PHYSICAL REVIEW D **82,** 044044 (2010)

# On the evolution of linear waves in cosmological plasmas

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(Received 18 June 2010; published 27 August 2010)

The scalings for basic plasma modes in the Friedmann-Robertson-Walker model of the expanding Universe are revised. Contrary to the existing literature, the wave collisionless evolution must comply with the action conservation theorem. The proper steps to deduce the action conservation from *ab initio* analytical calculations are presented, and discrepancies in the earlier papers are identified. In general, the cosmological wave evolution is more easily derived from the action conservation in the collisionless limit, whereas when collisions are essential, the statistical description must suffice, thereby ruling out the need for using dynamic equations in either case.

DOI: 10.1103/PhysRevD.82.044044 PACS numbers: 04.40.Nr, 52.35.−g, 95.30.Qd, 98.80.Jk

## I. INTRODUCTION

As the Universe has been expanding after the big bang, the wave fluctuations seeded by the primordial thermal noise have been evolving accordingly, adjusting to the changing metric. Scalings that would describe the modification of the wave amplitudes for various types of oscillations have been of interest for decades [1–9]. However, our finding here is that many of those reported earlier are, in fact, erroneous, the reason being that the underlying plasma hydrodynamics was applied incorrectly.

Used in the aforementioned literature is the collisionless hydrodynamics. Contrary to Ref. [4], without solving dynamic equations the wave evolution *can* be predicted in this case, specifically, from the action conservation theorem (ACT). The latter is well known from the geometrical optics [10–17] and was also independently rederived *ab initio* for a variety of oscillations [10,11,18–24], confirming the general treatment. For prescribed curved spacetime, ACT was also proved explicitly in Ref. [25] and can therefore serve as an independent test for *ab initio* theories based on collisionless models [26].

In fact, ACT, being robust, may be preferable for deducing the wave evolution over solving dynamic equations. This is because, even for basic waves, dynamic equations are often incomparably cumbersome yet cannot be simplified significantly without losing the Hamiltonian nature of the waves, and thereby introducing unphysical dissipation; cf., e.g., Refs. [18,27]. As we explain below, those are such inaccurate approximations that caused the earlier calculations to result in erroneous scalings. For example, oversimplified dispersion relations were used that would not hold even in a fixed metric. Also, collisionless equations were utilized while collisional effects were nonetheless implied and, unless the action is conserved, should have been explicitly accounted for.

Thus, our main point here is that the cosmological wave evolution is more easily derived from ACT in the collisionless limit, whereas when collisions are essential, the statistical description [28] must suffice, thereby ruling out the need for using dynamic equations in either case. However, it is still instructive and, perhaps, more persuasive to demonstrate how ACT works in the former case. Hence, using basic modes in nonrelativistic plasma as examples, we show the proper steps to derive hydrodynamic equations which, albeit approximate, nevertheless do comply with ACT for an arbitrary expansion rate. For relativistic temperatures, though, a full Vlasov treatment would be needed, so it is not discussed here. Yet, a sufficiently general formalism suitable to perform such calculations was developed recently in our Ref. [29] (in addition, see Ref. [4]), which also covers the general theory underlying the present work; however, notice the amended notation.

The paper is organized as follows. In Sec. II, we introduce the spacetime geometry by adjusting the corresponding discussion in Ref. [29] for the synchronous metric, which we will assume after Refs. [1–8]. In Sec. III, we discuss ACT in application to this metric and consider the evolution of several plasma modes, showing that those do comply with ACT. In Sec. IV, we discuss discrepancies in the related literature in more detail. In Sec. V, we summarize our main results. Supplementary materials, including Maxwell's equations and a discussion on the particle momentum and temperature evolution in an expanding metric, are given in the appendixes.

## II. SPACETIME GEOMETRY

### A. General relations

Suppose a general synchronous metric tensor $g_{\mu\nu}$, with Greek indexes henceforth spanning from 0 to 3, and the signature being $(-,+,+,+)$. Following Refs. [1–8], assume the expression for a spacetime interval in the form

$$ds^2 = -c^2 dt^2 + \eta_{ij} dx^i dx^j, \tag{1}$$

where $c$ is the speed of light, and $t \equiv x^0$ is time. Consider the constant-$t$ hypersurfaces $\Sigma_t$ as space; then, $x^i$ are spatial coordinates, and $\eta_{ij}$ plays a role of the spatial metric, with Latin indexes spanning from 1 to 3 [30]. Correspondingly,

 

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 \\ 0 & \eta_{ij} \end{pmatrix}, \tag{2}$$

yielding that [31]

$$g \equiv \det g_{\mu\nu} = -c^2\eta, \qquad \eta \equiv \det \eta_{ij}. \tag{3}$$

Regardless of the intrinsic curvature associated with $\eta_{ij}$, the spatial hypersurfaces $\Sigma_t$ can have a nonzero extrinsic curvature [29,32]

$$K_i^j = -\frac{\eta^{j\ell}}{2}\frac{\partial \eta_{\ell i}}{\partial t}, \tag{4}$$

where $\eta^{j\ell}$ is the tensor inverse to $\eta_{j\ell}$; particularly, $\theta \equiv -\mathrm{Tr}\,\hat{\mathbf{K}}$, being the volume expansion rate, reads as [see Eqs. (4.7.4)–(4.7.6) in Ref. [33] ]

$$\theta = \frac{1}{2}\mathrm{Tr}\left(\hat{\boldsymbol{\eta}}^{-1}\cdot\frac{\partial \hat{\boldsymbol{\eta}}}{\partial t}\right) = \frac{1}{2}\frac{\partial}{\partial t}\ln\eta = \frac{1}{\sqrt{\eta}}\frac{\partial\sqrt{\eta}}{\partial t}, \tag{5}$$

where we use a caret for abstract notation of rank-two tensors. Hence, consider the hyperplane $\mathcal{S}$ tangent to $\Sigma_t$. For any four-vector $X^\mu \equiv (X^0, X^i)$, its projection $(0, X^i)$ on $\mathcal{S}$ is also a four-vector. On the other hand, since the time component of the latter is zero anyway, one can consider the remaining part, $\mathbf{X} \equiv (X^1, X^2, X^3)$, as a spatial *three*-vector; thus, $\mathcal{S}$ is a three-vector space.

The vector algebra in $\mathcal{S}$ is as usual, so the common differential operators are defined as (Sec. 4.7 in Ref. [33])

$$\nabla\cdot\mathbf{X} = \frac{1}{\sqrt{\eta}}\frac{\partial}{\partial x^i}(\sqrt{\eta}X^i), \qquad (\nabla\times\mathbf{X})^i = \epsilon^{ijk}\nabla_j X_k,$$

where the former is connected with the four-divergence $X^\mu{}_{;\mu}$ as [see Eq. (3)]

$$\begin{aligned} X^\mu{}_{;\mu} &= \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(\sqrt{-g}X^\mu) \\ &= \frac{1}{\sqrt{\eta}}\frac{\partial}{\partial t}(\sqrt{\eta}X^0) + \nabla\cdot\mathbf{X} \\ &= \frac{\partial X^0}{\partial t} + \theta X^0 + \nabla\cdot\mathbf{X}, \end{aligned} \tag{6}$$

$\nabla_i \equiv \partial/\partial x^i$, and $\epsilon^{ijk}$ is the permutation pseudotensor, connected with the permutation symbol $[ijk]$ as

$$\epsilon^{ijk} = \eta^{-1/2}[ijk], \qquad \epsilon_{ijk} = \eta^{+1/2}[ijk]. \tag{7}$$

On the other hand, with $\eta_{ij}(t)$, the vector time derivative can be introduced in two different ways, particularly, as follows. Consider an arbitrary three-vector field $\mathbf{X}$ and its representation in a spatial basis, $\mathbf{X} = \mathbf{e}_i X^i$. The so-called Fermi-Walker derivative of $\mathbf{X}$ with respect to $t$, to be denoted as $\partial\mathbf{X}/\partial t$, applies to both the basis vectors $\mathbf{e}_i$ and the vector components $X^i$, as usual:

$$\frac{\partial\mathbf{X}}{\partial t} = \frac{\partial\mathbf{e}_i}{\partial t}X^i + \mathbf{e}_i\frac{\partial X^i}{\partial t}. \tag{8}$$

The projection of this on the $j$th axis is

$$\left(\frac{\partial\mathbf{X}}{\partial t}\right)^j = \mathbf{e}^j\cdot\frac{\partial\mathbf{e}_i}{\partial t}X^i + \frac{\partial X^j}{\partial t} = \frac{\partial X^j}{\partial t} - K_i^j X^i, \tag{9}$$

where we used that [29]

$$\mathbf{e}^j\cdot\frac{\partial\mathbf{e}_i}{\partial t} = -K_i^j. \tag{10}$$

Thus, Eq. (9) can be put in the following vector form:

$$\frac{\partial\mathbf{X}}{\partial t} = \mathbf{e}_j\left(\frac{\partial\mathbf{X}}{\partial t}\right)^j = \mathbf{e}_j\frac{\partial X^j}{\partial t} - \hat{\mathbf{K}}\cdot\mathbf{X}. \tag{11}$$

Hence, we further rewrite this as

$$\frac{\partial\mathbf{X}}{\partial t} = \mathcal{L}_t\mathbf{X} - \hat{\mathbf{K}}\cdot\mathbf{X}, \tag{12}$$

where $\mathcal{L}_t$ is the alternative, so-called Lie derivative with respect to time, which is defined here as applying only to the vector contravariant components:

$$\mathcal{L}_t\mathbf{X} = \mathbf{e}_j\frac{\partial X^j}{\partial t}. \tag{13}$$

For details, see, e.g., Refs. [32,34] or Sec. 21.5 in Ref. [35].

### B. Model metric

Following Refs. [1–8], let us henceforth assume the particular Friedmann-Robertson-Walker (FRW) model, which describes homogeneous isotropic spacetime (Sec. 8.1 in Ref. [33], Sec. 1.3 in Ref. [36]). Assuming also that the Universe is flat, Eq. (1) rewrites, in Cartesian coordinates, as

$$ds^2 = -c^2dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{14}$$

where $a(t)$ is the so-called expansion factor. Then,

$$\eta_{jk} = a^2\delta_{jk}, \qquad \eta = a^6, \tag{15}$$

with $\delta_{jk} = \mathrm{diag}(1, 1, 1)$ being the Euclidean metric or the covariant Kronecker symbol. Hence, for any $X^i$,

$$X_i = \eta_{ij}X^j = a^2X^i, \tag{16}$$

so, in addition to the contravariant and the covariant components, $X^i$ and $X_i$, it is convenient to introduce the orthonormal components

$$\bar{X}^i \equiv \bar{X}_i = aX^i = X_i/a, \qquad \sqrt{\bar{X}_i\bar{X}_i} = \sqrt{X^iX_i} \equiv X. \tag{17}$$

These represent the vector components in the local Euclidean metric, so, unlike $X^i$ and $X_i$, they are "measurable" in the usual sense, i.e., invariant to the normalization of $a(t)$. Also, the extrinsic curvature tensor now is

$$K_i^j = -\frac{\dot{a}}{a}\delta_i^j, \qquad \theta = \frac{3\dot{a}}{a}, \tag{18}$$

so the commonly introduced shear tensor $\hat{\boldsymbol{\sigma}}$ [29,34] is zero here, and, finally, Eq. (12) rewrites as

$$\frac{\partial \mathbf{X}}{\partial t} = \mathcal{L}_t \mathbf{X} + \frac{\theta}{3}\mathbf{X}. \tag{19}$$

## III. ACTION CONSERVATION

As derived in Ref. [25], ACT for collisionless plasmas reads as $\mathcal{J}^{\mu}{}_{;\mu} = 0$, where $\mathcal{J}^{\mu}$ is the action four-current density. Using Eq. (6), one can rewrite this as [37]

$$\frac{1}{\sqrt{\eta}}\frac{\partial}{\partial t}(\sqrt{\eta}J) + \nabla\cdot(\mathbf{v}_g J) = 0 \tag{20}$$

(cf., e.g., Refs. [10,15]), where $J \equiv \mathcal{J}^0$ is the wave action density, and $\mathbf{v}_g$ is the group velocity. For homogeneous waves, the divergence term can be omitted, so one gets

$$J\sqrt{\eta} = \text{const} \tag{21}$$

in the general form and, for the particular metric (15),

$$Ja^3 = \text{const}. \tag{22}$$

These equalities can be specified using that

$$J = \mathfrak{E}/\omega, \tag{23}$$

where $\mathfrak{E}$ is the wave energy density (Sec. 4.2 in Ref. [38]):

$$\mathfrak{E} = \frac{1}{16\pi}\delta\mathbf{E}^* \cdot \frac{\partial(\hat{\boldsymbol{\varepsilon}}\omega)}{\partial\omega}\cdot\delta\mathbf{E} + \frac{|\delta B|^2}{16\pi}, \tag{24}$$

$\delta\mathbf{E}$ and $\delta\mathbf{B}$ are the wave electric and magnetic fields, $\hat{\boldsymbol{\varepsilon}}$ is the dielectric tensor, and $\omega$ is the frequency, all of which can be approximated to the *zeroth* order in $\theta$. Hence, it is instructive to arrive at the general theorem, Eq. (22), through *ab initio* calculations, that is, by independently deriving the fields $\delta\mathbf{E}$ and $\delta\mathbf{B}$ from Maxwell's equations for particular waves. Below, a number of such examples is considered, showing the proper steps to recover Eq. (22) for specific plasma modes. Hence, collisionless theories which are at variance with ACT are ruled out.

### A. Electromagnetic waves in vacuum

First, let us consider how vacuum electromagnetic waves transform in the metric (15). Start out with the corresponding Maxwell's equations (Appendix A)

$$(\partial/\partial t + \theta)E^i = c\epsilon^{ijk}\nabla_j B_k, \tag{25}$$

$$-(\partial/\partial t + \theta)B^i = c\epsilon^{ijk}\nabla_j E_k \tag{26}$$

($E^i = \delta E^i$, $B^i = \delta B^i$). These allow for a solution

$$E^j = \mathcal{E}^j e^{i\psi}, \qquad B^j = \mathcal{B}^j e^{i\psi}, \tag{27}$$

$$\psi(x^\ell, t) = ik_\ell x^\ell - \int^t [i\omega(t') + \theta(t')]dt', \tag{28}$$

where $\mathcal{E}^j$ and $\mathcal{B}^j$ are fixed, so $\omega(t)$ has the meaning of the local frequency, and $k_\ell$ are the wave-vector covariant components, assumed constant. (Having in mind that $\hbar\mathbf{k}$ is the photon momentum, cf. Appendix B.) This yields

$$-\omega\boldsymbol{\mathcal{E}} = c\mathbf{k}\times\boldsymbol{\mathcal{B}}, \qquad \omega\boldsymbol{\mathcal{B}} = c\mathbf{k}\times\boldsymbol{\mathcal{E}}, \tag{29}$$

so the usual dispersion relation is obtained: $\omega^2 = c^2k^2$. On the other hand, $k^2 = k^i k_i$, where $k^i = k_i/a^2$. (In terms of the orthonormal components, this means that $a\bar{k}_i = \text{const}$, which is understood from the fact that the wavelength must grow linearly with $a$.) Thus, $\omega \sim a^{-1}$, and for the field amplitudes one gets, using Eq. (18):

$$E^i, B^i \sim \exp\left[-\int^t \theta(t')dt'\right] \sim a^{-3}. \tag{30}$$

Hence, the invariant, orthonormal components satisfy

$$a^2\bar{E}_i = \text{const} \tag{31}$$

(and similarly for $\bar{B}_i$), which is an *exact* result for any $a(t)$, contrary to Ref. [1]. This means that

$$\mathfrak{E} = |E|^2/(8\pi) \sim a^{-4}, \tag{32}$$

so the action density (23) scales in agreement with Eq. (22), as predicted.

### B. Electromagnetic waves in plasma

Suppose now that a transverse electromagnetic wave is propagating in plasma with nonzero electron density $N$. In this case, to find how the amplitude evolves with time, Maxwell's equations must be complemented with the corresponding kinetic [29,39] or hydrodynamic equations for electrons (ions can be considered motionless [40]), and it is the latter approach that we choose here. Start out with the continuity equation [29,34], which expresses the particle conservation similarly to Eq. (20):

$$\frac{\partial N}{\partial t} + \theta N + \frac{\partial(NU^i)}{\partial x^i} = 0, \tag{33}$$

where we used that $\eta$ is independent of $x^i$ and introduced the electron flow velocity $\mathbf{U}$; then, the average (over the wave period) density $N_0$ satisfies

$$\mathcal{N} \equiv N_0 a^3 = \text{const}. \tag{34}$$

Assume a nonrelativistic plasma and also that the effect of collisions on the electron average momentum $\boldsymbol{\mathcal{P}}$ is negligible, following Refs. [1–8]. Hence, the equation for $\mathbf{U}$ is derived from that for $\boldsymbol{\mathcal{P}}$ [29]:

$$\left[\frac{\partial}{\partial t} + (\mathbf{U}\cdot\nabla)\right]\mathbf{U} = \frac{q}{m_e}\left(\mathbf{E} + \frac{1}{c}\mathbf{U}\times\mathbf{B}\right) + \hat{\mathbf{K}}\cdot\mathbf{U} - \frac{\nabla\cdot\hat{\mathbf{P}}}{m_e N}, \tag{35}$$

where $q < 0$ and $m_e$ are the electron charge and mass, and $\hat{\mathbf{P}}$ is the electron pressure tensor. The magnetic Lorentz force can be neglected due to $B^i \lesssim E^i$ and $U^i \ll c$. Thus, in the coordinate form,

$$\frac{\partial U^j}{\partial t} + U^i \frac{\partial U^j}{\partial x^i} = \frac{q}{m_e} E^j - \frac{2\theta}{3} U^j - \frac{1}{m_e N} \frac{\partial P^{ij}}{\partial x^i}, \quad (36)$$

now with the factor of 2 in front of the extrinsic curvature term, due to Eq. (19).

For transverse waves, the effect of pressure is negligible [40]; thus, Eqs. (34) and (36) form a closed set, when combined with Maxwell's equations (Appendix A)

$$(\partial/\partial t + \theta)E^i = c\epsilon^{ijk}\nabla_j B_k - 4\pi q N U^i, \quad (37)$$

$$-(\partial/\partial t + \theta)B^i = c\epsilon^{ijk}\nabla_j E_k. \quad (38)$$

Like before, search for a wave with fixed $k_\ell$:

$$E^j = \frac{1}{a^3}\tilde{\mathcal{E}}^j e^{ik_\ell x^\ell}, \qquad B^j = \frac{1}{a^3}\tilde{\mathcal{B}}^j e^{ik_\ell x^\ell}, \quad (39)$$

and also substitute

$$U^j = \frac{1}{a^2} u^j e^{ik_\ell x^\ell}. \quad (40)$$

Then, after linearization,

$$\dot{\tilde{\mathcal{E}}}{}^i = ic\epsilon^{ijk}k_j\tilde{\mathcal{B}}_k - 4\pi q\mathcal{N}u^i/a^2, \quad (41)$$

$$\dot{\tilde{\mathcal{B}}}{}^i = -ic\epsilon^{ijk}k_j\tilde{\mathcal{E}}_k, \quad (42)$$

$$\dot{u}^i = q\tilde{\mathcal{E}}^i/(m_e a). \quad (43)$$

Differentiate the former with respect to time, using that

$$\frac{d}{dt}(\epsilon^{ijk}k_j\tilde{\mathcal{B}}_k) = k_j\frac{d\epsilon^{ijk}}{dt}\tilde{\mathcal{B}}_k + \epsilon^{ijk}k_j\frac{d}{dt}(a^2B^k), \quad (44)$$

with $\dot{\epsilon}^{ijk} = -\theta\epsilon^{ijk}$ [Eq. (7)], $da^2/dt = (2\theta/3)a^2$, and Eq. (42) for $\dot{\tilde{\mathcal{B}}}{}^k$. Then, one obtains

$$\ddot{\tilde{\mathcal{E}}}{}^i = -\frac{\theta}{3}\dot{\tilde{\mathcal{E}}}{}^i - \omega^2\tilde{\mathcal{E}}^i + \frac{\theta}{3}\frac{m_e}{q}\omega_p^2 a u^i, \quad (45)$$

where $u^i$ is yet to be found from Eq. (43), and

$$\omega^2(t) = \omega_p^2 + c^2k^2, \qquad \omega_p^2 = 4\pi N q^2/m_e, \quad (46)$$

with $\omega_p$ being known as the plasma frequency. At zero $\theta$, one gets $\tilde{\mathcal{E}}^i = \text{const} \times e^{-i\omega t}$, and the same for $u^i$. Suppose now that the metric expansion is present but slow, such that $\theta$ is much smaller than the local frequency $\omega$. Then, since the third term on the right-hand side in Eq. (45) is already of the order of $\theta$, we can evaluate the factor $u^i$ to the zeroth order in $\theta$, using $\dot{u}^i \approx -i\omega u^i$ in Eq. (43):

$$u^i \approx iq\tilde{\mathcal{E}}^i/(m_e a\omega). \quad (47)$$

Since, due to $a^2k^\ell = \text{const}$ and Eq. (34), we also have

$$\dot{\omega} = -\frac{\theta}{3}\left(\omega + \frac{\omega_p^2}{2\omega}\right), \quad (48)$$

one can eventually put Eq. (45) in the form

$$\ddot{\tilde{\mathcal{E}}}{}^i + \frac{\theta}{3}\dot{\tilde{\mathcal{E}}}{}^i + \left[\omega^2 + 2i\omega\frac{d\ln(a\omega)}{dt}\right]\tilde{\mathcal{E}}^i = 0. \quad (49)$$

Then, its asymptotic solution reads as

$$\tilde{\mathcal{E}}^i = \sqrt{a\omega}\exp\left(-i\int^t \omega(t')dt'\right) \times \text{const}. \quad (50)$$

From Eq. (39), one gets then $E^i \sim \sqrt{\omega/a^5}$, so the field orthonormal components scale as

$$\bar{E}_i \sim \sqrt{\omega/a^3}. \quad (51)$$

Let us compare this result with ACT. Since, to the zeroth order in $\theta$, one has $|B|^2 = (c^2k^2/\omega^2)|E|^2$, and the dielectric constant equals $\varepsilon = 1 - \omega_p^2/\omega^2$, as usual [40], we obtain (Sec. 4.2 in Ref. [38])

$$\mathfrak{E} = |E|^2/(8\pi) \sim \omega/a^3 \quad (52)$$

for the wave energy density. Thus, the action density (23) scales, again, in agreement with Eq. (22).

### C. Electron Langmuir waves

Let us also study the evolution of electrostatic oscillations, so that $B^i = 0$, and

$$E^i = -\nabla^i\varphi = -ik^i\tilde{\varphi}e^{ik_\ell x^\ell}. \quad (53)$$

Assume that ions are approximately motionless and consider the electron Langmuir oscillations [18]; then, the electrostatic potential satisfies the Poisson equation [flowing from Eq. (A1)] in the form:

$$k^2\tilde{\varphi} = 4\pi q n N_0, \quad (54)$$

where we substituted

$$N = N_0(1 + ne^{ik_\ell x^\ell}). \quad (55)$$

Unlike in Sec. III B, let us keep the pressure term in Eq. (36), so an expression is needed for $\hat{\mathbf{P}}$. Instead of postulating a scalar pressure, as in the ideal-fluid approximation [6,32,34,41], one can deduce a tensor equation for $\hat{\mathbf{P}}$ from the Vlasov equation [29] by considering the second-order velocity moment of the latter, as usual. Namely, employing also Eqs. (33) and (36), one obtains

$$\frac{\partial P^{jk}}{\partial t} + U^i\frac{\partial P^{jk}}{\partial x^i} + P^{ki}\frac{\partial U^j}{\partial x^i} + P^{ji}\frac{\partial U^k}{\partial x^i} + P^{jk}\nabla\cdot\mathbf{U} + (7\theta/3)P^{jk} + (\nabla\cdot\hat{\mathbf{Q}})^{jk} = 0 \quad (56)$$

(cf. Ref. [18]), where $\hat{\mathbf{Q}}$ is the heat flux. For the unperturbed pressure tensor $P_0^{ij}$ in homogeneous plasma ($\partial/\partial x^i \equiv 0$), Eq. (56) yields $\dot{P}_0^{ij} = -(7\theta/3)P_0^{ij}$, so, with Eq. (18) for $\theta$, one obtains

$$\Pi^{ij} \equiv a^7 P_0^{ij} = \text{const}, \quad (57)$$

in agreement with Appendix B. As for the wave perturbation to $P_0^{ij}$, notice the following. The Langmuir oscillations are strongly damped, i.e., do not exist, unless the electron thermal velocity is small compared to the wave phase velocity (Chap. 8 in Ref. [38]). On the other hand, in the latter regime, one has $(\nabla \cdot \hat{\mathbf{Q}})^{jk} \ll \partial P^{jk}/\partial t$. Thus, the heat flux can be neglected, yielding a hydrodynamic closure; see also Refs. [18,42–44].

After linearization, one then obtains the following:

$$\dot{n} = -ik_\ell u^\ell / a^2, \tag{58}$$

$$\dot{u}^i = -i(k^i/k^2)\omega_p^2 a^2 n - ik_j \pi^{ji}/(m_e \mathcal{N} a^2), \tag{59}$$

$$\dot{\pi}^{jk} = -i(\Pi^{jk} k_\ell u^\ell + \Pi^{k\ell} k_\ell u^j + \Pi^{j\ell} k_\ell u^k)/a^2, \tag{60}$$

where we substituted

$$P^{jk} = P_0^{jk} + \frac{1}{a^7}\pi^{jk} e^{ik_\ell x^\ell}. \tag{61}$$

At fixed $a$, or zero $\theta$, a solution is found in the $e^{-i\omega t}$ form, requiring that $\mathbf{k}$ be an eigenvector of $P_0^{ij}$ [18]; then,

$$\omega^2 = \omega_p^2 + 3k^2 v_T^2, \qquad v_T^2 \equiv P_0^{ij} k_i k_j/(m_e N_0 k^2). \tag{62}$$

Now let us consider how this solution evolves at nonzero $\theta \ll \omega$. First, take a time derivative of Eq. (58) and substitute Eq. (59):

$$\ddot{n} = -\frac{2\theta}{3}\dot{n} - \omega_p^2 n - \frac{\pi^{jk} k_j k_k}{m_e N_0 a^7}. \tag{63}$$

To rewrite the latter term, multiply Eq. (60) by a constant factor $k_j k_k$ to get

$$\dot{\pi}^{jk} k_j k_k = 3\Pi^{jk} k_j k_k \dot{n}. \tag{64}$$

Then $\pi^{jk} k_j k_k = 3\Pi^{jk} k_j k_k n$, so Eq. (63) is put as

$$\ddot{n} + \frac{2\theta}{3}\dot{n} + \omega^2 n = 0, \tag{65}$$

with $\omega(t)$ from Eq. (62). In this form, the equation for $n$ is equivalent to that for compressional gravitational waves (Sec. 15.9 in Ref. [33]), which is different only in the sign of $\omega_p^2$ due to the electrostatic repulsion being replaced with the gravitational attraction.

The asymptotic solution of Eq. (65) is given by

$$n = \frac{1}{a\sqrt{\omega}} \exp\left(-i\int^t \omega(t')dt'\right) \times \text{const}. \tag{66}$$

From Eqs. (53) and (54), one then obtains

$$\bar{E}_i \sim \frac{1}{a^3\sqrt{\omega}}, \tag{67}$$

and, thus, the energy density [18] scales like

$$\mathfrak{E} = \frac{\omega^2}{\omega_p^2}\frac{|E|^2}{8\pi} \sim \frac{\omega}{a^3}. \tag{68}$$

Hence, the evolution of the action density (23) agrees with Eq. (22); i.e., the number of plasmons is conserved, just as when the plasma is compressed mechanically in a fixed metric [18]. Particularly, when the thermal corrections are neglected ($kv_T \ll \omega_p$), combining Eq. (68) with Eq. (34) yields

$$|E|^2 \sim N_0^{3/2}, \tag{69}$$

again in agreement in Ref. [18].

### D. Magnetohydrodynamic waves

Finally, let us consider the evolution of low-frequency magnetohydrodynamic waves in the presence of a static magnetic field $\mathbf{B}_0$ (Sec. 4.3 in Ref. [45]). Start out with Faraday's law [Eq. (A5)] and substitute

$$\mathbf{E} = -\mathbf{U} \times \mathbf{B}/c, \tag{70}$$

which is obtained, as usual, from the momentum equation [Eq. (35)] for electrons under the assumption of negligible electron inertia ($m_e \to 0$); then,

$$(\mathcal{L}_t + \theta)\mathbf{B} = \nabla \times (\mathbf{U} \times \mathbf{B}). \tag{71}$$

Under the same assumption, the electron fluid velocity equals the ion fluid velocity; hence, from the ion momentum equation, we get

$$\left[\frac{\partial}{\partial t} + (\mathbf{U}\cdot\nabla)\right]\mathbf{U} = \frac{\mathbf{j}\times\mathbf{B}}{m_i N c} + \hat{\mathbf{K}}\cdot\mathbf{U}, \tag{72}$$

where $m_i$ is the ion mass, and the pressure term is neglected. The current density $\mathbf{j}$ is found from Ampere's law, under the quasistatic approximation: $\mathbf{j} = (c/4\pi)\nabla \times \mathbf{B}$. Then, using Eq. (19), one obtains

$$m_i N\left[\mathcal{L}_t + \frac{2\theta}{3} + (\mathbf{U}\cdot\nabla)\right]\mathbf{U} = \frac{1}{4\pi}(\nabla\times\mathbf{B})\times\mathbf{B}. \tag{73}$$

After linearization, Eqs. (71) and (73) read as

$$[\mathcal{L}_t + \theta]\delta\mathbf{B} = \nabla \times (\mathbf{U}\times\mathbf{B}_0), \tag{74}$$

$$[\mathcal{L}_t + (2\theta/3)]\mathbf{U} = (\nabla\times\delta\mathbf{B})\times\mathbf{B}_0/(4\pi m_i N_0), \tag{75}$$

where we introduced $\delta\mathbf{B} = \mathbf{B} - \mathbf{B}_0$ as a small quiver perturbation to $\mathbf{B}_0$. Like before, assume a wave with fixed $k_\ell$ and substitute Eq. (40) and also

$$\delta\mathbf{B} = \frac{1}{a^3}\tilde{\mathcal{B}} e^{ik_\ell x^\ell}. \tag{76}$$

Then, Eqs. (74) and (75) take the form

$$\mathcal{L}_t \tilde{\mathcal{B}} = ia[\mathbf{k}\times(\mathbf{u}\times\mathbf{B}_0)], \tag{77}$$

$$\mathcal{L}_t \mathbf{u} = i[(\mathbf{k}\times\tilde{\mathcal{B}})\times\mathbf{B}_0]/(4\pi m_i N_0 a), \tag{78}$$

which, using that $a^3 B_0^i = \text{const}$, one can put as

$$\mathcal{L}_t^2 \tilde{\mathcal{B}} + (2\theta/3)\mathcal{L}_t \tilde{\mathcal{B}} + V_A^2 \mathbf{W} = 0, \tag{79}$$

where $V_A = B_0/\sqrt{4\pi m_i N_0}$ is called the Alfvén speed,

$$\mathbf{W} \equiv \mathbf{k} \times (\mathbf{b} \times (\mathbf{b} \times (\mathbf{k} \times \tilde{\mathcal{B}}))), \tag{80}$$

and $\mathbf{b} \equiv \mathbf{B}_0/B_0$ is the unit vector along $\mathbf{B}_0$.

From $\nabla \cdot \delta\mathbf{B} = 0$, one gets $\mathbf{k} \cdot \tilde{\mathcal{B}} = 0$, so

$$\begin{aligned}\mathbf{W} &= \mathbf{k} \times (\mathbf{b} \times [\mathbf{k}(\tilde{\mathcal{B}} \cdot \mathbf{b}) - \tilde{\mathcal{B}}(\mathbf{k} \cdot \mathbf{b})]) \\ &= (\tilde{\mathcal{B}} \cdot \mathbf{b})[\mathbf{k} \times (\mathbf{b} \times \mathbf{k})] + (\mathbf{k} \cdot \mathbf{b})[\mathbf{k} \times (\tilde{\mathcal{B}} \times \mathbf{b})] \\ &= (\tilde{\mathcal{B}} \cdot \mathbf{b})[\mathbf{b}k^2 - \mathbf{k}(\mathbf{k} \cdot \mathbf{b})] + \tilde{\mathcal{B}}(\mathbf{k} \cdot \mathbf{b})^2. \end{aligned} \tag{81}$$

It is then convenient to separate the transverse and the longitudinal (with respect to $\mathbf{B}_0$) parts of $\tilde{\mathcal{B}}$ and $\mathbf{k}$ as

$$\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_\perp + \tilde{\mathcal{B}}_\parallel \mathbf{b}, \qquad \mathbf{k} = \mathbf{k}_\perp + k_\parallel \mathbf{b}. \tag{82}$$

Hence, $\mathbf{W}$ rewrites as

$$\mathbf{W} = \tilde{\mathcal{B}}_\perp k_\parallel^2 + (\mathbf{b}k^2 - \mathbf{k}_\perp k_\parallel)\tilde{\mathcal{B}}_\parallel, \tag{83}$$

and one also obtains

$$\mathcal{L}_t \tilde{\mathcal{B}} = \mathcal{L}_t \tilde{\mathcal{B}}_\perp + [\dot{\tilde{\mathcal{B}}}_\parallel - (\theta/3)\tilde{\mathcal{B}}_\parallel]\mathbf{b}, \tag{84}$$

$$\mathcal{L}_t^2 \tilde{\mathcal{B}} \approx \mathcal{L}_t^2 \tilde{\mathcal{B}}_\perp + [\ddot{\tilde{\mathcal{B}}}_\parallel - (2\theta/3)\dot{\tilde{\mathcal{B}}}_\parallel]\mathbf{b}, \tag{85}$$

where we neglected terms of the order of $\theta^2$. Then,

$$\begin{aligned}&\mathcal{L}_t^2 \tilde{\mathcal{B}}_\perp + (2\theta/3)\mathcal{L}_t \tilde{\mathcal{B}}_\perp + k_\parallel^2 V_A^2 \tilde{\mathcal{B}}_\perp - \mathbf{k}_\perp k_\parallel V_A^2 \tilde{\mathcal{B}}_\parallel \\ &\quad + \mathbf{b}(\ddot{\tilde{\mathcal{B}}}_\parallel + k^2 V_A^2 \tilde{\mathcal{B}}_\parallel) = 0,\end{aligned} \tag{86}$$

with the projections reading as

$$\ddot{\tilde{\mathcal{B}}}^i_\perp + (2\theta/3)\dot{\tilde{\mathcal{B}}}^i_\perp + k_\parallel^2 V_A^2 \tilde{\mathcal{B}}^i_\perp - k^i_\perp k_\parallel V_A^2 \tilde{\mathcal{B}}_\parallel = 0, \tag{87}$$

$$\ddot{\tilde{\mathcal{B}}}_\parallel + k^2 V_A^2 \tilde{\mathcal{B}}_\parallel = 0. \tag{88}$$

This shows that two different modes can exist, corresponding to different $\tilde{\mathcal{B}}_\parallel$.

*Shear waves.* If $\tilde{\mathcal{B}}_\parallel = 0$, Eq. (87) reads as

$$\ddot{\tilde{\mathcal{B}}}^i_\perp + (2\theta/3)\dot{\tilde{\mathcal{B}}}^i_\perp + k_\parallel^2 V_A^2 \tilde{\mathcal{B}}^i_\perp = 0. \tag{89}$$

The latter is formally equivalent to Eq. (65); thus,

$$\tilde{\mathcal{B}}^i_\perp = \frac{1}{a\sqrt{\omega}} \exp\left(-i \int^t \omega(t')dt'\right) \times \text{const}, \tag{90}$$

except now $\omega^2 = k_\parallel^2 V_A^2 \sim a^{-3}$. These are called shear Alfvén waves; for them [Eq. (76)], $\delta\bar{B}_i \sim \sqrt{\omega}/a^3 \sim a^{-9/4}$.

*Compressional waves.* If $\tilde{\mathcal{B}}_\parallel$ is nonzero, Eq. (88) yields

$$\tilde{\mathcal{B}}_\parallel = \omega^{-1/2} \exp\left(-i \int^t \omega(t')dt'\right) \times \text{const}, \tag{91}$$

where $\omega^2 = k^2 V_A^2 \sim a^{-3}$. These are called compressional Alfvén waves, and one gets

$$\delta\bar{B}_\parallel = \mathbf{b} \cdot \delta\mathbf{B} \sim \tilde{\mathcal{B}}_\parallel/a^3 \sim a^{-9/4} \tag{92}$$

for the orthonormal component of $\delta\mathbf{B}$ along $\mathbf{B}_0$. The driven oscillations of $\tilde{\mathcal{B}}^i_\perp$, as described by Eq. (87), provide that the magnetic field retains zero divergence. Thus, $\delta\bar{B}^i_\perp \sim (\bar{k}_\perp/\bar{k}_\parallel)\delta\bar{B}_\parallel \sim a^{-9/4}$, so the same scaling also holds for the total perturbation field $\delta\bar{B}_i$.

In other words, for both shear and compressional waves, one equally has

$$\delta\bar{B}_i \sim \frac{1}{a^3\sqrt{\omega}} \sim \frac{1}{a^{9/4}}. \tag{93}$$

Then (Prob. 6.6–8 in Ref. [38]),

$$\mathfrak{E} = |\delta B|^2/(8\pi) \sim a^{-9/2}, \tag{94}$$

and, since $\omega \sim a^{-3/2}$ also in both cases, the action density (23) again satisfies Eq. (22).

## IV. DISCUSSION

As guaranteed by the general theorem [25], within the model adopted here, the action must be conserved also for any other linear waves, including those in relativistic plasmas. However, among the corresponding *ab initio* analytic calculations that were reported [1–8], a number of scalings are at variance with ACT. This is resolved twofold, particularly as follows.

First of all, some models used in the earlier calculations are oversimplified, so they are inadequate even for a fixed metric. For example, comparing Ref. [1] with Refs. [28,46,47] shows that incorrect dispersion relations were derived in the former for transverse electromagnetic waves and sound waves in relativistic pair plasmas. (In fact, the sound waves could not exist in such plasmas [7,47].) The misconception was caused by applying the hydrodynamic approach to study collisionless phenomena within the ideal-fluid approximation. Already the lowest-order thermal corrections cannot be derived consistently in this case, requiring that the whole pressure tensor be calculated rather than postulated as diagonal (compare our Sec. III C with, e.g., Refs. [4–7]). At relativistic temperatures, though, a full Vlasov treatment is needed, and it is only recently that a suitable and sufficiently general formalism was developed to attack the problem [29] (but see also Refs. [4,39,48–51]). Thus, correct *ab initio* calculations for relativistic plasmas in variable metric are yet to be performed. On the other hand, for understanding the wave evolution driven solely by metric expansion, such calculations are unnecessary due to the existence of ACT.

The second type of error, common to the earlier papers, is caused by treating essentially collisional effects within the collisionless approximation. For example, in Refs. [4–7,9], it is assumed that the plasma temperature $T$ can be mediated by collisions with a thermostating buffer gas (e.g., photon gas), thereby affecting $\omega_p$ and $V_A$ through the relativistic mass shift; hence, the plasma parameters, as functions of time, are prescribed differently than would flow from the free-particle model (Appendix B). Since the effective collision rate $\nu$ is assumed less than the wave frequency $\omega$, it is then argued that the plasma can be considered collisionless also for calculating the wave amplitude evolution. However, this statement is erroneous. Although $\nu/\omega$ being small is sufficient to neglect the collisions locally, those yet influence the particle momentum $\mathbf{p}$ on the time scale comparable to that of $T(\langle p \rangle)$ or, if quasielastic, even faster; thus, $\nu$ must be included in the equation for the fluid momentum $\mathcal{P} \equiv \langle \mathbf{p} \rangle$ whenever thermal equilibration is assumed. Similarly, pair production and annihilation and also ionization and recombination do affect the wave amplitude evolution, contrary to Ref. [1]; cf., e.g., Refs. [52–54].

In general, when studying the wave amplitude evolution, any collisions affecting $\mathcal{P}$ at the rate $\nu \gtrsim \theta$ must be included explicitly, regardless of $\nu/\omega$. This limits the applicability of the collisionless approximation more severely than assumed in the earlier calculations. Hence, it remains an open question whether any cosmological plasmas of interest actually satisfy this approximation on the Universe expansion time scale. (Alternatively, though, such an approximation, and therefore ACT also, could be useful for studying wave evolution near local massive objects which collapse or expand more rapidly.) It is not our purpose to answer this question here; however, consider the following. Compared to *ab initio* calculations, the cosmological wave evolution is more easily deduced from ACT at $\nu \ll \theta$, as illustrated by our Sec. III, and the statistical description at $\nu \gg \theta$, as in Ref. [28]. Thus, using dynamic equations may be impractical in either case, and it is only the transitional regime ($\nu \sim \theta$) where collisional models are necessary.

## V. CONCLUSIONS

In this paper, the scalings for basic plasma modes in the FRW model of the expanding Universe are revised. Contrary to the existing literature, the wave collisionless evolution must comply with the action conservation theorem. The proper steps to deduce the action conservation from *ab initio* analytical calculations are presented, and discrepancies in the earlier papers are identified. In general, the cosmological wave evolution is more easily derived from the action conservation in the collisionless limit, whereas when collisions are essential, the statistical description must suffice, thereby ruling out the need for using dynamic equations in either case.

While the contribution here has been limited to the posing of the correct equations, it is hoped that this effort will stimulate the drawing of the appropriate cosmological inferences.

## ACKNOWLEDGMENTS

This work was supported by the NNSA under the SSAA Program through DOE Research Grant No. DE-FG52-08NA28553.

## APPENDIX A: MAXWELL'S EQUATIONS

From Eqs. (3.4) in Ref. [34], Maxwell's equations in the spatial metric $\eta_{ij}$ take the following form. Gauss's laws for the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are similar to those in the Minkowski spacetime:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0, \tag{A1}$$

where $\rho$ is the charge density. However, Ampere's law and Faraday's law are now given by

$$(\partial/\partial t + \theta)\mathbf{E} + \hat{\mathbf{K}} \cdot \mathbf{E} = c\nabla \times \mathbf{B} - 4\pi\mathbf{j}, \tag{A2}$$

$$-(\partial/\partial t + \theta)\mathbf{B} - \hat{\mathbf{K}} \cdot \mathbf{B} = c\nabla \times \mathbf{E}, \tag{A3}$$

where $\mathbf{j}$ is the current density. Yet it is more convenient here to rewrite these using the Lie derivative [Eq. (12)]:

$$(\mathcal{L}_t + \theta)\mathbf{E} = c\nabla \times \mathbf{B} - 4\pi\mathbf{j}, \tag{A4}$$

$$-(\mathcal{L}_t + \theta)\mathbf{B} = c\nabla \times \mathbf{E} \tag{A5}$$

(cf. Ref. [41]). Then, in agreement with Ref. [55], the coordinate form is yielded as

$$(\partial/\partial t + \theta)E^i = c\epsilon^{ijk}\nabla_j B_k - 4\pi j^i, \tag{A6}$$

$$-(\partial/\partial t + \theta)B^i = c\epsilon^{ijk}\nabla_j E_k. \tag{A7}$$

## APPENDIX B: PARTICLE MOMENTUM AND TEMPERATURE IN THE EXPANDING METRIC

Consider the motion of a free particle in the flat isotropic metric (15). As shown in Refs. [29,34], the particle relativistic momentum $\mathbf{p} = m\gamma\mathbf{v}$, with $\gamma = (1 - v^2)^{-1/2}$ being the Lorentz factor, in this case is governed by

$$\dot{\mathbf{p}} = \hat{\mathbf{K}} \cdot \mathbf{p} = -(\theta/3)\,\mathbf{p}. \tag{B1}$$

Using Eq. (19), one obtains $\dot{p}^i = -(2\theta/3)p^i$; hence, also with $p_i = \eta_{ij}p^j = a^2 p^i$, one further gets

$$p_i = \text{const}. \tag{B2}$$

The latter is understood from the fact that $p_i$ (unlike $p^i$) is also a *canonical* momentum [29,56]; hence, $p_i$ is conserved when the corresponding Hamiltonian is independent of $x^i$, which is exactly the case for a free particle in a homogeneous metric considered here. Accordingly, the contravariant momentum $p^i = p_i/a^2$ scales as

$$a^2 p_i = \text{const}. \tag{B3}$$

Then, for the orthonormal components, one gets

$$a\bar{p}_i = \text{const}, \tag{B4}$$

which is similar to how, in a fixed metric, the momentum would evolve for a particle trapped in a box with the length $a$ varying adiabatically [57]. Hence, the temperature $T$ of an isotropic ultrarelativistic gas, $T \sim \langle \bar{p} \rangle c$, scales as

$$aT = \text{const} \tag{B5}$$

and, for a nonrelativistic gas, $T \sim \langle \bar{p}^2 \rangle/(2m)$, yielding

$$a^2 T = \text{const} \tag{B6}$$

(cf. Ref. [50]).

Finally, let us show how Eqs. (B5) and (B6) relate to the evolution of the pressure tensor $P_0^{ij} = N_0 \langle p^i v^j \rangle$ (with zero average momentum and flow velocity in isotropic plasma) that was introduced in Ref. [29] and also used in our Sec. III C. By definition, the temperature tensor is $T^{ij} = P_0^{ij}/N_0$, where $N_0$ is the density governed by Eq. (34). Then, the *scalar* temperature $T$ must be defined through the trace of the rank-(1, 1) tensor $T^i_j$, invariant with respect to the spatial metric:

$$T = \frac{1}{3N_0} \eta_{ij} P_0^{ij} = \frac{1}{3} \langle \mathbf{p} \cdot \mathbf{v} \rangle. \tag{B7}$$

At ultrarelativistic energies, $\mathbf{p} \cdot \mathbf{v} \approx \bar{p}c \sim a^{-1}$; hence, Eq. (B5) is recovered. At nonrelativistic energies, $\mathbf{p} \cdot \mathbf{v} \approx \langle \bar{p}^2 \rangle / m$; hence, Eq. (B6) is recovered. On the other hand, $T^{ij} \sim T/a^2 \sim a^{-4}$ in the latter case, so $P^{ij} \sim a^{-7}$, in agreement with Eq. (57).


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