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Comparisons between nonlinear kinetic modelings of simulated Raman scattering using envelope equations

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In this paper, we compare two recent models [N. A. Yampolsky and N. J. Fisch, Phys. Plasmas **16**, 072104 (2009); D. Bénisti, D. J. Strozzi, L. Gremillet, and O. Morice, Phys. Rev. Lett. **103**, 155002 (2009)] introduced to predict the nonlinear growth of stimulated Raman scattering in the kinetic regime, and providing moreover a nonlinear description of the collisionless, Landau-like, damping rate of the driven electron plasma wave. We first recall the general theoretical framework common to these two models, based on the derivation of the imaginary part of the electron susceptibility, χ_i, and then discuss in detail why the two approaches differ. By comparing the theoretical predictions for χ_i to those derived from test particle or Vlasov simulations, we moreover discuss the range of validity of the two models. © *2012 American Institute of Physics*. [doi:10.1063/1.3677264]

I. INTRODUCTION

Predicting the nonlinear growth of stimulated Raman scattering (SRS) in the kinetic regime, when the electron distribution function may be greatly modified due to the growth of a large amplitude plasma wave, has remained a challenge for decades. Yet, this is a problem of great importance for inertial confinement fusion[1] or Backward Raman amplification,[2] two applications that motivated the theoretical works of Refs. 3 and 4 which we further discuss here. These two papers placed a particular emphasis on the nonlinear reduction of the collisionless, Landau-like, damping rate of the laser-driven electron plasma wave, an effect that may greatly enhance Raman reflectivity, as invoked to explain the experimental results of Ref. 5. Actually, since these experimental results have been published, several "reduced" models (see for example Refs. 6–11), relying on the hypothesis that the electric field amplitudes may be considered as slowly varying envelopes, have been introduced in order to recover the so-called "kinetic inflation" of Raman reflectivity described in Ref. 5. However, it is not quite easy to understand how all these models differ, what their ranges of validity are, and which model should be used depending on the physics conditions. It is the aim of this paper to partially fill this gap by carefully discussing, and comparing, the theoretical approaches of Refs. 3 and 4, as well as the precision of the results deduced from these theories.

A key parameter to derive how efficiently an electron plasma wave (EPW) may be laser-driven, and therefore to quantify Raman growth, is the nonlinear electron susceptibility, χ. Indeed, the ratio between the amplitudes, E_d, of the electromagnetic drive and, E_p, of the plasma wave is proportional to the imaginary part, χ_i, of the electron susceptibility. In this paper, we carefully and unambiguously define the amplitudes E_d and E_p, as well as χ, so as to show that

$$\chi_i E_p \approx E_d \cos(\delta\varphi), \tag{1}$$

where $\delta\varphi$ is the phase mismatch between the driving and plasma fields. We then discuss how to derive a simple theoretical expression for χ_i, and how Eq. (1) may be cast into an envelope equation. More precisely, we restrict here to the situation when E_p only depends on time, and grows with time, and compare the theoretical developments of Refs. 3 and 4 showing that Eq. (1) may be written as

$$dE_p/dt + \nu E_p \propto E_d \cos(\delta\varphi), \tag{2}$$

where ν is what we call the nonlinear Landau damping rate of the driven electron plasma wave. We focus here on the nonlinear variations of χ_i, or on the terms in the envelope equation for E_p, and will not discuss the values of the nonlinear phase mismatch $\delta\varphi$. This is because $\delta\varphi$ is induced by the nonlinear frequency shift $\delta\omega_p$ of the plasma wave, and the relevance of the various theoretical models for $\delta\omega_p$ was already discussed in Refs. 12 and 13.

For the two theoretical works of Refs. 3 and 4 we henceforth focus on, the derivation of Eq. (2) from Eq. (1) is actually quite different. In Ref. 3, one makes use of a Taylor expansion for χ to find, $\chi_i(\omega + i\gamma) \approx \chi_i(\omega + i0) + \gamma\partial_\omega\chi_r$, where $\gamma \equiv E_p^{-1}(dE_p/dt)$ and χ_r is the real part of the electron susceptibility. Then, a quasilinear value for χ_r is used while $\chi_i(\omega + i0) \equiv \nu\partial_\omega\chi_r$ is derived from energy conservation. By contrast, in Ref. 4, χ_i is derived directly from the electron dynamics and it is found that, for small enough values of γ, $\chi_i \approx \partial_\omega\chi_r^{\text{env}}[\gamma + \nu]$, where χ_r^{env} is some *effective* real susceptibility. In the linear regime, $\chi_r^{\text{env}} = \chi_r$ while, in the nonlinear regime once ν has dropped to nearly 0, it is found in Ref. 4 that $\partial_\omega\chi_r^{\text{env}} \gg \partial_\omega\chi_r$. Moreover, as is obvious from the results published in Refs. 3 and 4, the nonlinear decrease of ν towards 0 is much more abrupt in the work by Bénisti *et al.* than in the approach of Yampolsky and Fisch. Hence, a first discrepancy is quite apparent between the two models in the Taylor-like expansion of χ_i used to cast Eq. (1) into the

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envelope Eq. (2). However, despite this discrepancy, the actual values of χ_i found from the two theories may happen to be similar, leading to a similar description of Raman growth. This is what we will discuss in detail by comparing the theoretical values of χ_i with those derived from Vlasov and test particle simulations.

This article is organized as follows. In Sec. II, we first detail the theoretical framework common to the models of Refs. 3 and 4. In particular, we introduce the electron susceptibility and discuss how it differs from the previous definitions related to the Laplace transform of the fields. Then, we briefly recall the derivations of χ_i given in Refs. 3 and 4 leading to the envelope Eq. (2) for E_p, and carefully explain how the two models differ. In Sec. III, we compare the theoretical values of χ_i to those found from Vlasov and test particle simulations, which allows us to estimate the range of validity of the theoretical expressions for χ_i. Finally, Sec. IV summarizes and concludes this work.

II. THEORY

A. Common theoretical framework

In this paper, we address stimulated Raman scattering within the framework of the three wave model where the total electric field is

$$\vec{E}_{\rm tot} = -i\hat{x}(E_p/2)e^{i\varphi_p} + \hat{y}[(-i(E_l/2)e^{i\varphi_l} + (E_s/2)e^{i\varphi_s}] + c.c., \tag{3}$$

$E_{p,l,s}$ being the slowly varying amplitudes of the plasma, laser, and scattered waves, which are chosen to be real and positive, while the wave number and frequencies of these waves are, respectively, $k_{p,l,s} \equiv \partial_x \varphi_{p,l,s}$ and $\omega_{p,l,s} \equiv -\partial_t \varphi_{p,l,s}$. We, moreover, restrict here to the situation when each amplitude E_w, where $w = p, l$ or s, does not vary spatially and grows slowly with time, $E_w^{-1}(dE_w/dt) \ll \omega_w$. Finally, we also assume that the phase mismatch between the three waves, $\delta\varphi \equiv \varphi_p + \varphi_s - \varphi_l$, varies slowly with time.

As is well known (see, for example, Ref. 8), the electron motion along the direction of propagation of the waves, x, is given by

$$\frac{d^2x}{dt^2} = \frac{-e}{m}\left[\frac{-iE_p}{2} + \frac{E_d}{2}e^{-i\delta\varphi}\right]e^{i\varphi_p} + c.c., \tag{4}$$

$-e$ being the electron charge, m its mass, and

$$E_d \equiv (ek_pE_lE_s)/(2m\omega_l\omega_s) \tag{5}$$

is the ponderomotive field amplitude. Newton equation (4) may also be written as

$$\frac{d^2x}{dt^2} = \frac{ie}{m}E_0e^{i\psi} + c.c., \tag{6}$$

where $E_0 \equiv \sqrt{E_p^2 + E_d^2 - 2E_dE_p\sin(\delta\varphi)}$ and, clearly,

$$E_0e^{i(\psi-\varphi_p)} = E_p + iE_de^{-i\delta\varphi}. \tag{7}$$

$E \equiv E_0e^{i\psi} + c.c.$ may be seen as the total *effective* electrostatic field inducing the charge density, ρ (which is only due to the electron motion, the ions being treated as a neutralizing background), so that one may write ρ as

$$\rho \equiv (\rho_0/2)e^{i\psi} + c.c., \tag{8}$$

where ρ_0 is a slowly varying complex amplitude unambiguously defined by the requirement (derived from Gauss law) that,

$$k_pE_p = (\rho_0/\varepsilon_0)e^{i(\psi-\varphi_p)}. \tag{9}$$

Let us now introduce

$$\chi(t) \equiv -\frac{\rho_0}{\varepsilon_0 k_p E_0}, \tag{10}$$

which we define as the electron susceptibility. Then, Gauss law, Eq. (9), is

$$k_pE_p = -k_p\chi(E_p + iE_de^{-i\delta\varphi}). \tag{11}$$

In particular, the imaginary part of Eq. (11) yields,

$$\chi_iE_p = E_d\left[-\chi_r\cos(\delta\varphi) - \chi_i\sin(\delta\varphi)\right] \approx E_d\cos(\delta\varphi), \tag{12}$$

since the plasma wave dispersion relation is $\chi_r \approx -1$, while $\chi_i \ll 1$ because, as shown in Ref. 14, it is either on the order of the plasma wave growth rate or of the Landau damping rate, normalized to the plasma frequency, which are supposed to be small quantities. We thus derived Eq. (1) of the introduction, which we now need to write as an envelope equation for the plasma wave. (The envelope equations for the electromagnetic fields will not be discussed in this paper since they are the same for all the models we know of).

Before proceeding, we want to make clear how our definition for χ differs from the one introduced by Cohen and Kaufman in Ref. 15, where the Laplace representation of the fields is used. More precisely, if the total effective electrostatic field and charge density amplitudes may be written as,

$$E_0e^{i\psi} = e^{ik_px}\int_{\mathcal{L}} -i\tilde{E}(\Omega)e^{-i\Omega t}d\Omega, \tag{13}$$

$$\rho_0e^{i\psi} = e^{ik_px}\int_{\mathcal{L}} \tilde{\rho}(\Omega)e^{-i\Omega t}d\Omega, \tag{14}$$

where $\mathcal{L}$ is the Laplace contour, located in the upper half of the complex plane, then the electron susceptibility is defined in Ref. 15 by

$$\tilde{\chi}(\Omega) \equiv \frac{-\tilde{\rho}(\Omega)}{\varepsilon_0 k_p\tilde{E}(\Omega)}. \tag{15}$$

The difference between the definitions Eqs. (10) and (15) for the electron susceptibility is quite clear. χ as defined by Eq. (10) is a function of time, which varies with the wave amplitudes, while $\tilde{\chi}$ as defined by Eq. (14) is only a function of the complex frequency Ω and is therefore independent of the wave amplitudes. Using the definition (15) for $\tilde{\chi}(\Omega)$, one easily finds from Gauss law the following relation between the Laplace components $\tilde{E}_p(\Omega)$ and $\tilde{E}_d(\Omega)$ of the plasma wave and driving electric fields,

$$[1+\tilde{\chi}(\Omega)]\tilde{E}_p(\Omega) = -i\tilde{\chi}\tilde{E}_d(\Omega), \tag{16}$$

from which deriving a nonlinear envelope equation is, however, not straightforward. For example if, when integrating Eq. (16) over Ω, one assumes that $\tilde{E}_d$ and $\tilde{E}_p$ only have significant values about a central frequency, $\Omega \approx \Omega_0$, so that one may use the Taylor expansion $\tilde{\chi}(\Omega) \approx \tilde{\chi}(\Omega_0) + (\Omega - \Omega_0) d\tilde{\chi}/d\Omega$, one would find (see Ref. 15)

$$[1+\tilde{\chi}(\Omega_0)]E_p + i\frac{d\tilde{\chi}}{d\Omega}\bigg|_{\Omega=\Omega_0}\frac{dE_p}{dt} \approx -i\tilde{\chi}(\Omega_0)E_d, \tag{17}$$

which is not quite the nonlinear envelope equation we are looking for! Indeed, all the coefficients of Eq. (17) are fixed, independent of the wave amplitudes. Therefore, the previous equation would only be useful in the linear limit. This is why we will henceforth use the function χ defined by Eq. (10) and will work on Eq. (12) to derive an envelope equation for the plasma wave.

Now, χ is a function of time, whose actual value depends on the central *real* frequency of the plasma wave, ω_p (which also is a function of time, $\omega_p \equiv -\partial_t \varphi_p$), and on the growth rate $\gamma \equiv E_p^{-1}(dE_p/dt)$. In order to derive an envelope equation from Eq. (12), one may think of performing the following Taylor-like expansion, $\chi \approx \chi^a + \gamma\partial_\omega \chi^a$, where χ^a is calculated in the adiabatic limit, $\gamma \to 0$. However, proving such an expansion is not straightforward since χ is not directly a function of $(\omega_p + i\gamma)$, but it is a function of time which, under certain conditions, may be expressed in terms of ω_p and γ. This is one difference with $\tilde{\chi}(\Omega)$ since, when the imaginary part Ω_i of Ω is much less than its real part, Ω_r, it is quite clear that $\tilde{\chi}(\Omega) \approx \tilde{\chi}(\Omega_r) + i\Omega_i\partial_\Omega\tilde{\chi}(\Omega_r)$. How to perform a Taylor-like expansion of χ in the limit $\gamma \ll \omega_p$, in order to write Eq. (12) as an envelope equation, is one of the central points that will be discussed in this paper.

Finally, let us note that, in the linear limit when the wave amplitudes grow exponentially with time, the Laplace expansions (13) or (14) only have one nonzero component. Then, in this limit, it is clear that $\chi = \tilde{\chi}$. However, in the nonlinear regime, enforcing an exponential growth for the wave amplitudes does not necessary entail the same exponential growth for the charge density so that, in general, $\chi \neq \tilde{\chi}$.

B. Model by Bénisti *et al.*

In this subsection, we briefly recall the method used by Bénisti *et al.* in Refs. 4 and 8 to derive χ_i and cast Eq. (12) into the form of the envelope equation (2). Since the model of Yampolsky and Fisch is only valid when the wave amplitudes grow, and since this paper is mainly devoted to model comparisons, we restrict here growing waves.

A first estimate of χ_i, valid only for small wave amplitudes, is obtained by Bénisti *et al.* from a perturbation analysis. Moreover, using an argument of symmetry they show that for slowly growing waves, $\gamma_0 \ll \omega_p$ where $\gamma_0 \equiv E_0^{-1}(dE_0/dt)$, deeply trapped electrons contribute very little to χ_i. As a result, Bénisti *et al.* only include in χ_i the contribution of electrons such that $|v_0 - v_\phi| > V_l$, where v_0 is the *initial* electron speed, v_ϕ is the plasma wave phase velocity, and $V_l = (4/\pi)(\sqrt{eE_0/mk_p} - 3\gamma_0/2k_p)$ (see Ref. 8). The value chosen for V_l may be understood as follows. The inequality $|v_0 - v_\phi| \leq 4\sqrt{eE_0/mk_p}/\pi$ is the condition for trapping as derived by assuming adiabatic electron motion, while requiring $|v_0 - v_\phi| \leq (4/\pi)(\sqrt{eE_0/mk_p} - 3\gamma_0/2k_p)$ amounts to demanding that an electron has completed about one half of its trapped orbit in order to be considered as "deeply trapped." At first order in the perturbation analysis, and at 0-order in the time variations of the growth rate γ_0, Bénisti *et al.* then find

$$\chi_i = -\omega_{pe}^2 \int_{|v-v_\phi|>V_l} f_M(v)\frac{2\gamma_0(k_p v - \omega_p)}{[\gamma_0^2 + (k_p v - \omega_p)^2]^2}dv, \tag{18}$$

where ω_{pe} is the plasma frequency and f_M is the unperturbed distribution function, chosen here to be a Maxwellian. When $V_l = 0$, which corresponds to the linear limit, one recovers the usual linear value for χ_i, that may be found for example in Ref. 16.

It is actually convenient to write Eq. (18) for χ_i the following way:

$$\begin{aligned}\chi_i = &-\omega_{pe}^2 \int_{|v-v_\phi|>V_l}\Big[f_M(v) \\ &- (v-v_\phi)f'_M(v_\phi)\Big]\frac{2\gamma_0(k_p v - \omega_p)}{[\gamma_0^2 + (k_p v - \omega_p)^2]^2}dv, \\ &- \omega_{pe}^2 \int_{|v-v_\phi|>V_l}(v-v_\phi)f'_M(v_\phi) \\ &\times \frac{2\gamma_0(k_p v - \omega_p)}{[\gamma_0^2 + (k_p v - \omega_p)^2]^2}dv,\end{aligned} \tag{19}$$

because, unlike the right-hand side of Eq. (18), the first term in the right-hand side of Eq. (19) is well behaved in the limit $\gamma_0 \to 0$, even when $V_l = 0$. Then, for small enough values of γ_0, this first term is well approximated by replacing $\gamma_0^2 + (k_p v - \omega_p)^2$ with $(k_p v - \omega_p)^2$. As for the second term in the right-hand side of Eq. (19), it can be explicitly calculated, so that one finds,

$$\begin{aligned}\chi_i \approx &-\frac{2\gamma_0\omega_{pe}^2}{k_p^3}\int_{|v-v_\phi|>V_l}\frac{f_M(v) - (v-v_\phi)f'_M(v_\phi)}{(v-v_\phi)^3}dv \\ &-\frac{\omega_{pe}^2 f'(v_\phi)}{k_p^2}\left[\pi - 2\tan^{-1}\left(\frac{k_p V_l}{\gamma_0}\right) + \frac{2\gamma_0 k_p V_l}{\gamma_0^2 + (k_p V_l)^2}\right].\end{aligned} \tag{20}$$

In the domain of validity of the perturbation analysis, when V_l is much less than the thermal velocity v_{th}, the first term in the right hand side of Eq. (20) is very close to $\partial_\omega \chi_r^{\text{lin}}$, where χ_r^{lin} is the linear value of the real part of the electron susceptibility calculated in the limit $\gamma_0 \to 0$. Hence, if one denotes

$$\nu^{(1)} \equiv \frac{-\omega_{pe}^2 f'(v_\phi)}{k_p^2 \partial_\omega \chi_r^{\mathrm{lin}}} \left[\pi - 2\tan^{-1}\left(\frac{k_p V_l}{\gamma_0}\right) + \frac{2\gamma_0 k_p V_l}{\gamma_0^2 + (k_p V_l)^2} \right], \tag{21}$$

Equation (20) just is

$$\chi_i \approx \left[\gamma_0 + \nu^{(1)}\right] \partial_\omega \chi_r^{\mathrm{lin}}. \tag{22}$$

In particular, in the linear regime when $V_l = 0$, $\nu^{(1)}$ is nothing but the (linear) Landau damping rate ν_L derived in Ref. 17, in the limit when $\nu_L/\omega_{pe} \ll 1$. Therefore, in the linear regime, we indeed managed to derive a Taylor-like expansion for χ. Nevertheless, in the general case when $V_l \neq 0$, due to the complex dependence of $\nu^{(1)}$ upon γ_0, and since V_l may *a priori* vary from $V_l \ll \gamma_0/k_p$ to $V_l \gg \gamma_0/k_p$, it is not quite clear that any simple expression of χ_i as a function of γ_0, resembling to a Taylor-like expansion, may actually be extracted from Eq. (22). This point will, however, be clarified in a few lines.

Before coming to this point we first remark that, directly from Eq. (18), when $V_l \gg \gamma_0/k$,

$$\chi_i \approx -2\gamma_0 \omega_{pe}^2 \int_{|v - v_\phi| > V_l} \frac{f_M(v)}{(k_p v - \omega_p)^3}. \tag{23}$$

Let us now denote by χ_r^{eff} the "effective" real part of the susceptibility obtained in the limit $\gamma_0 \to 0$ by neglecting the contribution of the deeply trapped electrons, those such that $|v_0 - v_\phi| < V_l$. The first order perturbative estimate, $\chi_r^{\mathrm{eff},1}$, of χ_r^{eff} is easily found to be,

$$\chi_r^{\mathrm{eff},1} = -\omega_{pe}^2 \int_{|v - v_\phi| > V_l} \frac{f_M(v)}{(k_p v - \omega_p)^2}. \tag{24}$$

Hence, at first order in the perturbation analysis, and when $V_l \gg \gamma_0/k_p$,

$$\chi_i \approx \gamma_0 \partial_\omega \chi_r^{\mathrm{eff},1}. \tag{25}$$

Moreover, as shown in Ref. 8, at any order n one would find similarly that when $V_l \gg \gamma_0/k_p$

$$\chi_i \approx \gamma_0 \partial_\omega \chi_r^{\mathrm{eff},n}, \tag{26}$$

where $\chi_r^{\mathrm{eff},n}$ is the n^{th} order approximation of χ_r^{eff}. This led Bénisti *et al.* to the conclusion that, when $V_l \gg \gamma_0/k_p$, the following nonlinear, non perturbative estimate of χ_i,

$$\chi_i \approx \gamma_0 \partial_\omega \chi_r^{\mathrm{eff}}, \tag{27}$$

should hold. Moreover, a theoretical expression of χ_r^{eff} was provided in Refs. 8 and 14 where it was shown that $\partial_\omega \chi_r^{\mathrm{eff}} \gg \partial_\omega \chi_r^{\mathrm{lin}}$.

Hence, the previous analysis provided a perturbative estimate of χ_i valid for small wave amplitudes, and a nonlinear non perturbative approximation of χ_i which is accurate when $V_l \gg \gamma_0/k_p$. Now, in Ref. 8, when comparing the theoretical values of χ_i with those derived from test particle simulations (similar to those described in Sec. III), it was found that the perturbative estimate of χ_i was precise up to values of V_l large enough for Eq. (27) to already be quite accurate. Then, in order to derive a nonlinear expression of χ_i valid whatever $k_p V_l/\gamma_0$, Bénisti *et al.* just connected the perturbative estimate of χ_i to the nonlinear non perturbative expression (27) the following way:

$$\chi_i = \chi_i^{\mathrm{per}} \times [1 - Y(k_p V_l/\gamma_0)] + \gamma_0 \partial_\omega \chi_r^{\mathrm{eff}} \times Y(k_p V_l/\gamma_0), \tag{28}$$

where $Y(x)$ is a function rising from 0 to unity as x increases, and χ_i^{per} is the perturbative estimate of χ_i. At order n, $\chi_i^{\mathrm{per}} \equiv \partial_\omega \chi_r^{(n)}[\nu^{(n)} + \gamma_0]$ where $\nu^{(n)}$ and $\partial_\omega \chi_r^{(n)}$ are, respectively, the n^{th} order counterparts of $\nu^{(1)}$ defined by Eq. (21) and of the first term in Eq. (20). Then, Eq. (28) becomes

$$\chi_i = [\nu + \gamma_0] \partial_\omega \chi_r^{\mathrm{env}}, \tag{29}$$

with

$$\partial_\omega \chi_r^{\mathrm{env}} \equiv \partial_\omega \chi_r^{(n)} \times [1 - Y(k_p V_l/\gamma_0)] + \partial_\omega \chi_r^{\mathrm{eff}} \times Y(k_p V_l/\gamma_0), \tag{30}$$

$$\nu \equiv \nu^{(n)} \times \left[1 - Y(k_p V_l/\gamma_0)\right]. \tag{31}$$

Now, as shown again by results from test particle simulations, and as reported in Ref. 8, $\gamma_0 \partial_\omega \chi_r^{\mathrm{eff}}$ converges very abruptly towards χ_i when $k_p V_l/\gamma_0 \gtrsim 3$. Hence, one needs to choose a Heaviside-like function for $Y(x)$, rising very quickly form 0 to unity as x changes from a little less than 3 to a little more than 3. In Ref. 4, the function $Y(x) = \tanh^5[(e^{x/3} - 1)^3]$ was proposed. Moreover, it turns out that $\nu^{(n)}$ remains nearly constant when $k_p V_l/\gamma_0 \le 3$. Therefore, ν in Eq. (29) may be considered as independent of γ_0, at least over finite intervals of $k_p V_l/\gamma_0$, so that Eq. (29) may indeed be viewed as a Taylor-like expansion of χ_i.

Let us recall that in Eq. (29) $\gamma_0 \equiv E_0^{-1}(dE_0/dt)$, while one would need an expansion of χ_i in terms of $\gamma \equiv E_p^{-1}(dE_p/dt)$ in order to derive an envelope equation for E_p. Now, in the linear regime of SRS, all waves should grow similarly so that, in this regime, $\gamma_0 \approx \gamma$. Moreover, in the nonlinear regime, once ν has dropped to nearly 0 then, as discussed, for example in Ref. 13, $E_p \approx E_0$ so that, once again, $\gamma_0 \approx \gamma \equiv E_p^{-1}(dE_p/dt)$. Using this result and plugging Eq. (29) into Eq. (12) one finds

$$\frac{dE_p}{dt} + \nu E_p \approx \frac{E_d}{\partial_\omega \chi_r^{\mathrm{env}}} \cos(\delta\varphi), \tag{32}$$

where $\partial_\omega \chi_r^{\mathrm{env}}$ is defined by Eq. (30), and where ν given by Eq. (31) is the expression for the nonlinear Landau damping rate proposed by Bénisti *et al.* Note that this coefficient appears naturally from the expression of χ_i derived for a driven wave, and not from an estimate of the rate of energy gained by the electrons.

Let us now discuss Eq. (29) along lines similar to those of Ref. 18 in order to explain why the simple Taylor expansion $\chi_i = \partial_\omega \chi_r[\nu + \gamma]$ one would expect is not recovered. Actually, using a variational approach, like that developed by Whitham in Ref. 19 for an *undamped* wave, one would find the following envelope equation:

$$\partial_\omega \chi_r (dE_p/dt) = E_d \cos(\delta\varphi), \tag{33}$$

provided that collisionless dissipation is negligible. Moreover, in the linear regime, we previously showed that

$$\partial_\omega \chi_r^{\rm lin}(dE_p/dt) + \nu_L \partial_\omega \chi_r^{\rm lin} E_p = E_d \cos(\delta\varphi), \tag{34}$$

where ν_L is the (linear) Landau damping rate. Hence, in the nonlinear regime, one is naturally led to write the following envelope equation:

$$\partial_\omega \chi_r (dE_p/dt) + N(E_p) = E_d \cos(\delta\varphi), \tag{35}$$

where the operator $N(E_p)$ is the nonlinear counterpart of $\nu_L \partial_\omega \chi_r^{\rm lin} E_p$ and allows for collisionless dissipation in the nonlinear regime. Now, the only functional form for $N(E_p)$ consistent with the fact that the nonlinear Landau damping rate has vanished is $N(E_p)$ proportional to the time derivative of E_p, $N(E_p) \equiv \partial_\omega \chi_r'(dE_p/dt)$. Then, the envelope equation for E_p becomes,

$$\partial_\omega \chi_r^{\rm env}(dE_p/dt) = E_d \cos(\delta\varphi), \tag{36}$$

with $\partial_\omega \chi_r^{\rm env} \equiv \partial_\omega(\chi_r + \chi_r') \gg \partial_\omega \chi_r$. Hence, $\partial_\omega \chi_r^{\rm env}$ differs from $\partial_\omega \chi_r$ because of collisionless dissipation, and therefore all the more as ν_L is large, as was checked numerically in Ref. 18.

Let us try to clarify this point further by noting that, as the wave grows, there keeps on being a net transfer of momentum and energy from the wave to the electrons, even *after* the model by Bénisti *et al.* has predicted $\nu \approx 0$. Indeed, it is clear that while the electrostatic wave grows, it keeps on trapping new electrons whose phase mixing eventually leads to an increase in the electron kinetic energy. Similarly, the wave growth changes the orbits of the untrapped electrons, and therefore their energy and momentum. However, once the bounce frequency $\omega_B \equiv \sqrt{eE_0k_p/m}$ has become much larger than the wave growth rate, γ_0, one may assume that the electrons phase mix nearly instantaneously. In other words, the electron motion is nearly adiabatic and one may consider that the electron orbits, and electron energy $\mathcal{E}$, only depend on the instantaneous wave amplitude. Hence, whatever the time $\Delta t \gg 2\pi/\omega_B$ it takes for the wave amplitude to increase from E_0 to $E_0 + \Delta E_0$, the change in the electron energy is very close to $\Delta\mathcal{E}_{adia} \equiv \mathcal{E}(E_0 + \Delta E_0) - \mathcal{E}(E_0)$. Then, whatever $\Delta t \gg 2\pi/\omega_B$, the growth rate in the electron energy is just $\Delta\mathcal{E}_{adia}/\Delta t$ and is therefore inversely proportional to Δt. We therefore conclude that, when $\omega_B \gg \gamma_0$, the *rate* of energy transfer from the wave to the electrons is nearly proportional to the wave growth rate, γ_0. Hence, in the model by Bénisti *et al.*, $\nu \approx 0$ does not mean that the electron acceleration by the electrostatic field, responsible for Landau damping in the linear limit, has vanished, but this means that this acceleration, and therefore the collisionless dissipation of the EPW, is better modeled in the envelope equation for the plasma wave by a term proportional to the wave growth rate than by a genuine damping rate. Then, physically, $\partial_\omega \chi_r^{\rm env} \gg \partial_\omega \chi_r$ just because the energy transfer from the plasma wave to the electrons slows down the growth of the EPW induced by the laser drive.

C. Model by Yampolsky and Fisch

Yampolsky and Fisch provide in Ref. 3 a kinetic modeling of stimulated Raman scattering as simple as possible. They first simply assume that the usual Taylor expansion $\chi_i = \partial_\omega \chi_r(\nu + \gamma)$ holds, where ν is the nonlinear Landau damping rate of the driven plasma wave. This is clearly one major difference with the model by Bénisti *et al.* who found that such a Taylor-like expansion would not be accurate in the strongly nonlinear regime. Plugging the expression $\chi_i = \partial_\omega \chi_r(\nu + \gamma)$ into Eq. (12) straightforwardly yields the envelope equation

$$\frac{dE_p}{dt} + \nu E_p = \frac{E_d \cos(\delta\varphi)}{\partial_\omega \chi_r}. \tag{37}$$

Yampolsky and Fisch, moreover, use a quasilinear estimate for $\partial_\omega \chi_r$ which is actually just given by the first term of Eq. (20) with f_M replaced by the quasilinear distribution function f_{QL}. This distribution function solves the differential equation

$$\partial_t f_{QL} = \mathrm{Re}\left\{\frac{1}{2i}\left|\frac{eE}{m}\right|\partial_v \frac{\partial_v f_{QL}}{\omega_p + i\gamma - k_p v}\right\}, \tag{38}$$

with $f_{QL} = f_M$ at $t = 0$ for vanishing field amplitudes.

As for the Landau damping rate, ν, it is derived from conservation laws, namely, the conservation of energy and of the number of electrons. These read

$$\frac{d}{dt}\left(\frac{nm}{2}\int_{-\infty}^{+\infty} v^2\left[f_{QL}(v,t) - f_M(v)\right]dv\right) = 2\nu\omega_p\partial_\omega\chi_r\frac{\varepsilon_0 E_p^2}{4}, \tag{39}$$

$$\int_{-\infty}^{+\infty}\left[f_{QL}(v,t) - f_M(v)\right]dv = 0. \tag{40}$$

Equation (39) clearly relates the rate of variation of the electron kinetic energy (left-hand side of this equation) to the Landau damping of the electrostatic energy (right-hand side) so that, unlike in Bénisti *et al.*, ν is indeed related to the electron acceleration by the plasma wave.

In order to take advantage of Eqs. (39) and (40) to derive the nonlinear Landau damping rate, Yampolsky and Fisch make two more simplifying hypotheses,

1. f_{QL} differs from f_M only over a finite range in velocity, $|v - v_\phi| < \alpha\omega_B/k_p$, where α is a constant that is still to be

determined. This range is representative of the trapping domain.

2. In the domain $|v - v_\phi| < \alpha\omega_B/k_p$, one may use a Taylor expansion of the distribution functions, $f(v) \approx f(v_\phi) + (v - v_\phi)f'(v_\phi)$.

Using these approximations, Eq. (40) yields $f_{QL}(v_\phi) = f_M(v_\phi)$, so that Eq. (39) becomes

$$\frac{d}{dt}\left(\frac{nm}{2}\int_{-\alpha\omega_B/k_p}^{\alpha\omega_B/k_p} u(u^2 + 2uv_\phi + v_\phi^2)\left[f'_{QL}(v_\phi) - f'_M(v_\phi)\right]du\right) = \frac{\varepsilon_0\omega\nu}{2}\partial_\omega\chi_r E_p^2. \quad (41)$$

Using the quasilnear definition of ν,

$$\nu = -\frac{\pi\omega_{pe}^2 f'_{QL}(v_\phi)}{k_p^2\partial_\omega\chi_r}, \quad (42)$$

and approximating the linear Landau damping rate by $\nu_L \approx [-\pi\omega_{pe}^2 f'{}_M(v_\phi)]/[k_p^2\partial_\omega\chi_r]$, Eq. (41) is

$$\frac{d}{dt}\left[(\nu_L - \nu)\omega_B^3(t)\right] = \nu\frac{3\pi}{4\alpha^3}\omega_B^4(t), \quad (43)$$

which therefore needs to be solved together with Eq. (37).

As for the factor α, its value is obtained by matching the change in the total electron momentum with that calculated by Dewar in Ref. 21 by assuming adiabatic electron motion. It is then found, $\alpha = 32/(3\pi)$.

Moreover, as shown in Ref. 3, the quasilinear Landau damping rate is proportional to the wave growth rate, $\nu \propto \gamma$, at large plasma wave amplitudes $\int\omega_B dt \gg 1$. In this regime, the imaginary part of the electron susceptibility is proportional to the wave growth rate, $\chi_i \propto \gamma$, like in model the by Benisti *et al*. as described by Eq. (27).

III. COMPARISONS BETWEEN NUMERICAL AND THEORETICAL RESULTS

The comparisons will be on χ_i which, from Eq. (1), measures how efficiently an EPW may be laser driven and therefore has a very clear physical meaning. By contrast, that of ν is not as straightforward as for a freely propagating wave. Indeed, $-\nu$ it is not the rate of variation of the EPW amplitude, since this wave grows. Moreover, as explained, for example, in Ref. 4, there is no unique way to derive an envelope equation like Eq. (2) from the theoretical expression of χ_i. Hence, models providing similar expressions for χ_i would lead to similar predictions for the growth of the plasma wave amplitude, and therefore of SRS, although the terms of the envelope equation, and especially the damping rate ν, may be model dependent. This explains why we chose to focus here on the values found for χ_i.

A. Test particle simulations

The first set of comparisons we present are with test particle simulations where we numerically solve

$$\frac{d\xi}{dt} = v - v_\phi, \quad (44)$$

$$\frac{dv}{dt} = -\Phi(t)\sin(\xi), \quad (45)$$

where $\Phi(t)$ is a growing function of time. Then, as shown in Ref. 8,

$$\chi_i = \frac{-2\langle\sin(\xi)\rangle}{(k_p\lambda_D)^2\Phi}, \quad (46)$$

where λ_D is the Debye length and where $\langle\sin(\xi)\rangle$ is the statistical average of $\sin(\xi)$. Numerically, we only consider the situation when $\Phi(t)$ increases exponentially with time,

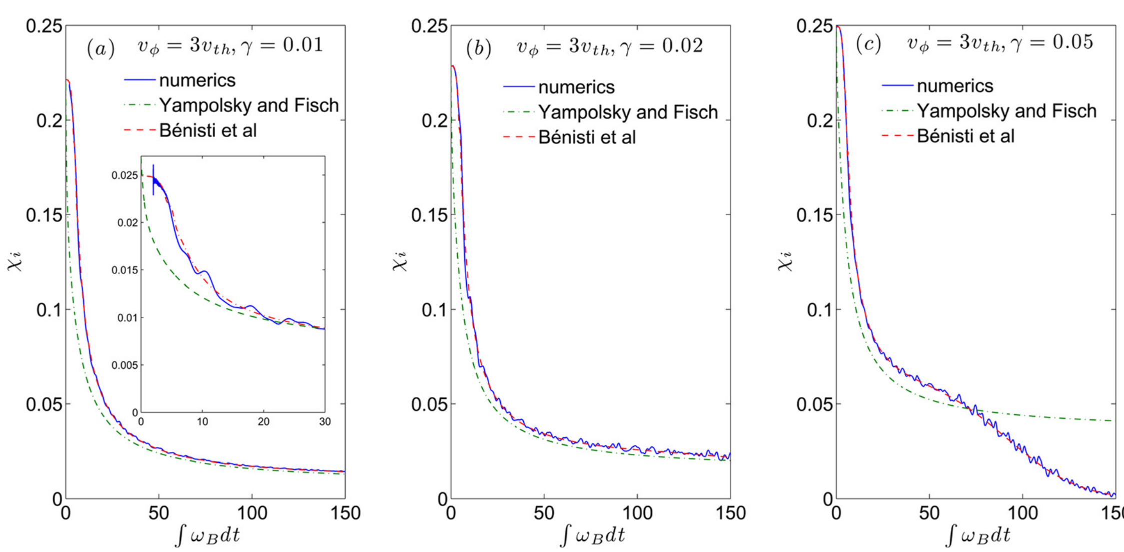

FIG. 1. (Color online) χ_i as calculated numerically using test particles simulations (blue solid line) or theoretically by either using the model by Yampolsky and Fisch (green dashed-dotted line) or the model by Bénisti *et al.* with $\chi_i^{\rm per}$ in Eq. (28) derived at order 11 (red dashed line), when $v_\phi = 3v_{th}$. Panel (a), $\gamma = 0.01$ (the inset of panel (a) is a close-up for small values of $\int\omega_B dt$); panel (b), $\gamma = 0.02$; panel (c), $\gamma = 0.05$.

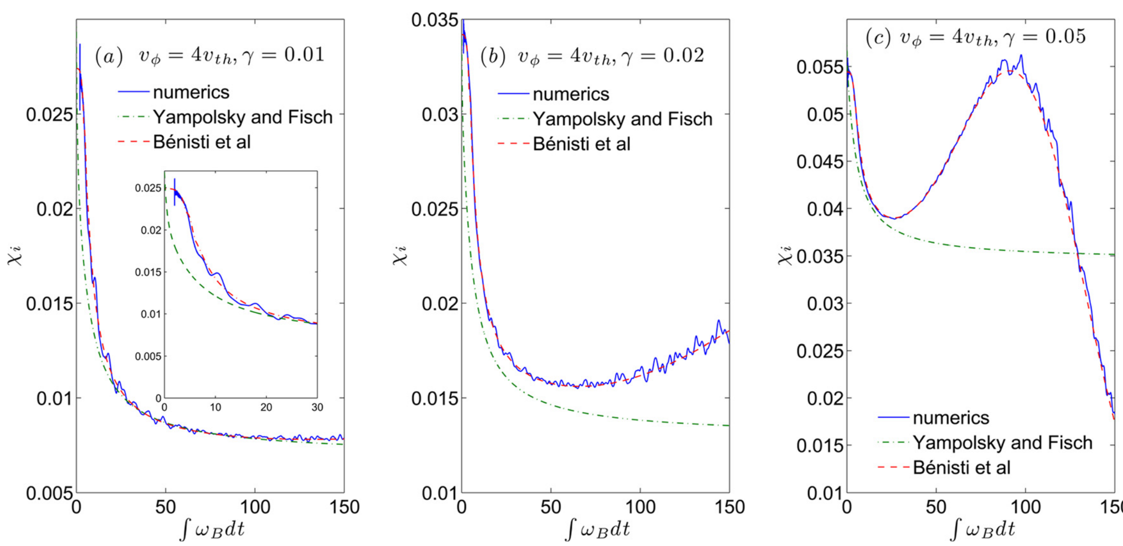

FIG. 2. (Color online) Same as Fig. 1 but with $v_\phi = 4v_{th}$.

$\Phi(t) \equiv \Phi_0 e^{\gamma t}$, and when v_ϕ remains constant. Equations (44) and (45) are numerically integrated for $N = 10^5$ particles with initial velocities distributed in a Maxwellian fashion, and the numerical estimate of $\langle \sin(\xi) \rangle$ is

$$\langle \sin(\xi) \rangle_{\text{num}} = \frac{1}{N} \sum_{i=1}^{N} \sin(\xi_i), \qquad (47)$$

where ξ_i is the position of the i^{th} particle.

Figs. 1 and 2 plot the variations of χ_i, calculated either numerically or theoretically, respectively, when $v_\phi = 3v_{th}$ and $v_\phi = 4v_{th}$, and when $\gamma = 10^{-2}, 2 \times 10^{-2}$ and 5×10^{-2}, as a function of $\int \omega_B dt = 2\sqrt{\Phi}/\gamma$. For all the cases we investigated, there was a very good agreement between the results from test particle simulations and from the model by Bénisti *et al.*, and usually a good agreement with the theoretical predictions of Yampolsky and Fisch. Nevertheless, χ_i as calculated by Yampolsky and Fisch systematically decreases more rapidly for small wave amplitudes than is observed numerically, as is made clear in the insets of Figs. 1(a) and 2(a). This is because, in their model, the damping rate changes as soon as particles get trapped, while, in reality, only after particles have experienced a large fraction of their trapped orbits (about half of it) does ν nonlinearly change. Moreover, for large wave amplitudes, the theoretical results of Yampolsky and Fisch may significantly differ from the numerical ones, as illustrated, for example, in Fig. 2(c). This is due to the use of a quasilinear expression for $\partial_\omega \chi_r$, which becomes clearly wrong in the strongly nonlinear regime. In this regime, the fully nonlinear expression $\partial_\omega \chi_r^{\text{env}}$ derived by Bénisti *et al*. is needed.

B. Vlasov simulations

The second set of numerical simulations are similar to those presented in Ref. 3, where Vlasov-Poisson equations are

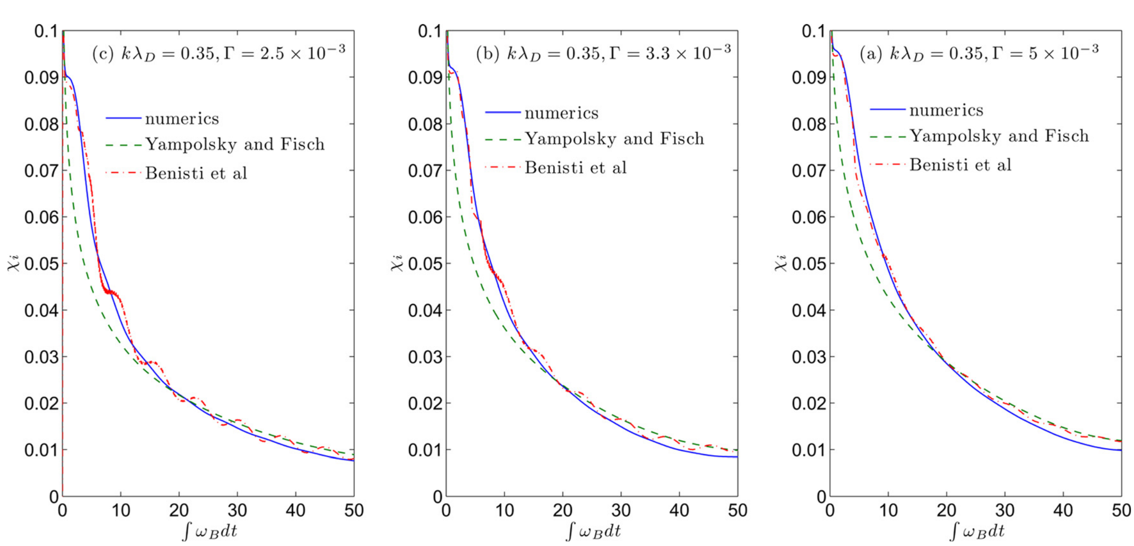

FIG. 3. (Color online) χ_i as calculated numerically using Vlasov simulations (blue solid line) or theoretically using the model by Yampolsky and Fisch (green dashed line) or by Bénisti *et al*. (red dashed-dotted line) when $v_{th} = 0.35$ and when the normalized growth rate of the drive is, panel (a), $\Gamma = 2.5 \times 10^{-3}$, panel (b), $\Gamma = 3.3 \times 10^{-3}$ and, panel (c), $\Gamma = 5 \times 10^{-3}$.

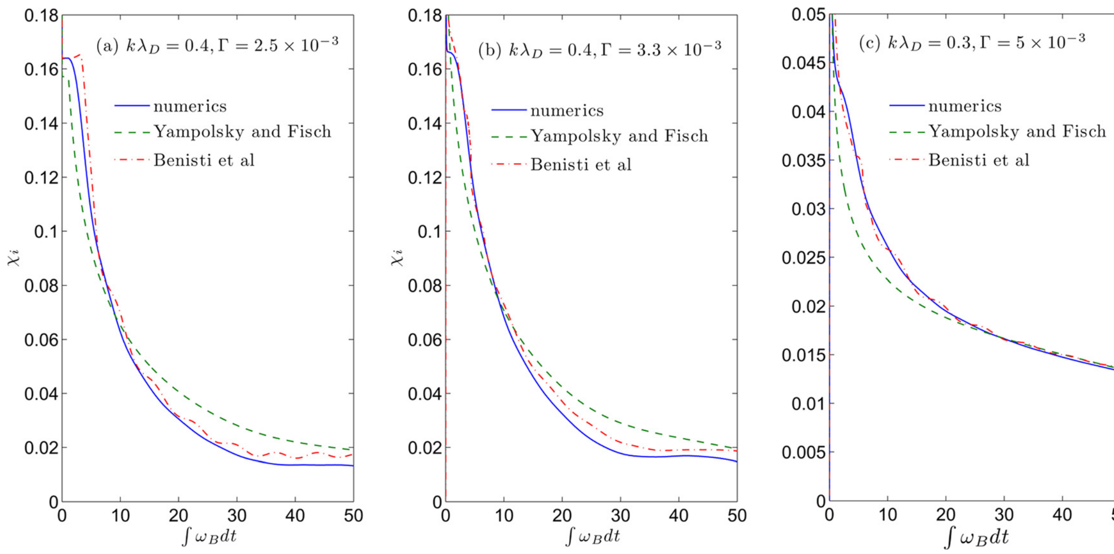

FIG. 4. (Color online) χ_i as calculated numerically using Vlasov simulations (blue solid line) or theoretically using the model by Yampolsky and Fisch (green dashed line) or by Bénisti *et al.* (red dashed-dotted line) panel (a), when $v_{th} = 0.4$ and $\Gamma = 2.5 \times 10^{-3}$, panel (b), when $v_{th} = 0.4$ and $\Gamma = 3.3 \times 10^{-3}$, panel (c) when $v_{th} = 0.3$ and $\Gamma = 5 \times 10^{-3}$.

solved inside of a uniform plasma, with periodic boundary conditions. The length of the simulation domain is the wavelength of the electron plasma wave which, in the dimensionless units used in the code, $e = m = k_p = \omega_{pe} = 1$, is just 2π. The EPW is driven inside an initially Maxwellian plasma, with normalized thermal velocity v_{th}, by an externally imposed field, $E_{\rm ext} \equiv E_0 e^{\Gamma t} \cos[x - \varphi_0(t)]$ (which is, therefore, not self-consistently calculated using Maxwell equations as should be the case for a realistic SRS simulation). The frequency of the drive, $\omega_0 \equiv \partial_t \varphi_0$, is chosen to decrease with the EPW amplitude in a fashion close to that theoretically calculated in Ref. 13.

In these simulations, the external field $E_{\rm ext}$ is, therefore, just the counterpart of the driving field $E_d \cos(\varphi_p - \delta\varphi)$ introduced theoretically so that, in order to numerically estimate χ_i, we first need to compute the dephasing $\delta\varphi$ between the driving and electrostatic fields, and then calculate the ratio $\chi_i^{\rm num} \equiv E_0 \cos(\delta\varphi)/E_p$. In order to compare this numerical estimate to the theoretical one, we also need to numerically compute the EPW growth rate, γ, and plug it into to the formulas, $\chi_i^{\rm YF} \equiv \partial_\omega \chi_r(\gamma + \nu)$ or $\chi_i^B \equiv \partial_\omega \chi_r^{\rm env}(\gamma + \nu)$, respectively derived by Yampolsky and Fisch and by Bénisti *et al.* [using for $\chi_i^{\rm per}$ in Eq. (28) a perturbative result at order 11]. These comparisons are plotted in Fig. 3 when $v_{th} = 0.35$ in normalized units (in physics units this would correspond to $k_p \lambda_D = 0.35$), and in Fig. 4 when $v_{th} = 0.4$ and $v_{th} = 0.3$.

Just like for the comparisons with test particle simulations, there is good agreement between the numerical and theoretical values of χ_i, although one can still notice an initial drop in $\chi_i^{\rm YF}$ more rapid than in $\chi_i^{\rm num}$. As for $\chi_i^{\rm B}$, it exhibits some oscillations which are representative of the oscillations in the numerical estimate in γ (in the model by Bénisti *et al.* ν drops more rapidly to 0 so that the value of χ_i is more sensitive to γ than in the model by Yampolsky and Fisch).

Despite these small discrepancies, the agreement between the theoretical and numerical values of χ_i is good for the moderate values of $\int \omega_B dt$ we investigated; $\int \omega_B dt \leq 50$ for the simulation results of Fig. 3, while we let $\int \omega_B dt$ go up to 150 in Figs. 1 and 2. Actually, for the Valsov simulations of Fig. 3, the dephasing $\delta\varphi$ between the driving and plasma fields gets very close to $\pi/2$ as the EPW amplitude increases, so that a small mistake in the numerical evaluation of $\delta\varphi$ may lead to a very bad estimate of $\chi_i^{\rm num} \equiv E_0 \cos(\delta\varphi)/E_p$. Hence, in Figs. 3 and 4, we chose to stop the comparisons between the theoretical and numerical values of χ_i when $\delta\varphi$ is so close to $\pi/2$ that a small relative error, of the order of 5%, in $\delta\varphi$ would entail a relative error close to 100% in $\chi_i^{\rm num}$, whose accuracy therefore becomes doubtful.

IV. CONCLUSION

In this paper, we compared the nonlinear kinetic modelings of stimulated Raman scattering derived in Refs. 3 and 4, respectively, by Yampolsky and Fisch and by Bénisti *et al.* Starting from Eq. (1) deduced directly from Gauss law, these papers provide a theoretical description of χ_i so as to derive the following envelope equation for the plasma wave amplitude, $dE_p/dt + \nu E_p \propto E_d \cos(\delta\varphi)$, where ν is the so-called nonlinear damping rate of the driven electron plasma wave. The derivation of this envelope equation is made in a completely different spirit in Ref. 3 as compared with Ref. 4. Indeed, Yampolsky and Fisch look for a very simple modeling and mainly resort to the quasilinear approximation, which is the most simple way to go beyond a perturbative analysis. By comparing the lengths of Subsections B and C, it is quite clear that their theory is much less complex than that of Bénisti *et al.*, who looked for the most accurate and general description of χ_i. This required connecting high order perturbative results with a totally nonlinear, and non perturbative, expression for χ_i. As a result, and as shown by comparing the numerical values of χ_i to the theoretical ones, the model by Bénisti *et al.* always seems quite accurate, whatever the physics conditions and the wave amplitude, while that of Yampolsky and Fisch is always good for moderate amplitudes, but may be inaccurate for very small or very large values of $\int \omega_B dt$. Moreover, the theory of Bénisti *et al*. may be generalized to allow for arbitrary space and time dependence of the wave amplitudes, as was done in Refs. 4, 11, 14, 18, and 21, while, for the moment, that of Yampolsky and Fisch only applies to growing waves. We, therefore, suggest that the model by Yampolsky and Fisch could be used in analytic estimates of collisionless dissipation for growing plasma waves, due to its simplicity. As for the model by Benisti *et al*., it provides higher accuracy and is currently implemented in an envelope code for quantitative predictions of Raman growth (see Ref. 14 for the one-dimensional version of that code which has recently been upgraded to allow for three-dimensional variations of the wave amplitudes).

In our comparisons, we really focused on χ_i which measures how efficiently an EPW may be laser driven, and found good agreements between both models for moderate values of $\int \omega_B dt$. However, when writing χ_i as, $\chi_i = \partial_\omega \chi_r(\gamma+\nu)$, or $\chi_i = \partial_\omega \chi_r^{\rm env}(\gamma+\nu)$, the relative nonlinear values of ν for each model may be quite different, just as $\partial_\omega \chi_r^{\rm env}$ may notably differ from $\partial_\omega \chi_r$. At this point, it should be noted that the physical meaning of ν is not as straightforward as for a freely propagating wave. Indeed, $-\nu$ it is not the rate of variation of the EPW amplitude, since this wave grows. Moreover, as shown, for example, in Ref. 4, the nonlinear envelope equation of a driven EPW is more complicated than $dE_p/dt + \nu E_p \propto E_d \cos(\delta\varphi)$, which may only be viewed as an "effective" equation, so that relating ν to some physics quantities is not that obvious. Nevertheless, such an envelope equation was shown, in Ref. 3 for the model by Yampolsky and in Ref. 14 for the model by Bénisti *et al*., to provide a description for Raman growth similar to that deduced from a Vlasov simulation. In this paper, we moreover showed that, for purely time growing waves, there is a range in (moderate) wave amplitudes where the predictions of both models as regards χ_i, and therefore the efficiency to laser drive a plasma wave, are similar.

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