Current drive in recombining plasma

P. F. Schmit and N. J. Fisch
Department of Astrophysical Sciences, Princeton University, Princeton, New Jersey 08544, USA

(Received 5 August 2011; accepted 12 September 2011; published online 12 October 2011)

The Langevin equations describing the average collisional dynamics of suprathermal particles in nonstationary plasma remarkably admit an exact analytical solution in the case of recombining plasma. The current density produced by arbitrary particle fluxes is derived including the influence of charge recombination. Since recombination has the effect of lowering the charge density of the plasma, thus reducing the charged particle collisional frequencies, the evolution of the current density can be modified substantially compared to plasma with fixed charge density. The current drive efficiency is derived and optimized for discrete and continuous pulses of current, leading to the discovery of a nonzero "residual" current density that persists indefinitely under certain conditions, a feature not present in stationary plasmas. © *2011 American Institute of Physics*. [doi:10.1063/1.3646745]

I. INTRODUCTION

The theory of wave-induced current drive in stationary plasmas is well-developed.[1] Among the various methods considered in the literature, much attention has been given to the creation of plasma current via resonant interactions between waves and fast particles.[2,3] Many techniques have been pursued to describe the fundamental dynamics of suprathermal, current-carrying particles in stationary plasma, including analytical treatments of the Fokker-Planck equations in the high-velocity limit[2–6] and numerical solutions of the Fokker-Planck equations.[7–10] Such work has almost always focused on describing current drive performance in the *steady state*, since the primary thrust of the research has been to support and expedite the development of a steady-state magnetic confinement fusion reactor.

Recently, however, a number of studies[11–15] of wave-particle interactions in *nonstationary* plasmas has revealed previously unexplored phenomenology and potentially useful mechanisms. Such phenomena are intrinsically non-steady-state, and hence require a modification of the methods typically used to analyze and describe the physics in stationary systems. In particular, Refs. 11–15 focus primarily on non-steady-state effects associated with expanding or compressing plasma. When a wave is embedded in such a nonstationary plasma and is undamped initially, modification of the bulk plasma parameters through the nonstationary processes changes the wave dynamics and can lead to an induced wave-particle resonance with the fast-particles on the tail of the bulk plasma velocity distribution.[12] If this interaction leads to an anisotropic distortion of the electron distribution function, an electric current can result, potentially producing useful magnetic energy. However, the evolution of this current through collisional relaxation also depends on the time-varying plasma parameters, and a complete analysis of the plasma current response requires an accurate description of these collisional dynamics.

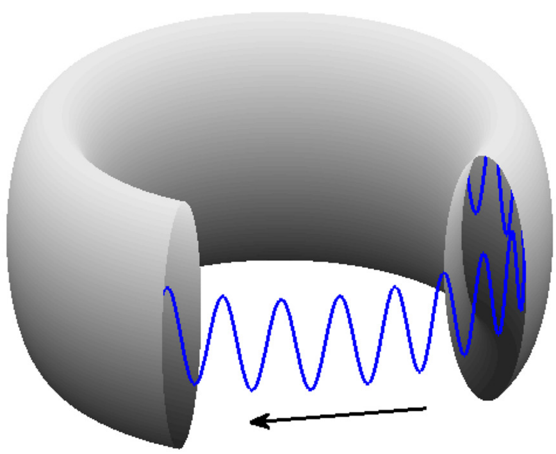
What sets apart the physics addressed in Refs. 11–14 from the work done on current drive schemes in other time-varying conditions, e.g., during plasma current ramp-up in tokamaks, where the current and magnetic field are time-varying,[16–21] is the direct influence of the time-variation on wave-particle processes in the plasma. Time variation in neither the application of rf power nor changes in the current or poloidal magnetic field alters the underlying wave dynamics or the particle collisionality. In contrast, the cooling, heating, and densification associated with, for example, expanding, compressing, and recombining plasmas, can change the wave amplitude due to plasmon conservation[11,12] or plasmon destruction,[22] change the wave phase velocity, and substantially alter the particle collisionality.

Thus, the objective here is to account for time-dependence in the bulk plasma parameters that, in particular, complicate the description of the collisional relaxation of fast-particles. For simplicity, the case of plasma undergoing charged-particle recombination is addressed, which remarkably admits an exact analytical description of the fast-particle average dynamics. Optimizations to maximize current drive performance are sought for both discrete impulses and continuous wave-particle resonances. One particularly interesting result is that certain regimes exist in which the plasma current saturates at a nonzero value time-asymptotically, a feature not present in stationary plasma.

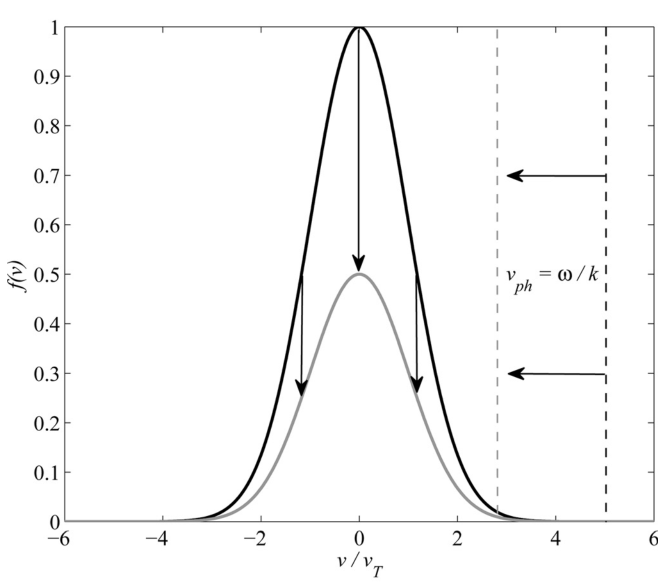
Section II describes the physical picture and introduces the Langevin equations describing fast-particle collisional dynamics, modified for the case of a temporally evolving bulk plasma. In Sec. III, a generalized fast-particle current-drive equation is derived that accounts for any number of time- and velocity-space-dependent particle fluxes, and the self-consistent inductive response of the plasma is considered. In Sec. IV, the plasma recombination model is introduced and the plasma current response is optimized for the case of discrete impulses, resulting in the discovery of a residual, time-asymptotic current density. Section V examines the realistic model of an embedded Langmuir wave undergoing a time-varying resonance in a recombining plasma. Section VI describes some limitations of the model. Finally, Sec. VII summarizes the main results. Certain details of the calculations are relegated to the appendices.

II. FAST PARTICLE DYNAMICS IN NONSTATIONARY PLASMA

Consider the case of wave-induced electrical currents in the presence of plasma recombination, where the current-carrying electrons are suprathermal. A conceptual picture explaining how these electrons can be produced is provided in Fig. 1. Figure 1(a) shows a toroidal plasma permeated by a traveling wave propagating toroidally within the plasma. Initially, the wave phase velocity $v_{\rm ph} = \omega/k \gg v_T$, where v_T is the electron thermal velocity, ω is the wave frequency, and k is the wavenumber. Assume, for example, that the traveling wave is a Langmuir wave, which implies $\omega \sim n^{1/2}$, where n is the electron number density. As the plasma undergoes recombination, ω tends to decrease, while k stays the same, causing $v_{\rm ph}$ to decrease accordingly.[11] However, v_T is not affected by the recombination, and thus after some time, $v_{\rm ph}$ becomes comparable to a few times v_T and collisionlessly damps on the resonant tail particles (cf. Fig. 1(b)).[12] An anisotropic fast particle distribution is produced subsequently, which results in a net electric current after some degree of collisional relaxation.



(a)



(b)

FIG. 1. (Color online) Conceptual picture of current drive in recombining plasma. (a) A toroidal plasma contains a fast traveling wave encircling the plasma toroidally. (b) Plasma recombination causes the wave phase velocity to slow down until it eventually intersects and resonates with the tail of the electron distribution, producing an asymmetric tail distribution and a corresponding electric current. The arrows indicate the temporal evolution of the wave phase velocity (dotted lines) and the electron distribution function (solid lines).

In a recombining plasma, not only does a wave change its phase velocity, but there is also plasmon destruction, i.e., nonresonant collisionless damping of the wave.[22] This can lead to production of electric current in the bulk plasma at the expense of the wave amplitude, but that is likely to be damped quickly compared to the current carried by suprathermal electrons. Therefore, we shall consider only the resonant, fast-particle contribution to the current. However, this topic will be revisited in Sec. V, which addresses plasmon destruction in the case of a continuously time-varying wave-particle resonance.

Following Ref. 2, the dynamical effects of the Boltzmann equation, written in the strict high-velocity limit (i.e., neglecting energy diffusion), and which describes the collisional relaxation of suprathermal electrons in magnetized plasma, can be expressed equivalently in a set of Langevin equations:[23,24]

$$\frac{dv}{dt} = -\left(\frac{\Gamma}{v^3}\right)v, \tag{1}$$

$$\frac{d\mu}{dt} = B(t), \tag{2}$$

where $v = |\mathbf{v}|$, $\mu = v_{||}/v$, $\Gamma = 4\pi ne^4 \ln \Lambda/m^2$, n is the charged particle number density, e is the elementary charge, $\ln \Lambda$ is the Coulomb logarithm, and $B(t)$ is a stochastic term responsible for pitch-angle scattering that is described statistically by the following properties:

$$\langle B(t)\rangle = -\left(\frac{\Gamma}{v^3}\right)(1+Z)\mu, \tag{3}$$

$$\langle B(t)B(t')\rangle = \left(\frac{\Gamma}{v^3}\right)(1+Z)(1-\mu^2)\delta(t-t'), \tag{4}$$

where Z is the charge state of the plasma ions, assuming there is only one ion species present. The brackets $\langle\ldots\rangle$ represent a statistical ensemble average, or expectation value, of the enclosed quantity. It is assumed that collisions with neutral particles are negligible in the regimes and time frames considered. (This would happen, for example, if following recombination the resulting neutral particles simply exit the device.) Note that for a plasma with fixed charge carrier density, Γ is a constant; however, in a recombining plasma, the dependence of Γ on the number density means that it can become time-dependent, i.e., $\Gamma = \Gamma(t)$, where the time-dependence is assumed to be prescribed. In any case, it can be seen that Eq. (1) is nonstochastic, meaning $v = \langle v \rangle$, while the ensemble average of Eq. (2) yields

$$\frac{d\langle \mu \rangle}{dt} = -\left(\frac{\Gamma}{v^3}\right)(1+Z)\langle \mu \rangle. \tag{5}$$

At this point, the notation will be simplified by eliminating the brackets around $\langle \mu \rangle \to \mu$ in the analysis that follows; the expectation values will be the implied dynamical quantities of interest. Combining Eqs. (1) and (5) then gives the following exactly integrable equation relating v and μ

$$\frac{d(\ln \mu)}{dt} = (1+Z)\frac{d(\ln v)}{dt}, \tag{6}$$

which then leads to the relation

$$\frac{\mu}{\mu_i} = \left(\frac{v}{v_i}\right)^{1+Z}, \tag{7}$$

where the subscript "i" henceforth refers to an initial condition. In time-explicit form, the solution to Eq. (1), obtained by direct integration, is

$$\frac{v}{v_i} = \left(1 - \frac{1}{t_{\rm st}}\int_{t_i}^{t} \tilde{\Gamma}(t')\, dt'\right)^{1/3} \Theta(t_d - t) \equiv \tilde{v}(t, t_i), \tag{8}$$

where $t_{\rm st} = v_i^3/3\Gamma_i$, $\tilde{\Gamma} = \Gamma/\Gamma_i$, and

$$G(t_d) = G(t_i) + t_{\rm st}, \tag{9}$$

with $G(t)$ the antiderivative of $\tilde{\Gamma}(t)$ with respect to t. Noting that t_d is the time at which $\tilde{v} \to 0$, the Heaviside step function, Θ, in Eq. (8) states explicitly that for $t > t_d$, the velocity $\tilde{v} = 0$. Without this step function, the solution $\tilde{v}$ would become negative for $t > t_d$, which is unphysical, since $\tilde{v}$ is defined as a magnitude. By setting $\tilde{\Gamma} = 1$, which represents the case of a stationary plasma, Eq. (9) yields $t_d = t_i + t_{\rm st}$. In other words, in the high velocity limit, $t_{\rm st}$ is the time needed after the initial impulse for a particle to thermalize, i.e., to pass to the $v \to 0$ limit.

Equation (7) then gives the results $\mu = \mu_i\, \tilde{v}^{1+Z}$ and $\langle v_\| \rangle = \mu v = \mu_i v_i\, \tilde{v}^{2+Z}$. Similarly, the particle kinetic energy $\mathcal{E} = mv^2/2 = \mathcal{E}_i\, \tilde{v}^2$. The exact determination of these ensemble-averaged quantities allows for the calculation of specific physical quantities of interest comprised of various combinations of v and μ. Note that Eq. (7) is exactly the same in steady-state plasmas,[2] except that μ and v have different time histories.

III. CURRENT-DRIVE IN NONSTATIONARY PLASMA

The time-evolution of the expectation value of the current carried by a single electron, $\langle q v_{||} \rangle = q\mu(t)v(t)$, can be calculated from the general solution of the Langevin equations, Eq. (8).[1] This result then can be used to determine the total plasma current induced by wave-particle interactions in a recombining plasma. Following Ref. 1, the notation is adopted such that $v_\| = v_\|(t, \mathbf{v})$, where $\mathbf{v}$ is the initial velocity of the electron at time $t = 0$. Henceforth, μ and v will, therefore, refer to the initial quantities μ_i and v_i, whose evolution is then described by applying the appropriate factor of $\tilde{v}$ (cf. Eq. (8) and subsequent discussion).

Suppose the energy $\Delta\mathcal{E}$ is expended to push an electron from velocity $\mathbf{v}$ to $\mathbf{v} + \boldsymbol{\Delta}$ through some particular wave-particle resonance, with

$$\Delta\mathcal{E} = m\left(\mathbf{v}\cdot\Delta + \frac{1}{2}|\Delta|^2\right). \tag{10}$$

The ensemble-averaged electric current difference at time t resulting from such a push at time t' is given by

$$\Delta j(t, t', \mathbf{v}, \Delta) = q\left\langle v_\|(t, t', \mathbf{v} + \Delta) - v_\|(t, t', \mathbf{v})\right\rangle, \tag{11}$$

where it is assumed the electron began with an identically counterpropagating, nonresonant "partner" electron that does not receive the impulse. The rate of pushing a density of electrons in this way is given by $P/\Delta\mathcal{E}$, where P is the power density expended in the resonant interaction. Thus, in a stationary plasma with a single stationary wave resonance, the total current density is given by

$$J(t) = \int_0^t dt' \frac{P(t')}{\Delta\mathcal{E}} \Delta j(t, t', \mathbf{v}, \Delta). \tag{12}$$

Extending Eq. (12) to allow for multiple discrete resonances, or even a continuum of resonances, becomes important when considering nonstationary plasma, since wave-particle resonance conditions can change dynamically as the bulk plasma changes.[12] To capture this behavior, first define the quantity $\Pi(t',\mathbf{v})$, which has units of power density per velocity volume and represents the power density expended pushing particles in an infinitesimal volume of velocity space neighboring $\mathbf{v}$ at time t'. Then, the generalization of Eq. (12) including any number of resonances is

$$J(t) = \int_0^t dt' \int d^3\mathbf{v} \frac{\Pi(t', \mathbf{v})}{\Delta\mathcal{E}(t', \mathbf{v})} \Delta j(t, t', \mathbf{v}, \Delta(t', \mathbf{v})). \tag{13}$$

The quantity $\boldsymbol{\Delta}(t',\mathbf{v})$, and hence also $\Delta\mathcal{E}(t',\mathbf{v})$ through Eq. (10), now exhibits both time- and velocity-space dependence, allowing for the inclusion of non-steady-state effects associated with embedded waves as suggested above. In the limit of infinitesimal incremental velocity-space displacements, i.e., $\boldsymbol{\Delta}(t',\mathbf{v}) \to 0$, Eq. (13) can be rewritten

$$\begin{aligned} J(t) &= \int_0^t dt' \int d^3\mathbf{v}\, \Pi(t', \mathbf{v}) \frac{\mathbf{S}_w(t', \mathbf{v}) \cdot \dfrac{\partial}{\partial \mathbf{v}} \left\langle q v_\|(t, t', \mathbf{v})\right\rangle}{\mathbf{S}_w(t', \mathbf{v}) \cdot \dfrac{\partial}{\partial \mathbf{v}} \mathcal{E}(\mathbf{v})} \\ &\equiv \int_0^t dt' \int d^3\mathbf{v}\, \Pi(t', \mathbf{v}) K(t, t', \mathbf{v}), \end{aligned} \tag{14}$$

where the vector $\mathbf{S}_w$, the wave-induced flux, is potentially time- and velocity-space dependent and points in the direction of velocity-space displacement of particles. Equation (14) corresponds to Eq. (1) of reference Ref. 8.

For a pure parallel push, i.e., $\mathbf{S}_w \sim \hat{i}_\|$, the kernel K of Eq. (14) can be calculated in a straightforward manner and is given by (cf. Eqs. (7) and (8))

$$K_\parallel = \frac{q}{m}\frac{\tilde{v}^{2+Z} + (2+Z)\mu^2(1-\tilde{v}^3)\tilde{v}^{-1+Z}}{\mu v}. \quad (15)$$

In full analogy with Eq. (5) of Ref. 2, the first term in the numerator of the kernel is associated with the parallel momentum transfer from the wave to the particles, while the second term is associated with the energy transfer to the particles. To see this, a similar expression for K can be calculated for a pure perpendicular wave impulse, i.e., $\mathbf{S}_w \sim \hat{i}_\perp$:

$$K_\perp = \frac{q}{m}\frac{(2+Z)\mu^2(1-\tilde{v}^3)\tilde{v}^{-1+Z}}{\mu v}, \quad (16)$$

which is identical to Eq. (15) except for the absence of the leading term in the numerator, since a perpendicular impulse involves no direct input of parallel momentum.

This formalism was first employed to determine steady-state current drive efficiencies in stationary plasmas.[1] In general, dynamic evolution of the plasma current density, $J_{\text{tot}}(t)$, in nonstationary systems results in an inductively driven electric field and an associated Ohmic counter-current, J_{Ohm}, opposing the wave-driven current. This paper shall focus primarily on the dynamics of the wave-induced fast-particle current, J_{rf}; for a discussion of the impact of the Ohmic counter-current, refer to Appendix A.

IV. IMPULSE RESPONSE IN RECOMBINING PLASMA

The formulation of the current drive problem above allows for the distinctive behavior of a recombining plasma (compared to a plasma with fixed n) to be characterized completely by solving $\tilde{v}(t,t')$, which, according to Eq. (8), is simply the time evolution of the magnitude of the fast-electron velocity. Suppose the electron-ion recombination rate, ν_R, is proportional to the product of the number density of each charge-carrier species, which in general leads to an exponential recombination rate and a scaling of the collision parameter $\Gamma(t)$ that goes like $\Gamma(t) = \Gamma_i e^{-\nu_R t}$, with $\nu_R > 0$. For this exponential recombination profile, Eq. (8) gives

$$\tilde{v}(t,t') = \left[1 + \frac{1}{\nu_R t_{\text{st}}}\left(e^{-\nu_R t} - e^{-\nu_R t'}\right)\right]^{1/3}\Theta(t_d - t). \quad (17)$$

Note that at $t = t'$, i.e., at the time of the initial impulse, $\tilde{v} = 1$. Then, according to Eqs. (14) and (15), a δ-function *parallel* impulse, i.e., $\Pi \sim \delta\,(t'-t_i)\delta^3(\mathbf{v}-\mathbf{v}_i)$ and $\mathbf{S}_w \sim \hat{i}_\parallel$, results in an immediate finite current density at the time of the impulse, which is due to the sudden increase of the parallel momentum of the resonant particles due to the wave. However, a similar kick in the perpendicular direction results in no immediate current, since $K_\perp = 0$ when $\tilde{v} = 1$, cf. Eq. (16).

One of the most interesting features of the current in an exponentially recombining plasma turns out to be that, in some scenarios, there exists a "residual" current, i.e., one that persists as $t \to \infty$. Following the initial impulse, $\tilde{v}$ tends to decrease toward zero as the velocity of the electron damps away. However, because the plasma density is decreasing with time due to recombination, the collision frequency decreases accordingly, and certain conditions will lead to a nonzero time-asymptotic value for $\tilde{v}$, and hence, a time-asymptotic current (when no other damping mechanisms, such as collisions with neutrals, are considered). Specifically, as $t \to \infty$, a residual current will persist under the condition

$$1 - \frac{1}{\nu_R t_{\text{st}}} e^{-\nu_R t'} \equiv \tilde{v}_\infty^3 > 0, \quad (18)$$

where $\tilde{v}_\infty$ is the saturated value of $\tilde{v}$ as $t \to \infty$; otherwise, the time required for a fast-particle to thermalize and for the associated current to decay is given by

$$t_d = t' - (1/\nu_R)\ln(1 - \nu_R t_{\text{st}} e^{\nu_R t'}), \quad (19)$$

which correctly approaches $t_d = t' + t_{\text{st}}$ in the limit as $\nu_R \to 0$ (cf. Eq. (9)). The persistence of a nonzero electric current time-asymptotically is unique to the recombining plasma, for a current carried by fast-particles will typically decay to zero in a stationary plasma of fixed density in the absence of a dc electric field.

With the electron average dynamics solved, it is mathematically straightforward to deal first with discrete impulses, where $\Pi = \sum_i w_i \delta(t'-t_i)\delta^3(\mathbf{v}-\mathbf{v}_i)$, and the w_i are the energy densities deposited with each discrete impulse. For example, it might be desirable to maximize the time-averaged current density for a single impulse, defined as $\langle J\rangle_T \equiv (1/T)\int_0^T J(t)\,dt$. For a $Z = 1$ plasma, the current density resulting from a single *parallel* impulse, $\Pi = w_1\delta(t'-t_1)\,\delta^3(\mathbf{v}-\mathbf{v}_1)$, made dimensionless by the normalization $\tilde{J}_1 = (mv_1/qw_1)J_1$, is given by

$$\tilde{J}_1 = \left[\frac{\tilde{v}^3 + 3\mu^2(1-\tilde{v}^3)}{\mu}\right]_1 \Theta(t_{d1} - t). \quad (20)$$

Since $\tilde{J}$ contains the quantity J/w, it can be seen as an "efficiency" stating how effectively wave energy is converted to current density.

Setting $T = t_{d1}$, we find from Eq. (20) that

$$\langle\tilde{J}\rangle_{t_d} = \frac{1}{\mu}\left[3\mu^2 + (1-3\mu^2)\left(\left(\frac{t_d - t_1}{t_d}\right)\tilde{v}_\infty^3 + \frac{1}{\nu_R t_d}\right)\right], \quad (21)$$

where the subscript "1" has been dropped to save space. When the condition in Eq. (18) is satisfied and the current saturates time-asymptotically, i.e., $t_d \to \infty$, Eq. (21) simplifies to

$$\langle\tilde{J}\rangle_\infty = \frac{1}{\mu}\left[3\mu^2 + (1-3\mu^2)\tilde{v}_\infty^3\right]. \quad (22)$$

This expression diverges as $\mu \to 0$, indicating there is a substantial benefit to pushing high-pitch-angle electrons in the parallel direction, which is similar to the result in the steady-state case.[25] However, there also exists a minimum efficiency with respect to variation in μ, when $\mu = \sqrt{\tilde{v}_\infty^3/3(1-\tilde{v}_\infty^3)} \equiv \mu_{\min}$. If $0 < \mu_{\min} < 1$, the impulse efficiency increases monotonically for values of $\mu > \mu_{\min}$, maximizing locally at $\mu = 1$. On the other hand, if $\mu_{\min} > 1$, or equivalently, if $\tilde{v}_\infty^3 > 3/4$, then the impulse efficiency decreases monotonically across the domain μ: (0,1), with an absolute minimum at $\mu = 1$. The same optimization analysis applies to the impulse efficiency when the time-asymptotic current goes to zero; comparing Eqs. (21) and (22), it is apparent that one need only make the replacement

$$\tilde{v}_\infty^3 \rightarrow \left(\left(\frac{t_d - t_1}{t_d}\right)\tilde{v}_\infty^3 + \frac{1}{\nu_R t_d}\right)$$

in the equality defining $\mu_{\min}$.

A similar analysis for a pure perpendicular δ-function impulse yields

$$\langle \tilde{J} \rangle_{t_d} = 3\mu\left(1 - \left(\frac{t_d - t_1}{t_d}\right)\tilde{v}_\infty^3 - \frac{1}{\nu_R t_d}\right) \tag{23}$$

and

$$\langle \tilde{J} \rangle_\infty = 3\mu\left(1 - \tilde{v}_\infty^3\right). \tag{24}$$

Both Eqs. (23) and (24) are well-behaved at $\mu = 0$, and they are maximized at $\mu = 1$ for fixed v. Thus, it holds qualitatively that the highest impulse efficiencies for perpendicular pushing occur when $\mu \approx 1$, while the highest impulse efficiencies for parallel impulses occur when $\mu \approx 0$. This is a reasonable result, considering that, in both cases, $\mathbf{v}\cdot\boldsymbol{\Delta} = 0$ in Eq. (10), implying a minimization of energy input per particle, while $\mathbf{v}$ can still be large, meaning the particles being pushed can already have intrinsically lower collision rates than thermal particles.

V. DYNAMICS OF AN EMBEDDED LANGMUIR WAVE

Consider now a power deposition profile that captures the effect of a pure wave mode damping collisionlessly on high-velocity particles

$$\Pi(t', \mathbf{v}) = P(t')\delta^3(\mathbf{v} - \mathbf{v}_r(t')). \tag{25}$$

The region of wave-particle resonance is highly localized in velocity space at any given moment, a consequence of the presumed narrow bandwidth of the interacting wave, and it also shifts continuously with time as the plasma density changes. At time $t = t'$, the wave deposits power density $P(t')$ into the particles in the infinitesimal neighborhood surrounding the resonant velocity $\mathbf{v}_r(t')$ in velocity space. For an embedded wave with finite initial energy density, $\int_0^\infty P(t')\,dt'$ must also be finite.

For example, consider the effect of an embedded Langmuir wave driving electrons in the parallel direction due to Landau resonance,[26] which can be described using the kernel $K_{||}$ (cf. Eq. (15)). To simplify the problem, take $\mu = 1$, which is a reasonable approximation when driving particles with high parallel velocities, and it reduces the problem to one effective velocity-space dimension. Additionally, take $Z = 1$. The phase velocity of a Langmuir wave changes proportionally to the square root of the plasma density, so the resonant velocity $|\mathbf{v}_r| \equiv v_r = v\exp\left(-\frac{\nu_R}{2}t'\right)$, where v is the initial wave phase velocity. Then, plugging Eq. (25) into Eq. (14), and including a Heaviside step function $\Theta(t_d(t') - t)$ in the kernel K (cf. Eq. (20) and subsequent discussion), the result is

$$\begin{aligned} J(t) &= \frac{q}{m}\int_0^t dt' P(t')\frac{3 - 2\tilde{v}^3(t,t')}{v_r(t')}\Theta(t_d(t') - t) \\ &= \frac{q}{m}\frac{1}{v}\int_0^t dt' P(t')\left[e^{\frac{\nu_R t'}{2}} + \frac{2}{\nu_R t_{\text{st}}}\right. \\ &\quad \times \left.\left(e^{\nu_R t'} - e^{-\nu_R t}e^{2\nu_R t'}\right)\right]\Theta(t_d(t') - t), \end{aligned} \tag{26}$$

where $t_{\text{st}} = v^3/3\Gamma_i$, as usual. It must be emphasized that the purpose of the step function Θ is to exclude unphysical contributions to the integral occurring where $\tilde{v}(t,t')$ takes on negative values in the kernel K, or equivalently, where t exceeds the damping time for current driven at time t', given by $t_d(t')$. The definition of $t_d(t')$ is restated as follows:

$$t_d(t') = t' - (1/\nu_R)\ln(1 - \nu_R T_{\text{st}} e^{\nu_R t'}),$$

where now the function T_{st} is the analog of the stationary plasma thermalization time t_{st} in the case where the resonant velocity $v_r(t')$ is shifting with time, or for the Langmuir waves considered in this example,

$$\begin{aligned} T_{\text{st}} &= \frac{v_r^3(t')}{3\Gamma_i} \\ &= t_{\text{st}} e^{-\frac{3\nu_R t'}{2}}. \end{aligned} \tag{27}$$

Then, we have

$$t_d(t') = t' - \frac{1}{\nu_R}\ln\left(1 - \nu_R t_{\text{st}} e^{-\frac{\nu_R t'}{2}}\right). \tag{28}$$

Equation (28) illustrates the complexity involved in correctly assessing the long-time behavior of the current, and three unique regimes exist in which the integral in Eq. (26) must be calculated differently. The proper method for handling the step function in calculating the integral is addressed in Appendix B.

The most general solution of Eq. (26) would describe the plasma response to an arbitrary complex Fourier mode in the place of $P(t')$

$$\begin{aligned} P(t') &= P_d\,\Re\left[e^{i(\omega t' + \eta) - \gamma t'}\right] \\ &= P_d\,e^{-\gamma t'}\cos(\omega t' + \eta), \end{aligned} \tag{29}$$

with P_d, ω, η, and γ all real constants, and γ is presumed positive so that $\int_0^\infty P(t')\,dt'$ remains finite. Then, the response to any smooth, continuous power deposition profile could be calculated by integrating (or summing) over the response functions corresponding to the appropriate Fourier modes comprising $P(t')$. However, due to the complexity of the result and its limited usefulness in the discussion that follows, the general solution can be found in Appendix C.

A more illuminating example makes the following simplification: take $\omega = 0$ and $\eta = 0$ in Eq. (29). Then, the total wave energy density deposited into particles is given by $W = \int_0^\infty P(t')\,dt' = P_d/\gamma$. This non-oscillating, decaying exponential, $P(t') = P_d e^{-\gamma t'}$, approximates the resonant linear wave power deposition profile in a recombining plasma, as will be explained below.

Suppose a linear wave resonant with the suprathermal tail of a Maxwellian distribution undergoes exponential collisionless damping at the effective rate γ_L.[26] Although a time-varying resonance could result in a change in γ_L as the wave samples different parts of the tail of the distribution, it will be assumed here that Landau damping is characterized by an average, constant timescale. Furthermore, the wave also undergoes collisional damping at some effective rate ν_c,[27] which is also assumed to be constant. Since the plasma is undergoing a reduction in the characteristic collision rates due to recombination, the assumption $\nu_c = \text{const}$ will result in an overestimate of the amount of collisional wave damping that occurs when the recombination rate is appreciable. Thus, the calculation of the residual current that follows will represent a lower bound. Since the wave energy density U_w is proportional to the square of the field amplitude,[12] we have $\delta U_w \sim -2(\gamma_L + \nu_c)U_w \delta t$.

Additionally, from Ref. 22, recombination results in the conservation of the invariant U_w/ω, so $\delta U_w \sim U_w(\delta\omega/\omega)$. Since $\omega \approx \omega_p \propto n^{1/2}$, one finds $\delta U_w \sim -(\nu_R/2)U_w \delta t$. Combining the effects of Landau damping, collisional damping, and recombination leads to

$$\frac{dU_w}{dt} = -\left(2(\gamma_L + \nu_c) + \frac{\nu_R}{2}\right)U_w, \tag{30}$$

which implies $\gamma = 2(\gamma_L + \nu_c) + \nu_R/2$. On the other hand, only the wave energy lost to Landau damping is deposited resonantly onto suprathermal particles, so $P(t') = 2\gamma_L U_w(t')$ implies $P_d = 2\gamma_L U_w(0)$, and thus,

$$\begin{aligned} W &= \frac{2\gamma_L}{2(\gamma_L + \nu_c) + \nu_R/2} U_w(0) \\ &= \frac{1}{1 + \frac{1}{4}\frac{2\nu_c + \nu_R}{\gamma_L}} U_w(0) \\ &\equiv \varepsilon U_w(0). \end{aligned} \tag{31}$$

Hence, ε represents the efficiency with which the wave energy density is channeled resonantly into suprathermal electrons compared to the energy lost to bulk electrons through nonresonant processes.

With the assumption $\omega = 0$ and $\eta = 0$ in Eq. (29), the expression for the total dimensionless current density $\tilde{J}$, which is normalized according to $\tilde{J} = (mv/qU_w(0))J$, is found to be

$$\begin{aligned} \tilde{J} = &-h(t; -\varepsilon\gamma) \\ &+ \frac{2\varepsilon}{\nu_R t_{st}}\left[\frac{1}{1-2\varepsilon} h(t; \gamma(1-2\varepsilon)) - \frac{e^{-\nu_R t}}{3-4\varepsilon} h(t; \gamma(3-4\varepsilon))\right], \end{aligned} \tag{32}$$

where

$$h(t; \Omega) = e^{\Omega t} + e^{\Omega t_l(t)\Theta(t - t_{d,\min})} - e^{\Omega t_u(t)\Theta(t - t_{d,\min})} - 1. \tag{33}$$

The Heaviside step functions in the two middle terms in Eq. (33) cause the terms to cancel when $t < t_{d,\min}$, allowing for a complete solution to be expressed in a single expression for all t: $[0, \infty]$. For a description of the notation used in Eq. (33), please refer to Appendix B.

For comparison, a similar expression for the total dimensionless current density for a stationary plasma, $\tilde{J}_{st}$, subject to the same exponential power deposition profile, $P(t') = P_d e^{-\gamma t'}$, can also be obtained from Eq. (26). In this case, $\nu_R = 0$, $\varepsilon = 1$, $\tilde{v}^3(t, t') = 1 - (t - t')/t_{st}$ (cf. Eq. (8)), and the damping time $t_d(t') = t' + t_{st}$ (cf. Eq. (19)), leading to the solution

$$\tilde{J}_{st} = \begin{cases} (1 - e^{-\gamma t}) + \dfrac{2}{\gamma t_{st}}(e^{-\gamma t} + \gamma t - 1), & t \le t_{st} \\ e^{-\gamma t}(e^{\gamma t_{st}} - 1) + \dfrac{2}{\gamma t_{st}}\left[e^{-\gamma t} + (\gamma t_{st} - 1)e^{-\gamma(t - t_{st})}\right], & t > t_{st} \end{cases}, \tag{34}$$

From Eq. (34), one observes that in the limit $t \to \infty$, the current density in a stationary plasma always decays to zero despite the fact that no wave energy is lost due to recombinational plasmon destruction. However, in a recombining plasma, there exists an interval of time, t: $[0, t_\infty]$, when the condition in Eq. (18) is satisfied, and any current driven during this interval saturates at some finite value (cf. Eq. (B1) in Appendix B for the definition of t_∞). In fact, this residual current can be extracted from Eq. (32) in the limit $t \to \infty$:

$$\begin{aligned} \lim_{t\to\infty} \tilde{J} &= 1 - (\nu_R t_{st})^{\frac{\varepsilon}{\varepsilon - 1}} + \frac{1}{\nu_R t_{st}} \frac{2\varepsilon}{1 - 2\varepsilon}\left((\nu_R t_{st})^{\frac{1-2\varepsilon}{1-\varepsilon}} - 1\right) \\ &\equiv \tilde{J}_\infty, \end{aligned} \tag{35}$$

where the condition $t_\infty > 0$ implies $\nu_R t_{st} > 1$ for this solution to be valid, as outlined in Appendix B. To arrive at this expression, note that as $t \to \infty$, $t_l \to t_\infty$ and $t_u \to \infty$ (cf. Fig. 3(a) in Appendix B).

Figure 2 shows $\tilde{J}_\infty$ plotted over the intervals ε: [0, 1] and $\nu_R t_{st}$: [1, 10]. One observes the threshold of the onset of

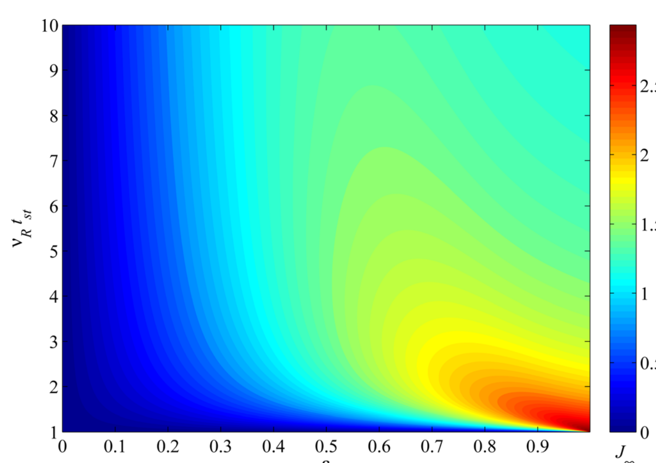


FIG. 2. (Color online) Plot of the time asymptotic normalized current density, $\tilde{J}_\infty$ (cf. Eq. (35)), vs. ε and $\nu_R t_{st}$.

nonzero $\tilde{J}_\infty$, when $\nu_R t_{st} = 1$. Additionally, as $\varepsilon \to 0$, signifying very fast recombination compared to the collisionless damping rate ($\nu_R \gg \gamma_L$, cf. Eq. (31)), enhanced plasmon destruction also eliminates the residual current. From Fig. 2, it is apparent that a regime exists in which $\tilde{J}_\infty$ is maximized. This regime calls for ε to be very close to its maximum value of 1, signifying a relatively strong collisionless damping rate compared to the collsional and recombination rates. On the other hand, values of $\nu_R t_{st}$ close to its minimum value of 1 are also ideal, implying that the recombination rate is comparable to the initial fast particle thermalization rate. In other words, driving maximum residual current requires $\nu_L \gg \nu_c \gg \nu_R \sim (t_{st})^{-1}$. This optimal regime is somewhat remarkable, considering the ideal value of $\nu_R t_{st}$ is very close to the region in which $\tilde{J}_\infty$ disappears altogether; indeed, as $\varepsilon, \nu_R t_{st} \to 1$, $\partial \tilde{J}_\infty / \partial(\nu_R t_{st})$ becomes infinitely steep. The maximum value of $\tilde{J}_\infty$ is found by taking the limit as both parameters go to 1:

$$\lim_{\varepsilon, \nu_R t_{st} \to 1} \tilde{J}_\infty = 3. \tag{36}$$

Considering values of $\nu_R t_{st} > 1$, it is interesting to note that the maximum residual current for a particular value of $\nu_R t_{st}$ is not produced simply by maximizing the resonant power deposition efficiency ε; indeed, the maximum residual current requires intermediate values of $\varepsilon < 1$. In other words, it turns out to be better to sacrifice some wave energy to plasmon destruction due to recombination such that the fast particles produced by the remaining wave energy experience slower damping as the collision frequency is dynamically reduced. It is also worth pointing out that the apparent singularity at $\varepsilon = 1/2$ in Eq. (35) is removable, as is evidenced by the smooth, continuous behavior of $\tilde{J}_\infty$ across the line $\varepsilon = 1/2$ in Fig. 2.

VI. DISCUSSION

The existence of a time-asymptotic current in Eq. (35) deserves further inspection. After a lapse of time of approximately ν_R^{-1}, the neutral particle density n_n produced through recombination is expected to be comparable to n_e, and collisions between electrons and neutral particles could become significant if neutrals are permitted to remain within the system. The model assumes that the collision frequency of electrons with other charged particles, ν_e, is greater than the electron-neutral collision frequency, ν_n. The ratio of these two quantities can be expressed approximately as

$$\frac{\nu_e}{\nu_n} \approx 8 \times 10^2 \frac{n_e}{n_n} \frac{\ln \Lambda}{T_e^2}, \tag{37}$$

with T_e the electron temperature (expressed in eV). Here, the electron-neutral collision frequency is estimated as $\nu_n \sim n_n(\pi a_0^2)\nu_T$, with v_T the electron thermal velocity and a_0 the Bohr radius.[28] Since $\ln \Lambda \sim \mathcal{O}(10)$, it is apparent that electron-neutral collisions in substantially recombined plasma, i.e., $n_e \sim n_n$, become important for electron temperatures above approximately 100 eV if all neutrals are retained within the system. Note that there is some reduction in the scattering cross section of electrons from neutrals at higher velocities, which will attenuate electron-neutral collisions to some extent as this approximate threshold is reached, since the current-carrying electrons are suprathermal.

In the event that neutrals are lost from the system quickly, as was originally presumed in Sec. II, then Eq. (35) is in fact the correct time-asymptotic fast particle current. For finite $\tau_g = L/R_{Sp}$, where L is the torus inductance and R_{Sp} is the torus resistance, the total plasma current will approach this value within time τ_g after J_{rf} has asymptotically approached its limiting value (cf. Appendix A). In the high-L limit, with $\tau_g \to \infty$, the time-asymptotic picture is more complicated. As the electron density falls off, the remaining electrons will be Ohmically accelerated to higher velocities to maintain the perfectly cancelling countercurrent. At sufficiently low electron density, it would no longer be appropriate to consider the plasma consisting of a thermal "bulk" plasma and a small fast particle population, but rather one consisting of two counterpropagating particle beams, which would then result in instability.

Nevertheless, for large enough recombination rates, i.e., $\nu_R t_{st} > 1$, charged-particle collisions are found to be insufficient to cause J_{rf} to disappear time-asymptotically. This analysis has critical implications for the alternate scenario in which electric current is carried by plasma undergoing expansion, which similarly results in a reduction of the plasma density with time without necessarily increasing the neutral density commensurately. In this case, there also exists the potential for a robust time-asymptotic current density, which will be the subject of a future publication; however, the model of a recombining plasma studied here offers great insight into the impact of densification on the dynamics of charged particle collisions and plasma current evolution.

VII. SUMMARY

In this paper, fast-particle collisional dynamics and current drive in nonstationary plasma has been calculated using the Langevin formalism to model the Boltzmann equation in the strict high-velocity limit. The particular model of a recombining plasma was chosen as a case of fundamental scientific interest and for its analytical tractability. Solutions were derived for the Langevin equations containing time-dependent parameters associated with plasma densification. A general expression for the current density produced by arbitrary time- and velocity-space-dependent particle fluxes was also derived.

Since recombination has the effect of lowering the charge density of the plasma, and, consequently, reducing the charged particle collision frequencies, the temporal evolution of the current density can be modified substantially compared to the case of a plasma with fixed charge density. Conditions for maximizing current drive efficiency for discrete and continuous wave-particle resonances were found, leading to the discovery of a nonzero time-asymptotic, "residual" current density in recombining plasma, corresponding to Eqs. (22) and (24) for discrete impulses and Eq. (35) for an embedded Langmuir wave. Maximizing this residual current turns out to require an optimal and unexpected compromise between wave energy loss due to recombination-driven plasmon destruction and more efficient current drive efficiency as the collisionality of the plasma is dynamically reduced.

ACKNOWLEDGMENTS

This work was supported by the U.S. DOE under Contract No. DE-AC02-09CH11466 and through the NNSA SSAA Program by DOE Research Grant No. DE-FG52-08NA28533. The authors thank Ilya Dodin for valuable discussions regarding the associated physics.

APPENDIX A: INDUCTIVE EFFECTS

Following Sec. V B in Ref. 1, the total plasma current density $J_{\rm tot}$ obeys the equation

$$\frac{dJ_{\rm tot}}{dt} = -\frac{J_{\rm tot} - J_{\rm rf}}{\tau_g}, \tag{A1}$$

where $J_{\rm rf}$ is the wave-driven fast particle current contribution, and $J_{\rm tot} = J_{\rm rf} + J_{\rm Ohm}$. Here, $\tau_g = L/R_{\rm Sp}$, where L is the torus inductance and $R_{\rm Sp}$ is the torus resistance. The resulting Ohmic electric field induced in the plasma is given by

$$E = \eta J_{\rm Ohm}, \tag{A2}$$

where $\eta \approx m\nu_{ei}/n_e e^2$ is the plasma resistivity, ν_{ei} is the electron-ion collision frequency, and n_e is the electron number density. Because L is determined by the geometry of the torus and the plasma resistivity is only weakly dependent on the charge-carrier density, it can be assumed that τ_g does not change dramatically even after a substantial percentage of charge-carriers have recombined.

The impact of this dc electric field on the particle trajectories in the Langevin equations is minimal, so $J_{\rm rf}$ can be treated as a prescribed term in Eq. (A1) once the wave-driven current in the absence of any dc fields, Eq. (14), is found. To see this, note that in the presence of a dc electric field E, the system of Langevin equations, Eqs. (1) and (5), is modified, becoming[1]

$$\frac{dv}{dt} = -\left(\frac{\Gamma}{v^3}\right)v + \left(\frac{qE}{m}\right)\mu, \tag{A3}$$

$$\frac{d\mu}{dt} = -\left(\frac{\Gamma}{v^3}\right)(1+Z)\mu + \frac{(qE/m)(1-\mu^2)}{v}. \tag{A4}$$

The collisional terms on the right-hand-side (RHS) of Eqs. (A3) and (A4) are generally much larger than the terms involving E. For example, noting that $\Gamma \propto \nu_{ei} v_T^3$, the ratio of the first and second terms on the RHS of Eq. (A3) goes like

$$\frac{\Gamma/v^2}{eE\mu/m} \sim \frac{\nu_{ei} v_T^3 m}{eEv^2\mu} \sim \frac{\eta e n_e v_T^3}{Ev^2\mu} = \frac{1}{\mu}\frac{en_e v_T}{J_{\rm Ohm}}\frac{v_T^2}{v^2} \gg 1, \tag{A5}$$

where Eq. (A2) was used along with the definition of the plasma resistivity. The first factor in the expression to the right of the equals sign in Eq. (A5), $1/\mu$, is always greater than 1. Since the current-carrying particles are suprathermal, the last factor is typically of the magnitude $1/a^2$, where $a \sim \mathcal{O}(1)$. However, the current density in the numerator of the middle factor is enormous compared to the Ohmic current density, since $J_{\rm Ohm}$ cannot be larger than $J_{\rm rf} \sim \delta n e v_T$, with $\delta n \ll n_e$. Thus, the product of the factors is large, and the factor involving E in Eq. (A3) is negligible. A similar conclusion can be deduced for Eq. (A4).

The general solution of Eq. (A1) is given by

$$J_{\rm tot}(t) = J_{\rm tot}(0)e^{-\frac{t}{\tau_g}} + \frac{1}{\tau_g}\int_0^t dt' e^{-\frac{t-t'}{\tau_g}} J_{\rm rf}(t'). \tag{A6}$$

While a closed form solution of Eq. (A6) is not easy to obtain, there are limits where the result is simple. For example, when $\tau_g \to 0$, the inductive response of the plasma is weak. Then, Eq. (A1) states $J_{\rm tot} \approx J_{\rm rf}$, and the inductive counter-current disappears. Similarly, in the opposite limit, $\tau_g \to \infty$, one has $dJ_{\rm tot}/dt \approx 0$. So for a plasma starting with zero current, $J_{\rm rf}$ is exactly cancelled by $J_{\rm Ohm}$; however, according to Eq. (A2), there still exists a dc electric field, and thus, a loop voltage exists around the plasma.

APPENDIX B: THE STEP FUNCTION

Equation (28) illustrates the difficulty one encounters in keeping track of the long-time behavior of the current, and three unique regimes exist in which different intervals of the domain of integration in Eq. (26) are excluded by the step function, as will be explained below. The critical dimensionless parameter in this particular problem is $\nu_R t_{\rm st}$, which compares the *initial* stationary thermalization time to the density e-folding time due to recombination.

Case 1, $\nu_R t_{\rm st} > 1$: The behavior of $t_d(t')$ is uniquely determined by the magnitude of $\nu_R t_{\rm st}$. When $\nu_R t_{\rm st} > 1$, the condition in Eq. (18) is satisfied for all current driven during the times t': $[0, t_\infty]$, with t_∞ given by

$$t_\infty = \frac{2}{\nu_R}\ln(\nu_R t_{\rm st}). \tag{B1}$$

This implies that $t_d \to \infty$ and some time-asymptotic current remains for any current driven during this time period, as can be seen in Fig. 3(a), which depicts the main features of $t_d(t')$ in this regime.

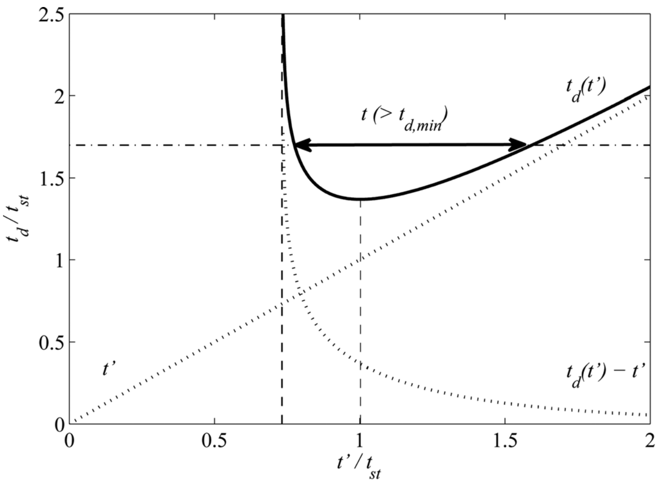
Equally important, for $t' > t_\infty$, t_d is initially decreasing and possesses an absolute minimum, designated as $t_{d,\rm min}$, at $t' = t_{\rm min}$, with

$$t_{\rm min} = \frac{2}{\nu_R}\ln\left(\frac{3}{2}\nu_R t_{\rm st}\right) \tag{B2}$$

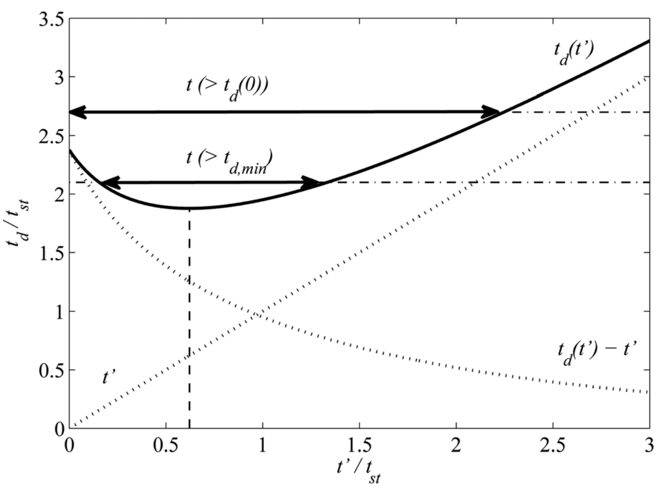
and

$$t_{d,\rm min} = \frac{2}{\nu_R}\ln\left(\frac{3\sqrt{3}}{2}\nu_R t_{\rm st}\right). \tag{B3}$$

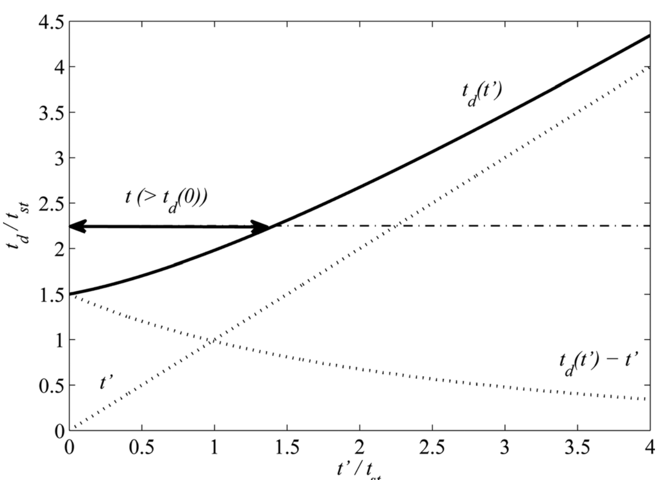
For all values of $t' > t_{\rm min}$, t_d is monotonically increasing. When the upper limit t of the integrand in Eq. (26) is such that $t > t_{d,\rm min}$, there exists an interval t': (t_l, t_u), marked by the double arrows in Fig. 3, for which



(a)$\nu_R t_{\rm st} = 3$. t_∞ and $t_{\min}$ shown (dashed).



(b)$\nu_R t_{\rm st} = 7/8$. $t_{\min}$ shown (dashed).



(c)$\nu_R t_{\rm st} = 7/12$

FIG. 3. Plots of the damping time $t_d(t')$ (solid) and related functions (dotted), normalized to $t_{\rm st}$, are plotted vs. t' for three unique cases defined by different values of $\nu_R t_{\rm st}$. Example values t for the upper limit of the integrand in Eq. (26) are shown (dashed-dotted) for several cases where part of the domain of integration must be excluded to eliminate unphysical contributions to the total current. The interval to be excluded from the domain for each particular t is spanned by the associated double arrow coinciding with the line $t_d = t$.

$t_d(t') < t$, and hence the current driven during these times has damped away by time t. Within this interval, the Heaviside step function $\Theta = 0$, and thus, it is this interval that is effectively removed from the domain of integration, so we have

$$\int_0^t dt' \to \left(\int_0^{t_l} dt' + \int_{t_u}^t dt' \right).$$

One can find the limits $t_{l,u}$ by solving the equation $t = t_d(t_{l,u})$, which has two unique solutions for all $t > t_{d,\min}$ when $\nu_R t_{\rm st} > 1$. Unfortunately, from Eq. (28), we see that this equation is transcendental, and thus, $t_{l,u}$ must be found numerically.

In summary, this regime is characterized both by a time-asymptotic current density resulting from current driven during the times t': $[0, t_\infty]$, and also by the exclusion of the interval t': (t_l, t_u) from the domain of integration in Eq. (26) when $t > t_{d,\min}$, corresponding to unphysical contributions from the kernel K when $t > t_d(t')$.

Case 2, $2/3 < \nu_R t_{\rm st} < 1$: In this regime, shown in Fig. 3(b), $t_\infty < 0$, meaning that the condition in Eq. (18) is no longer satisfied for any value of t' in the domain of integration of the integral in Eq. (26), and thus, no current persists time-asymptotically. On the other hand, there is still an absolute minimum $t_{d,\min}$ when $t' = t_{\min}$, cf. Eqs. (B2) and (B3). Note in Fig. 3(b) that within the interval t: $(t_{d,\min}, t_d(0))$, there still exists two unique positive solutions to the equation $t = t_d(t_{l,u})$, but for $t > t_d(0)$, one finds that the lower limit of the interval to be excluded from the integral in Eq. (26) is simply $t_l = 0$. One still must solve for the upper limit $t_u(t)$ numerically, since Eq. (28) is transcendental.

Case 3, $\nu_R t_{\rm st} < 2/3$: In this regime, depicted in Fig. 3(c), both t_∞, $t_{\min} < 0$ (cf. Eqs. (B1) and (B2)), meaning that, like case 2, this regime exhibits no time-asymptotic current density, and additionally, the absolute minimum of $t_d(t')$ is simply $t_{d,\min} = t_d(0)$. Thus, when $t > t_d(0)$, the interval t': $[0, t_u]$ is excluded from the domain of integration by the Heaviside step function $\Theta(t_d(t') - t)$.

APPENDIX C: CURRENT PRODUCED BY ARBITRARY COMPLEX FOURIER MODE

The solution to the arbitrary complex Fourier mode power deposition profile, Eq. (29), is presented. Plugging Eq. (29) into Eq. (26), the indefinite integral, which will be called $\mathcal{J}(t')$, is found to be

$$\mathcal{J}(t') = \frac{q}{m}\frac{P_d}{v}\Bigg[g(t'; \nu_R/2 - \gamma, \omega, \eta) + \frac{2}{\nu_R t_{\rm st}} \big(g(t'; \nu_R - \gamma, \omega, \eta) - e^{-\nu_R t} g(t'; 2\nu_R - \gamma, \omega, \eta) \big) \Bigg], \tag{C1}$$

anywhere that $\Theta \neq 0$, with

$$g(t'; \Omega, \omega, \eta) = \frac{e^{\Omega t'} [\Omega \cos(\omega t' + \eta) + \omega \sin(\omega t' + \eta)]}{\Omega^2 + \omega^2}. \tag{C2}$$

The general solution can be stated as

$$J(t) = \begin{cases} \mathcal{J}(t) - \mathcal{J}(0), & t \le t_{d,\min} \\ [\mathcal{J}(t) - \mathcal{J}(t_u(t))] + [\mathcal{J}(t_l(t)) - \mathcal{J}(0)], & t > t_{d,\min}, \end{cases} \tag{C3}$$

where $t_l(t)$ and $t_u(t)$ are the unique solutions to the equation $t_d(t_{l,u}) = t$, with $t_u > t_l$ (cf. Eq. (28) and Appendix B). In general, these two quantities must be obtained numerically. The parameter $t_{d,\min}$ is given by Eq. (B3) in Appendix B.

[1]N. J. Fisch, Rev. Mod. Phys. **59**, 175 (1987).
[2]N. J. Fisch and A. H. Boozer, Phys. Rev. Lett. **45**, 720 (1980).
[3]N. J. Fisch, Phys. Rev. Lett. **41**, 873 (1978).
[4]K. Borrass and A. Nocentini, Plasma Phys. Controlled Fusion **26**, 1299 (1984).
[5]V. Fuchs, R. A. Cairns, M. M. Shoucri, K. Hizanidis, and A. Bers, Phys. Fluids **28**, 3619 (1985).
[6]C. S. Liu, V. S. Chan, and Y. C. Lee, Phys. Rev. Lett. **55**, 2583 (1985).
[7]C. F. F. Karney and N. J. Fisch, Phys. Fluids **22**, 1817 (1979).
[8]N. J. Fisch and C. F. F. Karney, Phys. Rev. Lett. **54**, 897 (1985).
[9]C. F. F. Karney, Comput. Phys. Rep. **4**, 183 (1986), and references therein.
[10]K. Appert, T. Hellsten, O. Sauter, S. Succi, J. Vaclavik, and L. Villard, Comput. Phys. Commun. **43**, 125 (1986).
[11]I. Y. Dodin, V. I. Geyko, and N. J. Fisch, Phys. Plasmas **16**, 112101 (2009).
[12]P. F. Schmit, I. Y. Dodin, and N. J. Fisch, Phys. Rev. Lett. **105**, 175003 (2010).
[13]P. F. Schmit, C. R. Mooney, I. Y. Dodin, and N. J. Fisch, J. Plasma Phys. **77**, 629 (2011), available at http://journals.cambridge.org/action/displayAbstract?fromPage=online&aid=8362132.
[14]P. F. Schmit, I. Y. Dodin, and N. J. Fisch, Phys. Plasmas **18**, 042103 (2011).
[15]K. Avinash, Phys. Plasmas **17**, 123710 (2010).
[16]Y. Takase, A. Ejiri, H. Kakuda, Y. Nagashima, T. Wakatsuki, O. Watanabe, P. T. Bonoli, O. Meneghini, S. Shiraiwa, J. Wright, C. Moeller, H. Kasahara, R. Kumazawa, T. Mutoh, and K. Saito, Nucl. Fusion **51**, 063017 (2011).
[17]A. Schmidt, P. T. Bonoli, O. Meneghini, R. R. Parker, M. Porkolab, S. Shiraiwa, G. Wallace, J. C. Wright, R. W. Harvey, and J. R. Wilson, Phys. Plasmas **18**, 056122 (2011).
[18]K. K. Kirov, Y. Baranov, J. Mailloux, M. L. Mayoral, M. F. F. Nave, and J. Ongena, Nucl. Fusion **50**, 075003 (2010).
[19]N. J. Fisch, J. Plasma Phys. **76**, 627 (2010).
[20]M. Uchida, Y. Yoshinaga, H. Tanaka, and T. Maekawa, Phys. Rev. Lett. **104**, 065001 (2010).
[21]J. Garcia, G. Giruzzi, P. Maget, J. F. Artaud, V. Basiuk, J. Decker, G. Huysmans, F. Imbeaux, Y. Peysson, and M. Schneider, Nucl. Fusion **50**, 025025 (2010).
[22]I. Y. Dodin and N. J. Fisch, Phys. Plasmas **17**, 112118 (2010).
[23]S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).
[24]M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945).
[25]N. J. Fisch and C. F. F. Karney, Phys. Fluids **24**, 27 (1981).
[26]T. H. Stix, *Waves in Plasmas* (AIP, New York, 1992), Chap. 8.
[27]A. Lenard and I. B. Bernstein, Phys. Rev. **112**, 1456 (1958).
[28]G. Foster and W. Williamson, Jr., Phys. Rev. A **13**, 936 (1976).