s \sim v_{Te}\sqrt{m_e/m_i}$ is the ion sound speed. Hence, Eq. (72) rewrites as

$$L_\Omega \gtrsim (m_i/m_e)L_V. \tag{75}$$

Therefore, only weakly inhomogeneous plasma can be compressed adiabatically when k grows; otherwise a significant percentage of the wave energy is transformed into the particle thermal energy via Landau damping.

Suppose now that k decreases. In this case, the envelope approximation holds only on time

$$t \lesssim \nu_\nabla^{-1}, \tag{76}$$

after which the wavenumber becomes zero, and, thus, the wave action is no longer conserved. Therefore, adiabatic compression must satisfy $\nu_V \gtrsim \nu_\nabla$, or

$$V/L_V \gtrsim \upsilon_{\mathrm{ph}}/L_\Omega. \tag{77}$$

Assuming Eq. (74), this condition hence reads as

$$L_\Omega \gtrsim \frac{L_V}{(k\lambda_{\mathrm{D}})^2} \frac{m_i}{m_e}, \tag{78}$$

which requires that plasma be even more homogeneous than in the case when k increases [cf. Eq. (75)].

VI. CONCLUSIONS

In this paper, we show how a nondissipative Langmuir wave evolves adiabatically in warm unmagnetized inhomogeneous nonstationary plasma. The hydrodynamic calculation offered here is asymptotically precise at small temperatures ($k\lambda_{\mathrm{D}} \ll 1$) and, apart from missing Landau damping, just as accurate as the perturbative kinetic calculation in Refs. 44 and 45. On the other hand, it also accounts for the temperature anisotropy, which is expected at collisionless compression or rarefaction yet missed in Refs. 44 and 45. Therefore, our results complete those of Refs. 44 and 45 and finally reconcile the Langmuir wave *ab initio* theory with the action conservation theorem anticipated from general principles of Lagrangian dynamics.

Specifically, we derive the wave equation in the GO limit [Eq. (36)] and confirm that it yields a continuity equation [Eq. (61)] for the wave action density; hence, it is demonstrated that, under certain conditions, also found explicitly, the wave action is conserved. As a corollary, it is then shown that, in homogeneous plasma carrying Langmuir oscillations, the wave field amplitude universally scales with the electron density as $\widetilde{E} \propto N^{3/4}$. In addition, we find how the wavevector evolution varies depending on the wave geometry. Particularly, during compression $\mathbf{k}$ is conserved when aligned with the average velocity $\mathbf{V}(\mathbf{r}, t)$ at homogeneous density and temperature, but otherwise changes, with its absolute value following Eqs. (68)–(70). Also, the wave frequency ω is conserved only when the plasma is stationary, but otherwise evolves according to Eq. (41).

ACKNOWLEDGMENTS

APPENDIX A: AUXILIARY VECTOR IDENTITY

In this appendix we derive an alternative form of

$$\{\mathbf{A}, \mathbf{B}\} \equiv \nabla \cdot [(\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}]$$
$$- (\mathbf{A} \cdot \nabla)(\nabla \cdot \mathbf{B}) - (\mathbf{B} \cdot \nabla)(\nabla \cdot \mathbf{A}) \tag{A1}$$

for two arbitrary fields $\mathbf{A}$ and $\mathbf{B}$. First, the expression in the square brackets above rewrites as (Ref. 74, Sec. 5.5–2)

$$(\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} = \nabla(\mathbf{A} \cdot \mathbf{B})$$
$$- \mathbf{A} \times (\nabla \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{A}); \tag{A2}$$

thus, its divergence equals

$$\nabla \cdot [(\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}]$$
$$= \nabla^2(\mathbf{A} \cdot \mathbf{B}) - \nabla \cdot [\mathbf{A} \times (\nabla \times \mathbf{B})]$$
$$- \nabla \cdot [\mathbf{B} \times (\nabla \times \mathbf{A})]. \tag{A3}$$

The chain rule for the second term on the right-hand side yields

$$\nabla \cdot [\mathbf{A} \times (\nabla \times \mathbf{B})]$$
$$= \nabla \cdot [\underline{\mathbf{A}} \times (\nabla \times \mathbf{B})] + \nabla \cdot [\mathbf{A} \times (\underline{\nabla \times \mathbf{B}})], \tag{A4}$$

where underlined are the vectors to which the differentiation by the external ∇ applies. Because of the symmetry properties of the scalar triple product (Ref. 74, Sec. 5.2–8) (here of the vectors ∇, $\mathbf{A}$, and $\nabla \times \mathbf{B}$), this also rewrites as

$$\nabla \cdot [\mathbf{A} \times (\nabla \times \mathbf{B})]$$
$$= (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) - \mathbf{A} \cdot \nabla \times (\nabla \times \mathbf{B}). \tag{A5}$$

Further use that $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$ (Ref. 74, Sec. 5.5–2); thus, Eq. (A5) and a symmetric expression for the third term on the right-hand side of Eq. (A3) can be put in the form

$$\nabla \cdot [\mathbf{A} \times (\nabla \times \mathbf{B})] = (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B})$$
$$- (\mathbf{A} \cdot \nabla)(\nabla \cdot \mathbf{B}) + \mathbf{A} \cdot \nabla^2 \mathbf{B}, \tag{A6}$$

$$\nabla \cdot [\mathbf{B} \times (\nabla \times \mathbf{A})] = (\nabla \times \mathbf{B}) \cdot (\nabla \times \mathbf{A})$$
$$- (\mathbf{B} \cdot \nabla)(\nabla \cdot \mathbf{A}) + \mathbf{B} \cdot \nabla^2 \mathbf{A}. \tag{A7}$$

Substitution of these into Eq. (A3) and then Eq. (A3) into Eq. (A1) yields

$$\{\mathbf{A}, \mathbf{B}\} = \nabla^2(\mathbf{A} \cdot \mathbf{B}) - \mathbf{A} \cdot \nabla^2 \mathbf{B} - \mathbf{B} \cdot \nabla^2 \mathbf{A}$$
$$- 2(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}). \tag{A8}$$

In Cartesian coordinates, Eq. (A8) finally rewrites as (Ref. 74, Sec. 5.5–5)

$$\{\mathbf{A}, \mathbf{B}\} = 2(\nabla A_j) \cdot (\nabla B_j) - 2(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}), \tag{A9}$$

where summation over repeated indices is assumed.

APPENDIX B: PRESSURE TERMS IN THE DENSITY EQUATION

1. General case

In this appendix we find $\nabla \cdot \mathbf{\Pi}$ as a function of n to the first order in ϵ. Start off from Eq. (9) to get

$$\nabla \cdot \mathbf{\Pi} \approx \frac{1}{m_e N} \left(i n k_j \frac{\partial P_{j\ell}}{\partial x_\ell} + i h_j k_\ell \widetilde{P}_{j\ell} - \frac{\partial^2 \widetilde{P}_{j\ell}}{\partial x_j \, \partial x_\ell} \right). \tag{B1}$$

The first term here is already of the sought form, and, since $\mathbf{h}$ is of the order of ϵ, the second term is expressed using

$$\widetilde{P}_{j\ell} \approx n\left(P_{j\ell} + P_{js}\frac{k_s k_\ell}{k^2} + P_{\ell s}\frac{k_j k_s}{k^2}\right), \tag{B2}$$

as obtained from the homogeneous stationary plasma approximation [Eqs. (15) and (20)]. However, for calculating the third term Eq. (B2) is not sufficiently accurate,[75] so $\partial^2 \widetilde{P}_{j\ell}/\partial x_j \partial x_\ell$ is found as follows.

First, take $\partial^2/\partial x_j \partial x_\ell$ of Eq. (8), neglecting the terms of order higher than ϵ. This yields

$$\frac{\partial'}{\partial t}\left(\frac{\partial^2 \widetilde{P}_{j\ell}}{\partial x_j \partial x_\ell}\right) + 3P_{j\ell}\frac{\partial(\nabla\cdot\widetilde{\mathbf{V}})}{\partial x_j \partial x_\ell} = k_\ell n\Omega\left(3\frac{\partial P_{j\ell}}{\partial x_j} + \frac{k_\ell k_s}{k^2}\frac{\partial P_{js}}{\partial x_j}\right.$$
$$\left. + \frac{3k_j k_s}{k^2}\frac{\partial P_{js}}{\partial x_\ell}\right) + k_j k_\ell(\widetilde{P}_{j\ell}W_{ss} + 2\widetilde{P}_{js}W_{\ell s} + 2W_{js}\widetilde{P}_{s\ell}). \tag{B3}$$

The second term on the left-hand side allows an alternative representation via

$$\frac{\partial}{\partial x_j \partial x_\ell}(\nabla\cdot\widetilde{\mathbf{V}}) = n\Omega h_s k_s \frac{k_j k_\ell}{k^2} - \frac{\partial^2}{\partial x_j \partial x_\ell}\left(\frac{\partial' n}{\partial t}\right) \tag{B4}$$

[see Eq. (6)], and the latter term in Eq. (B4) also equals

$$\frac{\partial^2}{\partial x_j \partial x_\ell}\left(\frac{\partial' n}{\partial t}\right) = \frac{\partial'}{\partial t}\left(\frac{\partial^2 n}{\partial x_j \partial x_\ell}\right) - n(k_j k_s W_{s\ell} + k_\ell k_s W_{sj}). \tag{B5}$$

Hence, Eq. (B3) can be put in the form

$$\frac{\partial'}{\partial t}\left(\frac{\partial^2 \widetilde{P}_{j\ell}}{\partial x_j \partial x_\ell} - 3P_{j\ell}\frac{\partial^2 n}{\partial x_j \partial x_\ell}\right) = \Psi, \tag{B6}$$

$$\Psi = k_\ell n\Omega\left(3\frac{\partial P_{j\ell}}{\partial x_j} + \frac{k_\ell k_s}{k^2}\frac{\partial P_{js}}{\partial x_j} + 3N\frac{k_j k_s}{k^2}\frac{\partial T_{js}}{\partial x_\ell}\right) + \delta\Psi, \tag{B7}$$

and $\delta\Psi$ is given by

$$\delta\Psi = 3nk_j k_\ell(\partial'_t P_{j\ell}) - 3P_{j\ell}n(k_j k_s W_{s\ell} + k_\ell k_s W_{sj})$$
$$+ k_j k_\ell(\widetilde{P}_{j\ell}W_{ss} + 2\widetilde{P}_{js}W_{\ell s} + 2W_{js}\widetilde{P}_{s\ell}). \tag{B8}$$

Using the slow component of Eq. (3) in the form

$$\partial'_t P_{j\ell} = -P_{j\ell}W_{ss} - P_{js}W_{\ell s} - P_{s\ell}W_{js}, \tag{B9}$$

and also Eq. (B2), rewrite $\delta\Psi$ as

$$\delta\Psi = -4\,n\mathbf{k}\cdot(\widehat{\mathbf{W}}\widehat{\mathbf{G}}\widehat{\mathbf{P}}\mathbf{k}), \quad \widehat{\mathbf{G}} = \widehat{\mathbf{1}} - \mathbf{kk}/k^2. \tag{B10}$$

Since Ψ is a rapidly oscillating function with a slow envelope of order ϵ, Eq. (B6) is integrated as

$$\frac{\partial^2 \widetilde{P}_{j\ell}}{\partial x_j \partial x_\ell} = 3P_{j\ell}\frac{\partial^2 n}{\partial x_j \partial x_\ell} + \frac{i\Psi}{\Omega}. \tag{B11}$$

Substitute Eq. (B11) for the third term in Eq. (B1), together with Eq. (B2) for the second term; then one gets

$$\nabla\cdot\mathbf{\Pi} = -C_{j\ell}\frac{\partial^2 n}{\partial x_j \partial x_\ell} - ik_\ell n\frac{\partial C_{s\ell}}{\partial x_j}\left(\delta_{js} + \frac{k_j k_s}{k^2}\right)$$
$$+ \frac{in}{3}\left(\frac{4}{\Omega}\mathbf{k}\widehat{\mathbf{W}} - \mathbf{h}\right)\cdot\widehat{\mathbf{G}}\widehat{\mathbf{C}}\mathbf{k}. \tag{B12}$$

Since the wave is assumed propagating along a local principal axis of the temperature tensor,[64] one has $\widehat{\mathbf{G}}\widehat{\mathbf{C}}\mathbf{k} = \widehat{\mathbf{G}}\mathbf{k}$ $\times$ scalar. Yet $\widehat{\mathbf{G}}\mathbf{k} \equiv 0$, so Eq. (B12) is simplified, and, using $ink_\ell \approx \partial n/\partial x_\ell$, one finally obtains

$$\nabla\cdot\mathbf{\Pi} = -C_{j\ell}\frac{\partial^2 n}{\partial x_j \partial x_\ell} - \left(\delta_{js} + \frac{k_j k_s}{k^2}\right)\frac{\partial C_{s\ell}}{\partial x_j}\frac{\partial n}{\partial x_\ell}. \tag{B13}$$

2. Isotropic temperature

For the isotropic temperature case, Eq. (B13) gives

$$\nabla\cdot\mathbf{\Pi} = -3\,v_{Te}^2\nabla^2 n - 6\,\nabla n\cdot\nabla v_{Te}^2, \tag{B14}$$

which as well can be obtained by substituting Eq. (80) in Sec. IV.3 of Ref. 50 into our Eq. (B1).

Alternatively, Eq. (B14) can be derived using a phenomenological adiabatic law (cf., e.g., Ref. 76, Sec. 5.1 of Ref. 77, Sec. 3.5 of Ref. 52)

$$(\partial_t + \mathbf{V}_e\cdot\nabla)(p_e N_e^{-\gamma}) = 0, \tag{B15}$$

with p_e being the electron scalar pressure, including the slow and the quiver parts: $p_e = p + \widetilde{p}$. [Here $\gamma = 3$ (corresponding to one-dimensional adiabatic oscillations[70]) is an extrapolation from the homogeneous plasma case, for which the exact solution is known from a more rigorous hydrodynamic treatment, like in our Sec. III or Refs. 48–50, or the complete kinetic treatment (Ref. 52, Chap. 8 and Ref. 53).] To show this, introduce the plasma element Lagrangian displacement $\boldsymbol{\xi}$ such that[41,78,79]

$$\widetilde{N} + \nabla\cdot(\boldsymbol{\xi}N) = 0, \tag{B16}$$

$$\widetilde{p} + \boldsymbol{\xi}\cdot\nabla p = m_e a^2(\widetilde{N} + \boldsymbol{\xi}\cdot\nabla N), \tag{B17}$$

where $a^2 = 3v_{Te}^2$. Then, from Eq. (9) with $\widehat{\mathbf{P}} = p\widehat{\mathbf{1}}$ and $\widehat{\widetilde{\mathbf{P}}} = \widetilde{p}\widehat{\mathbf{1}}$, one gets

$$\mathbf{\Pi} = -\nabla(na^2) - [n\mathbf{q} + \nabla(\boldsymbol{\xi}\cdot\mathbf{q})]/N. \tag{B18}$$

Here $\mathbf{q} \equiv a^2\nabla N - \nabla p/m_e$ is of the order of ϵ, and therefore one can take $\partial'_t \boldsymbol{\xi} \approx \widetilde{\mathbf{V}}$ [cf., e.g., Eq. (4.6) in Ref. 79], so $\boldsymbol{\xi} \approx i n\mathbf{k}/k^2$, as follows from Eq. (B16). Hence,

$$\mathbf{\Pi} \approx -\nabla(na^2) - nN^{-1}\widehat{\mathbf{G}}\mathbf{q}, \tag{B19}$$

yielding

$$\nabla\cdot\mathbf{\Pi} \approx -\nabla^2(na^2) - inN^{-1}\mathbf{k}\widehat{\mathbf{G}}\cdot\mathbf{q}. \tag{B20}$$

Using that $\mathbf{k}\widehat{\mathbf{G}} \equiv 0$, one finally obtains

$$\nabla\cdot\mathbf{\Pi} \approx -a^2\nabla^2 n - 2\,\nabla n\cdot\nabla a^2, \tag{B21}$$

which is equivalent to Eq. (B14).

APPENDIX C: PONDEROMOTIVE PRESSURE

The conservation of the Langmuir wave action allows one to calculate the effective stress tensor due to the wave, which is done as follows (see also Refs. 67, 69, and 80). For simplicity, suppose homogeneous cold stationary plasma volume $\mathcal{V}$ and assume that it is adiabatically deformed as defined by an infinitesimal displacement field $\boldsymbol{\xi}(\mathbf{r})$, resulting in the strain tensor

$$\hat{\mathbf{w}} = [(\nabla\boldsymbol{\xi}) + (\nabla\boldsymbol{\xi})^{\mathrm{T}}]/2. \tag{C1}$$

Hence the stress tensor $\hat{\boldsymbol{\sigma}}$ is found:[81]

$$\sigma_{j\ell} = \frac{1}{\mathcal{V}}\frac{\partial\mathfrak{E}}{\partial w_{j\ell}}, \tag{C2}$$

where $\mathfrak{E}=\mathcal{V}J\omega$ is the wave total energy inside $\mathcal{V}$. Because $\mathcal{V}J$ is conserved, Eq. (C2) rewrites as

$$\sigma_{j\ell} = \frac{J\omega_p}{2N}\frac{\partial N}{\partial w_{j\ell}}, \tag{C3}$$

where we used that $\omega=\omega_p(N)$. The density perturbation due to the strain is $\delta N=-N\nabla\cdot\boldsymbol{\xi}$ [cf. Eq. (B16)]. Since $\nabla\cdot\boldsymbol{\xi}=w_{ss}$, and $\partial w_{ss}/\partial w_{j\ell}=\delta_{j\ell}$, this yields

$$\hat{\boldsymbol{\sigma}} = -(J\omega_p/2)\hat{\mathbf{1}}. \tag{C4}$$

Therefore the stress due to the wave field is isotropic and appears as an effective pressure $p_E=J\omega_p/2$. (Thermal correction would also yield an anisotropic component to the wave stress tensor.[67,69]) Using Eq. (64), one then recovers the expression [Eq. (66)] for the ponderomotive pressure in cold plasma carrying a field $\widetilde{\mathbf{E}}$ which oscillates at the frequency $\omega=\omega_p$.[69]

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[75] This point is missed in Appendix C of Ref. 46, where the conductivity of inhomogeneous nonstationary anisotropic unmagnetized plasma is calculated. Hence, the corresponding corrections due to the temperature do not comply with PCT.

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