Damping of linear waves via ionization and recombination in homogeneous plasmas

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An oscillation-center model is proposed that analytically describes transformation of an arbitrary homogeneous linear wave at gradual ionization and recombination in homogeneous plasma. For the case when either of the processes dominates, general adiabatic invariants are found, from which the wave energy is derived as a function of the frequency. © *2010 American Institute of Physics*. [doi:10.1063/1.3514166]

I. INTRODUCTION

The problem of wave transformation driven by gradual ionization and recombination (I-n-R) has been attracting attention for decades; see, e.g., Refs. 1–16 and, in particular, Refs. 4–7 for broader reviews. Unlike for plasma compression within the geometrical optics approximation, in which case the wave action conservation is anticipated from basic principles[8,17–21] and was also confirmed through independent calculations,[22–31] no scalings were reported for I-n-R that would describe the evolution of the wave amplitude in general (cf. Ref. 7). However, from *ad hoc* results reported in Refs. 1–6, one may expect that fundamental relations exist between the energy density $\mathcal{E}$ of a homogeneous wave and its frequency ω [also evolving through I-n-R (Refs. 4 and 32)], when ionization dominates over recombination and vice versa.

To find these fundamental relations is the purpose of this paper. In particular, we focus on the low-frequency limit of above-threshold photoionization,[33–35] so there are no external sources of energy and the dynamics of free charges can be described classically within the so-called simple-man model (SMM).[36–39] (Recombination is addressed similarly.) Instead of solving the Maxwell's equations directly, as in Refs. 1–6, it is then possible to employ a more abstract oscillation-center (OC) formalism developed in our Ref. 40, assuming that collisional damping is negligible. Specifically, we connect the reduction of the wave energy via I-n-R with the increase of the OC energy of free plasma particles [Eqs. (5) and (14)], which are then expressed through (yet not necessarily equals) the corresponding ponderomotive potentials [Eqs. (10) and (15)], and the latter are linked to the shift of ω [Eq. (11)] as explained in Ref. 40.

As a result, we find, without referring to the wave dispersion relation or a specific dielectric tensor, the general adiabatic invariants yielding $\mathcal{E}(\omega)$ for I-n-R when either ionization or recombination dominates. In particular, we show that, at given I-n-R rates, the corresponding relations are determined entirely by the linear polarizabilities of plasma particles $\hat{\alpha}_s$ (Ref. 40) projected on the wave polarization $\mathbf{e}$. This explains why any waves in nonmagnetized cold homogeneous plasma, where $\hat{\alpha}_s$ are isotropic, exhibit the same $\mathcal{E}(\omega)$,[1–5] whereas in magnetized plasma, where $\hat{\alpha}_s$ are essentially anisotropic, the scalings are $\mathbf{e}$-dependent.[6] Overall, our model unfolds, in simple terms yet quantitatively, the general physics behind the *ad hoc* calculations that were reported in Refs. 1–6 and reproduces their results as particular cases (Table I).

The paper is organized as follows. In Sec. II, we formulate our general approach to the problem. In Sec. III, we derive the adiabatic invariants and $\mathcal{E}(\omega)$ for I-n-R when ionization dominates over recombination or vice versa. In Sec. IV, we consider the combined effect of I-n-R on linear waves in plasmas. In Sec. V, we summarize our main results and discuss the possibility of further extension of the results reported here.

II. GENERAL APPROACH

Suppose a plasma with a traveling linear wave inside and consider an electron-ion pair produced through above-threshold ionization at some time $t=t_*$ and location $\mathbf{x}=\mathbf{x}_*$. Assume that the ionization potential (IP) is much smaller than the energies to be gained by the particles in the wave field. Then the electron-ion interaction is negligible.[41] However, according to SMM, it does determine the initial conditions for the free motion of the particles at $t>t_*$, specifically as follows. Suppose the pair is produced, say, from a neutral atom which had some velocity $\mathbf{v}_n$. Then, the electron and ion velocities $\mathbf{v}_s$ satisfy $\mathbf{v}_s(t_*)=\mathbf{v}_n$,[36,42–44] s being the species index ($s=e,i$). Yet, $\mathbf{v}_s$ can be decomposed into the OC velocity $\mathbf{V}_s$ and the quiver velocity

$$\tilde{\mathbf{v}}_s=(\omega/q_s)\mathrm{Im}(\hat{\alpha}_s\cdot\tilde{\mathbf{E}}), \tag{1}$$

where q_s is the charge, and $\hat{\alpha}_s$ can be understood as the polarizability tensor[45–48] determining the particle displacement from its average (OC) location in response to the wave electric field $\tilde{\mathbf{E}}=\mathbf{E}(\mathbf{x},t)e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}$, with $\mathbf{E}$ being the envelope. Therefore

$$\mathbf{V}_s=\mathbf{v}_n-\tilde{\mathbf{v}}_s(\mathbf{x}_*,t_*) \tag{2}$$

[Fig. 1(a)]. It is seen then that the induced OC current and temperature perturbations inherit the wave spatial and temporal periodicity. As a result, the wave itself is affected, thereby developing harmonics in both space and time. We will require, though, that the ionization be *gradual*, i.e., have

TABLE I. Adiabatic invariants conserved at ionization ($\mathrm{inv_i}$) and recombination ($\mathrm{inv_r}$) for different types of waves in cold plasma. Here $\mathcal{E}$ is the wave energy density, ω is the wave frequency, and Ω_s is the gyrofrequency of resonant species s. The I-n-R rates are assumed independent of the field phase.

Wave types	$\mathrm{inv_i}$	$\mathrm{inv_r}$
Langmuir	$\mathcal{E}\omega$	$\mathcal{E}\omega^{-1}$
Electromagnetic, nonmagnetized	$\mathcal{E}\omega$	$\mathcal{E}\omega^{-1}$
R and L, parallel propagation	$\mathcal{E}\lvert\omega\pm\Omega_s\rvert$	$\mathcal{E}\lvert\omega\pm\Omega_s\rvert^{-1}$
Alfvén, shear and compressional	$\mathcal{E}\omega^{-1}$	$\mathcal{E}\omega$

rate smaller than the least frequency gap that separates the wave from other modes. (In particular, this means that the initial density must be nonzero, unlike in Refs. 11–15.) Hence, other modes are not excited, and the harmonics are weak,[49] so the field exhibits well-defined frequency ω and wavevector $\mathbf{k}$, assuming that the envelope $\mathbf{E}$ is smooth in both space and time (cf. Ref. 4).

Suppose further that the plasma is cold,[50] namely

$$\frac{kv_{Ts}}{\omega} \ll 1, \quad \frac{k_\parallel v_{Ts}}{\omega-\ell\Omega_s} \ll 1, \quad \frac{k_\perp v_{Ts}}{\Omega_s} \ll 1, \tag{3}$$

where v_{Ts} is the thermal velocity of each species s, and the two latter inequalities apply at nonzero quasistatic magnetic field $\mathbf{B}_0$; in that case, Ω_s are the corresponding gyrofrequencies, ℓ are integers, and $k_\parallel$ and $k_\perp$ are the components of $\mathbf{k}$ parallel and perpendicular to $\mathbf{B}_0$, respectively. The conditions (3) provide that the time it takes for particles to traverse a field inhomogeneity is larger than any characteristic period of the particle oscillations.[51] Hence, the motion of all free species is adiabatic. This means that, both at $t<t_*$ and $t>t_*$, the plasma can be described by the OC Hamiltonian[40]

$$H=\mathfrak{E}+\mathfrak{H}+U, \tag{4}$$

where $\mathfrak{E}$ is the wave energy,[52] $\mathfrak{H}$ is the energy of quasistatic fields, and U is the sum over the OC kinetic energies K of all plasma particles, to which neutrals can also be added. On the other hand, since the IP is negligible, H equals the system total energy, which is conserved also at $t=t_*$. [Since $\mathbf{v}_s(t)$ are continuous, $\dot{\mathbf{v}}_s$ do not exceed those for free oscillations; hence, Larmor radiation is neglected.] In this case, the conservation of H yields that $\delta\mathfrak{E}$ caused by ionization is given by $\delta\mathfrak{E}=-\delta\mathfrak{H}-\delta U$.

Suppose further that the bulk plasma is homogeneous and so is $\mathbf{B}_0$, if present. In the latter case, assume also the presence of conducting walls conserving the magnetic flux. Hence, the magnetic field energy remains fixed, and there is no electric field, so $\delta\mathfrak{H}=0$. Then

$$\delta\mathfrak{E}=-\,\delta U, \tag{5}$$

which is a generalized formulation of the fact that, at negligible IP, wave damping at ionization is caused by heating of newly born plasma particles.[4,44,53–58] (Notice that this damping mechanism is missing in the equations of pulse propagation derived in Refs. 59 and 60.) Remarkably, the quiver motion energy that these particles gain does not contribute to $\delta\mathfrak{E}$ because the former, by definition, is a part of $\mathfrak{E}$ itself.

To find δU that enters Eq. (5), suppose that the wave envelope is homogeneous in the whole volume $\mathcal{V}$, so the OC canonical momenta are conserved for all free particles.[40] For plasmas that we consider, this means that the corresponding $\mathbf{V}_s$ are unaffected too, and, since $K_s=\frac{1}{2}m_sV_s^2$ in our case (yet not in general[40]), the same applies to K_s. (For magnetized particles, the energy of Larmor rotation is also assumed included in K_s.) Hence, δU caused by ionization is entirely due to participating particles, so it can be derived from Eq. (2) explicitly. In what follows, we offer such a calculation for plasmas where each ion species is allowed to have only one charge state, sacrificing generality for the sake of clarity. Notice, however, that a general calculation is also straightforward within the framework outlined here.

III. ADIABATIC INVARIANTS

A. Ionization damping

From Eq. (2), the OC energy gained by the electron at $t=t_*$ equals $K_e=\frac{1}{2}m_e(\mathbf{v}_n-\tilde{\mathbf{v}}_e)^2$; also, the ion gains $K_i=\frac{1}{2}m_i(\mathbf{v}_n-\tilde{\mathbf{v}}_i)^2$, and the neutral that disappears loses $K_n=\frac{1}{2}(m_e+m_i)V_n^2$, where $\mathbf{V}_n=\mathbf{v}_n$.[61] Then, within time δt, the variation of U can be put as $\delta U=\Sigma_s\delta U_s$, where summation is taken over all charged species, and

$$\delta U_s=\int d^3x'\int_t^{t+\delta t}dt'\left\langle\left(\frac{1}{2}m_s\tilde{v}_s^2-m_s\mathbf{v}_n\cdot\tilde{\mathbf{v}}_s\right)w_s\right\rangle_n,$$

where w_s is the probability of producing species s through ionization per unit time per unit volume, and $\langle\ldots\rangle_n$ denotes averaging over $\mathbf{v}_n$. Under the conditions (3), $\hat{\alpha}_s$ does not depend on $\mathbf{v}_n$,[45] and thus neither does $\tilde{\mathbf{v}}_s$. For above-threshold ionization, one can expect w_s to be independent of $\mathbf{v}_n$ too, particularly when $v_n\ll\tilde{v}_s$. Hence, the second term in the above equation averages to zero. (Of course, at small v_n, the latter can as well be neglected compared to the first term.) Then

$$\delta U_s=\frac{1}{2}\eta_s m_s\langle\tilde{v}_s^2\rangle\mathcal{V}\delta n_s, \qquad \eta_s\equiv\frac{\langle w_s\tilde{v}_s^2\rangle}{\langle w_s\rangle\langle\tilde{v}_s^2\rangle}. \tag{6}$$

Here $\langle\ldots\rangle$ denotes averaging over both $\mathbf{x}$ and t, δn_s is the average perturbation to the density n_s due to ionization during δt, and η_s reflects the dependence of w_s on the field phase; without such a dependence one has $\eta_s=1$.

It is convenient to express δU_s in terms of the ponderomotive potential Φ_s, which is given by[40,47]

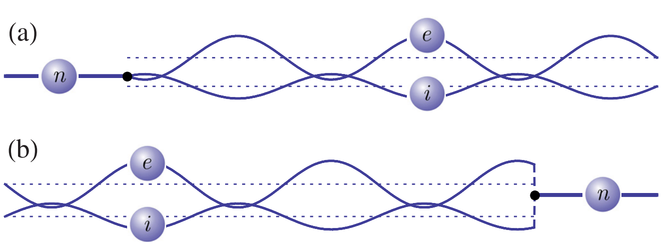

FIG. 1. (Color online) Schematic of the particle velocities $\mathbf{v}_s(t)$ at (a) ionization, (b) recombination. Dotted are the OC velocities $\mathbf{V}_s$.

$$\Phi_s = -\frac{1}{4}(\mathbf{E}^* \cdot \hat{\alpha}_s \cdot \mathbf{E}). \quad (7)$$

After multiplying and dividing Eq. (6) by Φ_s, one obtains, using Eq. (1), that

$$\delta U_s = \frac{\eta_s |\hat{\alpha}_s \cdot \mathbf{E}|^2}{a_s(\mathbf{E}^* \cdot \hat{\alpha}_s \cdot \mathbf{E})} \Phi_s \mathcal{V} \delta n_s, \quad (8)$$

where $a_s \equiv -q_s^2/(m_s \omega^2)$ can be understood as the polarizability of a free cold particle with mass m_s and charge q_s. Introducing the unit polarization vector $\mathbf{e}$, one can then rewrite Eq. (8) as $\delta U_s = \kappa_s \Phi_s \mathcal{V} \delta n_s$, where

$$\kappa_s = \frac{\eta_s |\hat{\alpha}_s \cdot \mathbf{e}|^2}{a_s(\mathbf{e}^* \cdot \hat{\alpha}_s \cdot \mathbf{e})}. \quad (9)$$

Therefore, Eq. (5) yields for $\mathcal{E} \equiv \mathfrak{E}/\mathcal{V}$ that

$$\delta\mathcal{E} = -\delta n_e \sum_s \kappa_s \Phi_s \vartheta_s, \quad (10)$$

where $\vartheta_s = \delta n_s/\delta n_e$ is the number of particles of type s produced through ionization per one electron. (When particles eventually leave the field, there is also an additional change of the wave energy; e.g., for stationary adiabatic $\mathbf{E}$, $\mathfrak{E}$ is further reduced by Φ_s per particle.[40])

If recombination is negligible, δn_e can be unambiguously expressed through the wave frequency shift, $\delta\omega = \delta n_e \Sigma_s (\partial\omega/\partial n_s)\vartheta_s$. On the other hand, $\partial\omega/\partial n_s$ can, in turn, be expressed through Φ_s, due to[40]

$$\Phi_s = (\mathcal{E}/\omega)(\partial\omega/\partial n_s). \quad (11)$$

This yields $\delta\omega = (\omega/\mathcal{E})\delta n_e \Sigma_s \Phi_s \vartheta_s$. Dividing Eq. (10) by the latter, one obtains then

$$\frac{d\mathcal{E}}{d\omega} = -\rho\frac{\mathcal{E}}{\omega}, \quad \rho = \frac{\Sigma_s(\mathbf{e}^* \cdot \hat{\alpha}_s \cdot \mathbf{e})\kappa_s \vartheta_s}{\Sigma_{s'}(\mathbf{e}^* \cdot \hat{\alpha}_{s'} \cdot \mathbf{e})\vartheta_{s'}}. \quad (12)$$

Therefore, at given ionization rates, $\mathcal{E}(\omega)$ is determined entirely by the plasma particle linear polarizabilities $\hat{\alpha}_s$ projected on the wave polarization $\mathbf{e}$.

Since the above result relies on the conservation of the OC energy of free charges, which are adiabatic invariants,[62] the constant of integration of Eq. (12) also can be called an *adiabatic invariant* of the ionization process, henceforth denoted $\mathrm{inv_i}$. In particular, when ρ is independent of the wave amplitude, $\mathrm{inv_i}$ can be put as

$$\mathcal{E} \exp\left[\int_{\omega_0}^{\omega} d\omega' \rho(\omega')/\omega'\right] = \mathrm{inv_i}, \quad (13)$$

where ω_0 is the initial frequency. This generalizes the results of Refs. 1–6 pertaining to the evolution of the homogeneous wave energy at gradual ionization (for the case when collisional damping is negligible[63]).

B. Recombination damping

Consider also wave damping through recombination. Accounting for the energy loss δW due to bremsstrahlung, Eq. (5) now rewrites as

$$\delta\mathfrak{E} = -\delta U - \delta W, \quad (14)$$

where we assume that the plasma is optically thin for the corresponding radiation. Assuming further that this radiation is emitted at frequencies much larger than ω, recombination can be treated as an instantaneous event; hence, δU and δW can be calculated classically as follows.

Consider recombination of an electron-ion pair producing a neutral with some velocity $\mathbf{v}_n$ at some $t = t_*$. Under our assumptions, the interaction between the electron and the ion is negligible at $t < t_*$, and at $t = t_*$ their velocities jump from $\mathbf{v}_s(t_* - 0) \equiv \mathbf{u}_s$ to $\mathbf{v}_s(t_* + 0) = \mathbf{v}_n$ [Fig. 1(b)]. Then, $\delta U = \Sigma_s\left(\frac{1}{2}m_s v_n^2 - K_s\right)$ and $\delta W = \Sigma_s\left(\frac{1}{2}m_s u_s^2 - \frac{1}{2}m_s v_n^2\right)$, where K_s are the OC initial kinetic energies of the charged particles participating in recombination. Substituting $\mathbf{u}_s = \mathbf{V}_s + \tilde{\mathbf{v}}_s$, one further gets $\delta\mathfrak{E} = -\Sigma_s\left(\frac{1}{2}m_s \tilde{v}_s^2 + m_s \mathbf{V}_s \cdot \tilde{\mathbf{v}}_s\right)$. Hence, averaging over time, space, and $\mathbf{V}_s$ yields [cf. Eq. (10)]

$$\delta\mathcal{E} = -|\delta n_e| \sum_s \bar{\kappa}_s \Phi_s \bar{\vartheta}_s, \quad (15)$$

where $\bar{\kappa}_s$ is given by Eq. (9), like κ_s, except the corresponding coefficient $\bar{\eta}_s$ is now a functional of the recombination probability $\bar{w}_s$ rather than w_s. Similarly, $\bar{\vartheta}_s$ replaces ϑ_s. Also, δn_e is now negative, so one obtains

$$\frac{d\mathcal{E}}{d\omega} = \bar{\rho}\frac{\mathcal{E}}{\omega}, \quad \bar{\rho} = \frac{\Sigma_s(\mathbf{e}^* \cdot \hat{\alpha}_s \cdot \mathbf{e})\bar{\kappa}_s \bar{\vartheta}_s}{\Sigma_{s'}(\mathbf{e}^* \cdot \hat{\alpha}_{s'} \cdot \mathbf{e})\bar{\vartheta}_{s'}}. \quad (16)$$

Therefore, analogously to Eq. (13), the adiabatic invariant conserved at recombination reads as

$$\mathcal{E} \exp\left[-\int_{\omega_0}^{\omega} d\omega' \bar{\rho}(\omega')/\omega'\right] = \mathrm{inv_r}, \quad (17)$$

assuming $\bar{\rho}$ is independent of the wave amplitude. This generalizes the results of Refs. 1–3, 5, and 6 pertaining to the evolution of the homogeneous wave energy at gradual recombination (for the case when collisional damping is negligible[63]).

C. Examples

Now let us find $\mathcal{E}(\omega)$ explicitly for some basic oscillations. Suppose plasma with single species of ions, $q_i = -q_e$; hence, $\vartheta_s = \bar{\vartheta}_s = 1$. Although precise I-n-R rates could be used,[10,64] for simplicity adopt a model $\eta_s = \bar{\eta}_s = 1$ (so $\bar{\rho} = \rho$); yet, it is still accurate for circularly polarized waves in particular.[4] First, consider Langmuir and electromagnetic waves in nonmagnetized plasma.[50] In this case, one can neglect the ion contribution, so $\rho = \kappa_e = 1$, where we used that

$$\hat{\alpha}_e \approx a_e \hat{\mathrm{I}}, \quad (18)$$

with $\hat{\mathrm{I}}$ being the unit tensor. [Correspondingly, one obtains the well-known ponderomotive potential $\Phi_e = q_e|E|^2/(4m_e\omega^2)$,[65] which, in this particular case, also happens to equal the average energy of the electron quiver motion, $\frac{1}{2}m_e\langle\tilde{v}_e^2\rangle$.] Thus, Eqs. (13) and (17) yield

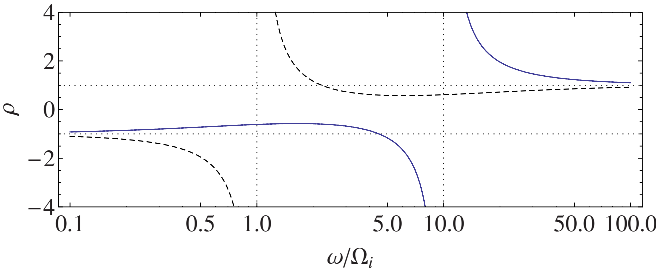

FIG. 2. (Color online) ρ [Eq. (22)] vs ω/Ω_i for $\mathbf{k}\|\mathbf{B}_0$ in plasma with single species of ions ($m_i=10m_e$). Solid: R-wave, dashed: L-wave, dotted: asymptotes [$\rho(\omega\to 0,\infty)\to\mp 1$].

$$\mathcal{E}\omega=\mathrm{inv}_{\mathrm{i}},\quad \mathcal{E}\omega^{-1}=\mathrm{inv}_{\mathrm{r}},\tag{19}$$

and, since $\partial\omega/\partial n_e>0$ for either waves, $\mathcal{E}$ always decreases, as expected. Notice that, unlike in Refs. 1–5, where these scalings are derived *ad hoc* and thus require modeling plasma dynamics, in our case Eq. (19) is general and flows uniformly for any waves in cold nonmagnetized homogeneous plasma. This is due to the fact that $\hat{\alpha}_e$ is isotropic in such plasma [Eq. (18)], and thus $\mathcal{E}(\omega)$ cannot depend on the wave polarization, so neither does it depend on the dispersion relation, as explained above.

As another example, consider a plasma in a static magnetic field $\mathbf{B}_0=\mathbf{z}^0B_0$ (we take $B_0>0$), in which case

$$\hat{\alpha}_s=a_s\begin{pmatrix}\dfrac{1}{1-b_s^2} & \dfrac{ib_s}{1-b_s^2} & 0\\ \dfrac{-ib_s}{1-b_s^2} & \dfrac{1}{1-b_s^2} & 0\\ 0 & 0 & 1\end{pmatrix},\tag{20}$$

where $b_s\equiv\Omega_s/\omega$. Suppose $\mathbf{k}$ is parallel to $\mathbf{B}_0$. Then, the polarization vector $\mathbf{e}$ must be an eigenvector of $\hat{\alpha}_s$

$$\mathbf{e}=\tfrac{1}{\sqrt{2}}(\mathbf{x}^0\pm i\mathbf{y}^0),\quad \alpha_{s\pm}=a_s(1\pm b_s)^{-1},\tag{21}$$

where $\alpha_{s\pm}$ are the associated eigenvalues, and the sign $\pm$ corresponds to R- and L-waves, thereby circularly polarized.[50] A straightforward calculation hence yields

$$\rho=\frac{\omega^2+|\Omega_i\Omega_e|}{(\omega\pm\Omega_i)(\omega\pm\Omega_e)}\tag{22}$$

(Fig. 2). Thus, in agreement with Ref. 6, the wave energy density satisfies

$$\mathcal{E}|(\omega\pm\Omega_i)(\omega\pm\Omega_e)|/\omega=\mathrm{inv}_{\mathrm{i}},\tag{23a}$$

$$\mathcal{E}\omega/|(\omega\pm\Omega_i)(\omega\pm\Omega_e)|=\mathrm{inv}_{\mathrm{r}}.\tag{23b}$$

This means that, close to the resonance $\omega\approx|\Omega_s|$ (here $s=e$ for R-wave and $s=i$ for L-wave), one gets[66]

$$\mathcal{E}|\omega\pm\Omega_s|=\mathrm{inv}_{\mathrm{i}},\quad \mathcal{E}|\omega\pm\Omega_s|^{-1}=\mathrm{inv}_{\mathrm{r}}.\tag{24}$$

Since $(\omega-|\Omega_s|)$ has the same sign as Φ_s, and Φ_s has the same sign as $\partial\omega/\partial n_s$ [Eq. (11)], ionization results in $\delta\omega$ being of the same sign as $(\omega-|\Omega_s|)$. Hence, ω moves *away* from the resonance in this case, thereby causing $\mathcal{E}$ to decrease. Similarly, at recombination ω moves *toward* the resonance, and $\mathcal{E}$ decreases as well (Fig. 3).

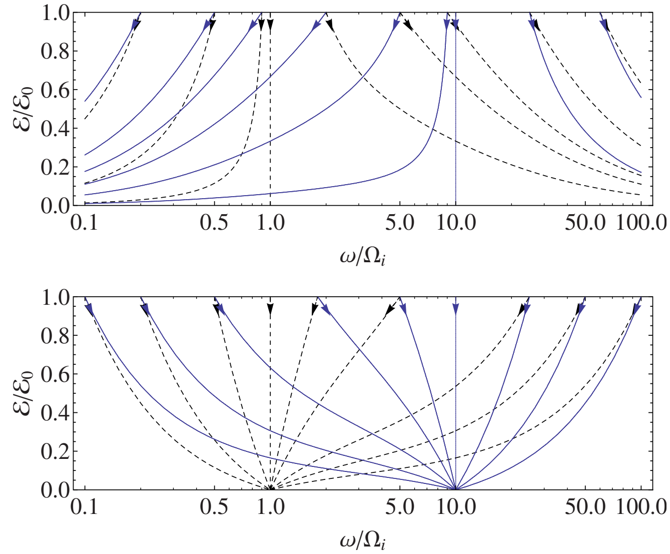

FIG. 3. (Color online) $\mathcal{E}$ [Eqs. (23)] vs ω/Ω_i for $\mathbf{k}\|\mathbf{B}_0$ in plasma with single species of ions ($m_i=10m_e$), for different initial ω. Top: ionization, bottom: recombination, solid; R-wave, dashed: L-wave. The arrows show in which direction the wave evolves; $\mathcal{E}_0$ is the wave initial energy density.

Other asymptotics of Eq. (23) are as follows. At $\omega\gg|\Omega_e|$, Eq. (19) is recovered. At $\omega\ll|\Omega_i|$, corresponding to Alfvén waves, inverse scalings are obtained;

$$\mathcal{E}\omega^{-1}=\mathrm{inv}_{\mathrm{i}},\quad \mathcal{E}\omega=\mathrm{inv}_{\mathrm{r}}.\tag{25}$$

Formally, Eq. (25) can also be shown to hold at nonzero angles between $\mathbf{k}$ and $\mathbf{B}_0$, when the wave polarization is no longer circular.[50] However, in this case I-n-R rates become essentially phase-dependent, so the calculation must be revised to account for nonunit η.

IV. IONIZATION AND RECOMBINATION COMBINED

Finally, consider I-n-R coexisting with comparable rates, ν_e and $\bar{\nu}_e$, defined through $\dot{n}_e=(\nu_e-\bar{\nu}_e)n_e$. In this case, $\mathcal{E}$ cannot be expressed as a function of ω, so we search for $\mathcal{E}(t)$ instead. From Eqs. (10) and (15), one gets

$$\dot{\mathcal{E}}=-\nu_e n_e\sum_s\kappa_s\Phi_s\vartheta_s-\bar{\nu}_e n_e\sum_s\bar{\kappa}_s\Phi_s\bar{\vartheta}_s.\tag{26}$$

After substituting Eq. (11), this yields $\dot{\mathcal{E}}=-\gamma\mathcal{E}$, where

$$\gamma=\nu_e n_e\sum_s\kappa_s\vartheta_s\frac{\partial\ln\omega}{\partial n_s}+\bar{\nu}_e n_e\sum_s\bar{\kappa}_s\bar{\vartheta}_s\frac{\partial\ln\omega}{\partial n_s},\tag{27}$$

and, for single species of ions with $n_e=n_i$ and $\eta_s=\bar{\eta}_s$,

$$\gamma=(\nu_e+\bar{\nu}_e)\sum_{s=e,i}\kappa_s\frac{\partial\ln\omega}{\partial\ln n_s}.\tag{28}$$

From our derivation, it follows that $\gamma>0$, in particular, even when the electron density is fixed on average ($\nu_e=\bar{\nu}_e$), I-n-R still cause the wave to decay, unless an additional source of energy is present. (This is a kinetic effect, which may not be captured by hydrodynamic models like in Refs. 67–69.) Nevertheless, I-n-R can render a wave unstable through nonlinear effects redistributing its energy in space; see, e.g., Refs. 70–73.

V. DISCUSSION

What we propose here is an analytical model which describes I-n-R damping of arbitrary linear waves in plasmas, assuming that I-n-R rates are sufficiently small. Specifically, we connect the reduction of the wave energy via I-n-R with the increase of the OC energy of free plasma particles [Eqs. (5) and (14)], which is then expressed through (yet not necessarily equals) the corresponding ponderomotive potentials [Eqs. (10) and (15)], and the latter are linked to the shift of ω [Eq. (11)] as explained in Ref. 40. As a result, the general adiabatic invariants or $\mathcal{E}(\omega)$ (when either ionization or recombination dominates) are derived without referring to the wave dispersion relation or a specific dielectric tensor. It is shown that, at given I-n-R rates, wave damping is entirely determined by the plasma particle linear polarizabilities $\hat{\alpha}_s$ projected on the wave polarization $\mathbf{e}$. This explains why any waves in nonmagnetized cold plasma, where $\hat{\alpha}_s$ are isotropic, exhibit the same $\mathcal{E}(\omega)$,[1–5] whereas in magnetized plasma, where $\hat{\alpha}_s$ are essentially anisotropic, the scalings are $\mathbf{e}$-dependent.[6]

The previously known expressions for $\mathcal{E}(\omega)$ (Refs. 1–6) are reproduced as particular cases under the assumption that I-n-R rates are fixed over the wave period [Table I; see also Eq. (23)]. For electromagnetic waves, this assumption is justified for circular polarization, and otherwise is adopted as a model. Notice, however, that the approach presented here permits further generalization to any polarization and arbitrary (yet small enough) I-n-R rates, like in Refs. 4, 10, and 64, as well as multiple charge states of ions. Finite-temperature plasmas can also be described, by introducing the velocity dependence in $\hat{\alpha}_s$ and modifying[40] the expressions for the OC kinetic energies K_s correspondingly. Similarly, relativistic effects can be modeled within the OC approach,[74,75] too. (Then the problem becomes somewhat similar to electron-positron pair production.[76]) However, notice that the wave dispersion may be non-negligible in this case even at low densities,[77] contrary to, e.g., Refs. 78 and 79.

ACKNOWLEDGMENTS

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