Breakdown of the Brillouin limit and classical fluxes in rotating collisional plasmas

J. M. Rax,[1,a)] A. Fruchtman,[2] R. Gueroult,[3] and N. J. Fisch[3]
[1]*Université de Paris XI and LOA, ENSTA–Ecole Polytechnique, 91128 Palaiseau, France*
[2]*HIT, Holon Institute of Technology, 52 Golomb St., Holon 58102, Israel*
[3]*PPPL, Princeton University, Princeton, New Jersey 08540, USA*

The classical collisionless analysis displaying the occurrence of slow and fast rigid body rotation modes in magnetized plasmas is extended to collisional discharges. Collisions speed up the fast mode, slow down the slow one, and break down the classical Brillouin limit. Rigid body rotation has a strong impact on transport, and a collisional radial transport regime, different from the classical Braginskii collisional flux, is identified and analyzed. © *2015 AIP Publishing LLC*.
[http://dx.doi.org/10.1063/1.4929791]

I. INTRODUCTION

Rotation in cylindrical magnetized quasineutral plasma columns can be (i) induced through the application and control of a large radial electric field with coaxial or rings electrodes, or (ii) spontaneous as a result of the azimuthal drift currents, due to the radial ambipolar electric field, supplemented by the magnetization currents associated with the density gradient.

Rotation in cylindrical magnetized quasineutral and non-neutral plasmas has been widely investigated within the framework of: (i) plasma centrifuges experiments,[1–8] (ii) non-neutral plasmas physics,[9–12] and (iii) thermonuclear magnetic confinement studies with homopolar devices[13–15] or rotating mirrors.[16–19] Besides these classical fields, the general problem of angular momentum conversion[20–22] between static magnetic field, wave helicity, and plasma vorticity has received considerable attention in other contexts such as (i) particle acceleration and magnetic field generation in plasma channel and plasma bubble,[23–25] (ii) resonant particle acceleration and isotope separation with magnetized cylindrical particles beams,[26,27] (iii) mass separation with rotating crossed fields configurations,[28–30] (iv) and plasma propulsion.[31,32]

Within the framework of most of these various studies, the problem of angular momentum dynamics in cylindrical magnetized plasmas has been mainly considered in the collisionless regime where rigid body rotations provide a simple and universal model;[9–12] although, in the collisional regime, resistive magnetohydrodynamic and perturbative models have been used to address the issue of collisional dissipation,[33,34] the collisional extension of the universality of rigid body rotations has never been explored. Rigid body rotation has been predicted and observed in collisionless plasmas experiments dedicated to isotope separation in vacuum-arc[35,36] and nonneutral plasmas studies.[9,10]

In this study, we will consider a two-fluid model and address the issue of the impact of collisions on the basis of the extension of the classical collisionless rigid body solutions; then, transport will be considered, and we will identify a large modification of the classical Braginskii[37] regime of transport. Such a new regime, of the neoclassical type, is expected as the particle motion displays a supplementary times scales, the full rotation period, beside the cyclotron one. When the collisionless motion displays two times scales, collisional transport displays several regimes because of the occurrence of two controls parameters: the product of the two collisionless times scales with the collision frequency. The main purpose of this study is to evaluate the impact of collisions on Brillouin rotation and the impact of rotation on collisional transport; the origin of the radial force balance and, more generally, the details of the plasma formation will not be considered here.

A precise evaluation of the radial and angular velocities fields is crucial to understand the dynamics of axially symmetric magnetized discharges, such as those used in mass separation[28–30] processes taking advantage of the Brillouin limit.[12] We address this issue and show that the canonical vorticity and the generalized Hall parameter, to be defined below, are key quantities controlling the interplay between magnetization, rotation, and collisions.

The main effect of collisions is that the second order algebraic equation whose small and large roots determine the slow and fast classical collisionless angular velocities[9–12] becomes a fourth order equation when collisions are taken into account; we solve and study this latter equation. We identify and analyze the neoclassical regime (we call this new transport regime *neoclassical* as the effect of rotation here is similar to the effect of rotational transform in screw pinches or in the Pfirsch-Schlüter regime of tokamaks) and compare it to the classical Braginskii regime.

The sum of the electrostatic potential energy $-e\phi$ plus the particles chemical potential $k_B T \ln(n/n_0)$ (n is the density, T the temperature, k_B the Boltzmann constant, and $-e$ the particle charge), the electrochemical potential or free energy per particle, provides a free energy content whose gradient is the source which drives spontaneous plasma rotations, but this classical collisionless angular momentum generation mechanism will appear to be supplemented by a second

a)Author to whom correspondence should be addressed. Electronic mail: jean-marcel.rax@polytechnique.edu

22, 092101-1

collisional process: part of the radial outward momentum flux, associated with the particles collisional flux, from the central part near the axis of the discharge toward the edge, is converted by the axial magnetic field into an azimuthal flow leading to a rotation, and part of the azimuthal flux is converted by the axial magnetic field into a radial flow leading to transport. That collisions must have an impact on classical collisionless rigid body rotations is obvious but the trend of this impact is difficult to access on a purely phenomenological basis. On the one hand, collisions induce a radial flux, and the magnetic force transforms this radial outward flow into an azimuthal flow and we can expect an increase of the rotation with respect to the collisionless regime, but, on the other hand, collisions damp the azimuthal flow and we are led to the conclusion that the angular velocity will decrease when the collision frequency increases; thus, a definite answer to the problem of the interplay between azimuthal rotation, radial escape, and collisions requires a deeper analysis which is provided by this paper and leads to the conclusion that the collisionless fast mode is accelerated and the slow one is decelerated; thus, both increasing and decreasing acceleration tendencies are at work.

The two-fluid model used to derive these new results about azimuthal rotation around the axial magnetic field and radial transport across the field is then compared with the single particle behavior. Both fluid and single particle analysis are shown to provide a unified and coherent picture of the azimuthal and radial dynamic in collisional, magnetized, axially symmetric, plasma discharges.

Two main classes of rotational motions can be considered for a cylindrical magnetized plasma column: (i) sheared rotations and (ii) rigid rotor rotations, the first class is typified by the so called Keplerian rotations where the azimuthal velocity v_θ decreases as the inverse of the radius $v_\theta(r) \sim v r_0 / r$. Keplerian rotations are well documented in neutral fluid dynamics as they describe singular vorticity fields. For rigid body rotation, the azimuthal velocity v_θ is proportional to the radius $v_\theta(r) \sim \Omega r$. Both Keplerian and rigid body rotations can be matched together to construct the so called *Rankine vortex*, which provides the most useful modeling of localized vorticity in fluid and plasma dynamics.

To describe a collisional magnetized plasmas column, all along this paper, we will consider the generic discharge model depicted in Fig. 1. Near the discharge axis, we will consider a rigid body type of rotation $v_\theta(r) \sim \Omega r$ as an approximate model for an edge sheared rotation. This approximation is relevant in the central part of the discharge, but obviously breaks down near the wall where viscosity damps the rotational motion in order to match the wall zero velocity.[38] This matching takes place in a viscous boundary layer; similarly, the quasineutral approximation breaks down near the wall and near the top and bottom plates, where the field lines end up, in order to insure the equality of positive and negative charge fluxes through a non-neutral sheath layer. Note that quasineutrality is in fact mostly insured by the fluxes along the field lines and the study of the global charge balance requires a full 3D model of the discharge. Despite the simplifying assumption of constant angular velocity near
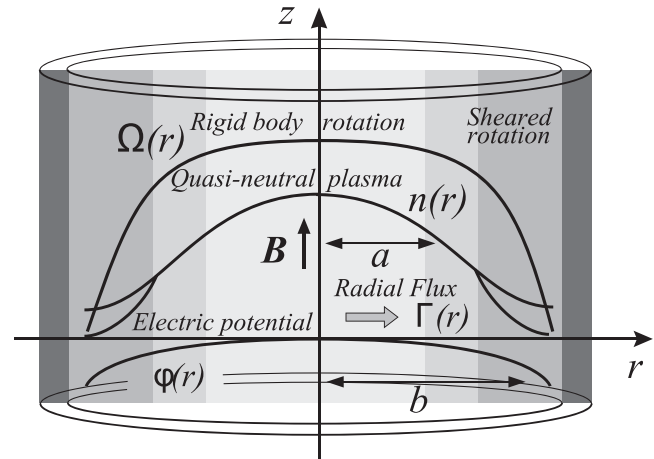

FIG. 1. Vorticity, density, and potential profiles in a magnetized discharge.

the center, away from the viscous boundary layer, the analysis of spontaneous rigid rotor rotations is less straightforward in the collisional case than in the collisionless case. As the problem is more intricate, the rotation fulfills a fourth order algebraic equation which is derived, solved, and analyzed in Sec. IV. The smallest root is the one which arises spontaneously in magnetized collisional discharges experiments.

Rigid body rotations described and analyzed in this paper, are expected to provide a relevant model of azimuthal flow near the axis of magnetized discharge where weakly ionized plasma behave as an inviscid fluid; near the wall, sheath non-neutral layer and viscous boundary layer breakdown both the quasineutral hypothesis and the rigid body assumption. The study presented here is restricted to the equilibrium of the central part of the discharge. The issue of the matching of the solid body rotation to the sheared rotation in the viscous layer is not addressed, as well as the issue of the matching of the quasineutral model to the sheath edge. The important issue of rotating plasmas instabilities channeling part of the free energy into electrostatic modes is also out of the task of the present study. Besides azimuthal flow in axially magnetized discharges, we analyze also the impact of rotation on the radial flow and show that, despite the inclusion of centrifugal forces in the Braginskii fluxes,[37] we do not recover classical collisional fluxes. The new results derived in this study will be relevant for discharges lifetimes larger than the collision time where the radial flux is to be evaluated with the rotational correction and for rotation velocity smaller or of the order of the collision frequency where the Brillouin relations are to be corrected with collisional effects.

This paper is organized as follows. In Sec. II, we review the definition of the canonical vorticity of the electron and ion populations in a weakly ionized magnetized discharge. The conservation of canonical (magnetized) vorticity is reviewed in this introductory section. In Sec. III, we review the classical rigid body rotation analysis of a collisionless plasma. Section IV is dedicated to a formal study of the fully collisional regime. The stream function and velocity potential, which usually clarify two dimensional fluid flow problems, are shown to be fairly simple for a

magnetized collisional plasma discharge. These stream functions and a velocity potential are used to establish the equation fulfilled by the ions and electrons angular velocities and the slow collisional rotation modes is analyzed. We show that collisions increase the fast rotation and decrease the slow rotation. The problem of (neoclassical) radial transport is addressed in Sec. V. We summarize our findings and conclude in the last section. All along this paper when we drop the index i or e, it means that the relation is valid for both electrons and ions. We use either the word *rotation* or *vorticity* for Ω although fluid vorticity 2Ω is two times the fluid rotation Ω.

II. CANONICAL VORTICITY CONSERVATION

Here, we will prove, starting from two different points of view, first from symmetry considerations and then through fluid equations summation, that the sum of the fluid vorticity plus the cyclotron pulsation obeys a global conservation law. This quantity is the *canonical vorticity*[21,22] of a given population. This canonical vorticity will be used to define a *generalized Hall parameter* α along the classical definition of the Hall parameter. The generalized Hall parameter will prove to be very useful for the study of rigid body collisional rotations.

Consider, in Fig. 1, a cylindrical weakly ionized plasma discharge immersed in a static homogeneous magnetic field $\mathbf{B} = B\mathbf{b} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential. The various vector fields are described with respect to a cylindrical basis $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z = \mathbf{b})$ whose axial direction is oriented along the magnetic field. Electron and ion cyclotron frequencies, ω_e and ω_i, are defined through relations: $\omega_e = eB/m = e\nabla \times \mathbf{A} \cdot \mathbf{b}/m$ and $\omega_i = eB/M = e\nabla \times \mathbf{A} \cdot \mathbf{b}/M$. For each population, electrons, ions, and neutrals, classical vorticities $2\Omega_e$, $2\Omega_i$, and $2\Omega_g$ are defined through the relation $2\Omega = \nabla \times \mathbf{v} \cdot \mathbf{b}$. The notations for velocities $\mathbf{v}$, charge e and masses m and M are standard. Before proving that the sum of the vorticity plus the cyclotron frequency is one of the main dynamical parameter of the discharge rotational dynamics, let us demonstrate that this quantity fulfills a local conservation law.

The ion and electron canonical momentum are defined as: $\mathbf{p}_i = M\mathbf{v}_i + e\mathbf{A}$ and $\mathbf{p}_e = m\mathbf{v}_e - e\mathbf{A}$. The vector $\nabla \times \mathbf{p}$ is named *canonical vorticity* and $\nabla \times \mathbf{v}$ *classical vorticity*. We call n_g, n_i, and n_e the neutrals' density (g), the ions' (i) density, and the electrons' (e) density, and $\mathbf{v}_g$, $\mathbf{v}_i$, and $\mathbf{v}_e$ are the velocities of these populations. The flux of ions, electrons, and neutrals axial canonical vorticity, $\nabla \times \mathbf{p} \cdot \mathbf{b}$, is given by

$$\sum_{j=g,i,e} n_j\mathbf{v}_j(\nabla \times \mathbf{p}_j \cdot \mathbf{b}) = 2n_g M\mathbf{v}_g\Omega_g + n_i M\mathbf{v}_i(2\Omega_i + \omega_i)$$
$$+ n_e\mathbf{v}m_e(2\Omega_e - \omega_e). \quad (1)$$

Because of axial symmetry, the axial canonical angular momentum is conserved and this conservation in a steady state discharge requires $\nabla \cdot \sum n_j\mathbf{v}_j(\nabla \times \mathbf{p}_j \cdot \mathbf{b}) = 0$, where the index j indicates the electron, ion, and neutral populations

$$\nabla \cdot [n_g M\mathbf{v}_g 2\Omega_g + n_i M\mathbf{v}_i(2\Omega_i + \omega_i) + n_e m\mathbf{v}_e(2\Omega_e - \omega_e)] = 0.$$
$$(2)$$

This general identity will be recovered through the analysis of the fluid linear momentum balance equations describing the linear momentum collisional exchanges between the three fluids. That a linear momentum balance provides an angular momentum conservation relation might look surprising at first sight; but, when no spin intrinsic angular momentum is involved in the dynamical coupling of a system, angular momentum balance does not provide any new information with respect to linear momentum balance as all angular momentums are of orbital nature.

In order to keep the inertial terms in the fluid momentum balance equations, we use a convenient form displaying explicitly the vorticity through the classical identity $(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla\mathbf{v}^2/2 + (\nabla \times \mathbf{v}) \times \mathbf{v}$. Considering the collisional coupling between the three populations,[39,40] the balances between inertial force, electric Coulomb force, magnetic Laplace force, pressure, and friction forces is

$$\nabla\left(\frac{\mathbf{v}_e^2}{2}\right) + (\nabla \times \mathbf{v}_e) \times \mathbf{v}_e = -\frac{e}{m}\mathbf{E} - \frac{e}{m}\mathbf{v}_e \times \mathbf{B} - \frac{\nabla P_e}{n_e m}$$
$$- \nu_e(\mathbf{v}_e - \mathbf{v}_g), \quad (3)$$

$$\nabla\left(\frac{\mathbf{v}_i^2}{2}\right) + (\nabla \times \mathbf{v}_i) \times \mathbf{v}_i = \frac{e}{M}\mathbf{E} + \frac{e}{M}\mathbf{v}_i \times \mathbf{B} - \frac{\nabla P_i}{n_i M}$$
$$- \nu_i(\mathbf{v}_i - \mathbf{v}_g), \quad (4)$$

$$\nabla\left(\frac{\mathbf{v}_g^2}{2}\right) + (\nabla \times \mathbf{v}_g) \times \mathbf{v}_g = -\frac{\nabla P_g}{n_g M} - \nu_{eg}(\mathbf{v}_g - \mathbf{v}_e)$$
$$- \nu_{ig}(\mathbf{v}_g - \mathbf{v}_i), \quad (5)$$

where $\mathbf{E}$ is the ambipolar or the applied electric field, P_e and P_i are the electrons and ions pressures, and ν_i, ν_e, ν_{eg}, and ν_{ig} are the momentum transfer collisions frequencies.

In this paper, we will assume that the neutral density is constant and neglect neutral depletion due to ionization[41–43] and rotation. The interplay between ionization depletion and centrifugal depletion requires a larger framework; the study of this interplay will not be addressed here where we keep the neutral Eq. (5) for the sole purpose of identifying global vorticity conservation through the exchange between the three populations.

The set of equations (3)–(5) describes a steady state plasma flow as a result of both the action of the electric field $\mathbf{E}$ and pressure gradient ∇P, these two forces induce (i) an azimuthal electric drift (the electron and ion Hall currents which cancels each other) plus (ii) a diamagnetic flow (the electron and ion Nernst currents which add up together), and these azimuthal ion and electron rotations ultimately drive a neutral rotation through collisional friction. We have neglected both plasma and gas viscosities; neglecting viscosity implies that the dynamical exchanges between the various species preserve the conservation of fluid linear momentum so that no dissipation takes place. Note that friction forces describe microscopic exchanges between the fluid velocities of the various populations and not dissipation associated with dispersion of energy and momentum among the microscopic degrees of freedom below the fluid level, as is the case with viscosity. Fluid momentum conservation breaks down near the wall

where the neutral gas and plasma develop a thin viscous boundary layer to accommodate their rotational motions with the fixed wall, resulting in momentum dissipation and ultimately heating, so we consider a two dimensional fluid flow near the central part of the discharge far from the top and bottom plates and far from the cylindrical side wall.

Considering an infinite extension along the axial direction, we restrict the study to a two dimensional fluid motion. Introducing vorticities and cyclotron frequencies, this leads to the following form of the momentum balance equations:

$$\nabla\left(\frac{v_e^2}{2}\right) + 2\Omega_e \mathbf{b} \times \mathbf{v}_e = \frac{e}{m}\nabla\phi - \omega_e \mathbf{v}_e \times \mathbf{b} - \frac{\nabla P_e}{m n_e}$$
$$- \nu_e(\mathbf{v}_e - \mathbf{v}_g), \tag{6}$$

$$\nabla\left(\frac{v_i^2}{2}\right) + 2\Omega_i \mathbf{b} \times \mathbf{v}_i = -\frac{e}{M}\nabla\phi + \omega_i \mathbf{v}_i \times \mathbf{b} - \frac{\nabla P_i}{M n_i}$$
$$- \nu_i(\mathbf{v}_i - \mathbf{v}_g), \tag{7}$$

$$\nabla\left(\frac{v_g^2}{2}\right) + 2\Omega_g \mathbf{b} \times \mathbf{v}_g = -\frac{\nabla P_g}{M n_g} - \nu_{eg}(\mathbf{v}_g - \mathbf{v}_e)$$
$$- \nu_{ig}(\mathbf{v}_g - \mathbf{v}_i), \tag{8}$$

where ϕ is the electrostatic potential, $\mathbf{E} = -\nabla\phi$. Local linear momentum conservation implies that the momentum losses by one population is gained by the others and vice versa; this constraint implies that $n_g M \nu_{eg} = n_e m \nu_e$ and $n_g M \nu_{ig} = n_i M \nu_i$.

Taking the sum of the three equations (6)–(8), multiplied by their associated masses and densities, then applying the curl operator $\nabla\times$ on both sides, we get the local canonical vorticity conservation law

$$\nabla \times \left[M n_g \Omega_g \mathbf{b} \times \mathbf{v}_g + M n_i \left(\Omega_i + \frac{\omega_i}{2}\right) \mathbf{b} \times \mathbf{v}_i \right.$$
$$\left. + m n_e \left(\Omega_e - \frac{\omega_e}{2}\right) \mathbf{b} \times \mathbf{v}_e \right] = 0. \tag{9}$$

The vectorial identity $\nabla \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\nabla \cdot \mathbf{c})$ is fulfilled as we assume no axial variations of the discharges parameters, so Eq. (9) is the very same conservation relation described by Eq. (2). This conservation law displays clearly the fact that the quantity of interest to study magnetized plasma rotation is the canonical vorticity, and in view of the importance of this parameter, we define the generalized Hall parameters α for electrons e and ions i as

$$\alpha_{e/i} = \pm \frac{\omega_{e/i} \mp 2\Omega_{e/i}}{\nu_{e/i}}. \tag{10}$$

In order to study the impact of collisions on rigid body rotation and radial transport and to provide a deeper insight into the spontaneous and induced collisional rotations problem, let us first briefly review the main results of the spontaneous and induced rigid rotor collisionless problem.

III. COLLISIONLESS RIGID ROTOR EQUILIBRIUM

The occurrence of rigid body rotations in magnetized collisionless plasma columns was established theoretically and experimentally within the framework of nonneutral plasmas studies[9–12] and vacuum arc centrifuge isotope separation studies.[3–8] We review here briefly the main results of these studies.

Both ions and electrons are to be considered, but we will restrict the study to the ion population, the electron case being very similar. For collisionless ions, the balance between inertia, electric Coulomb force, magnetic Laplace force, and centrifugal and pressure stresses can be written as Eq. (7) with $\nu_i = 0$

$$(\omega_i + 2\Omega_i)\mathbf{v}_i \times \mathbf{b} = \nabla\left(\frac{e}{M}\phi + \frac{v_i^2}{2} + \frac{k_B T_i}{M}\ln\frac{n_i}{n_0}\right). \tag{11}$$

The electron equation is similar up to a sign. The cross product of this relation with the axial unit vector $\mathbf{b}$ leads to the expressions of the ions velocity $\mathbf{v}_i$

$$(\omega_i + 2\Omega_i)\mathbf{v}_i = -\nabla\left(\frac{e}{M}\phi + \frac{v_i^2}{2} + \frac{k_B T_i}{M}\ln\frac{n_i}{n_0}\right) \times \mathbf{b}. \tag{12}$$

But Eq. (12) does not provide an expression of the velocity $\mathbf{v}_i$ as Ω_i is a function of $\mathbf{v}_i$. Equation (12) can be easily interpreted in terms of electric drift $-\nabla\phi \times \mathbf{b}$, inertial drift $-\nabla v_i^2 \times \mathbf{b}$, and diamagnetic flow $-\nabla n_i \times \mathbf{b}$; the main difference with respect to classical drift theory is the occurrence of the denominator $\omega_i + 2\Omega_i$ rather than simply ω_i as a result of Coriolis force. As we are interested by the canonical vorticities $\omega_i + 2\Omega_i$, we take the curl ($\nabla \times \mathbf{v}_i \cdot \mathbf{b} = 2\Omega_i$ and $\partial/\partial z = 0$) of Eq. (12) to obtain

$$(\omega_i + 2\Omega_i)2\Omega_i = \Delta\left(\frac{e}{M}\phi + \frac{v_i^2}{2} + \frac{k_B T_i}{M}\ln\frac{n_i}{n_0}\right). \tag{13}$$

This last relation can be further simplified as, for the collisionless case, no radial flow develops so that the velocity is purely azimuthal and the Laplacian of this azimuthal component is simply given by $\Delta v_i^2 = 4\Omega_i^2$. The final result

$$\Omega_i^2 + \Omega_i \omega_i = \frac{1}{2M}\Delta\left(e\phi + k_B T_i \ln\frac{n_i}{n_0}\right) \tag{14}$$

is a second order algebraic equation whose solutions give the two rigid rotor modes as a function of both the electrostatic potential and the density, the two components of the rotation being, respectively, associated with the electric drift Hall current and the diamagnetic Nernst current around the magnetic field. The rigid body rotation hypothesis implies that the right hand side of Eq. (14) must be independent of the radial position. For the case of uniform non neutral cold ion ($T_i = 0$) plasma equilibrium, we recover the classical Brillouin solution[12] as $k_B T_i \ln n_i/n_0 = 0$ and as Poisson equation relates directly $\Delta\phi$ and ω_{pi} the ion plasma pulsation: $e\phi = -M\omega_{pi}^2 r^2/4$. It is worthwhile to note that for a given population the adiabatic Boltzmann equilibrium: $n_i = n_0 \exp -e\phi/k_B T_i$ is associated with a zero angular velocity solution as no free energy is available at equilibrium for rotation. When rotation is present, four radial forces are to be taken into account: (i) electrostatic and (ii) pressure forces (whose balance gives rise to Boltzmann equilibrium in

unmagnetized irrotational discharges), and (iii) centrifugal force plus (iv) Laplace magnetic force. Thus, a simple balance between (i) electrostatic and (ii) pressure forces cannot provide a rotating equilibrium solution. Rather than the electrostatic potential ϕ, let us consider the full electrochemical potential, the free energy per particle, defined as the sum $e\phi + k_B T_i \log n_i/n_0$; this quantity must display a parabolic profile in order to get a rigid body rotation equilibrium as $\Delta[e\phi(r) + k_B T_i \log n_i(r)/n_0]$ must be independent of the radial position r on the right hand side of Eq. (14) as the left hand side does not depend of the radius under the homogeneous magnetic field and rigid body rotation hypothesis.

We define ω_e^* and ω_i^* such that the density profiles are given by generalized Boltzmann distributions

$$n_i = n_{0i} \exp - \frac{M\omega_i^{*2} r^2}{2 k_B T_i} \exp - \frac{e\phi(r)}{k_B T_i}, \quad (15)$$

$$n_e = n_{0e} \exp - \frac{m\omega_e^{*2} r^2}{2 k_B T_e} \exp \frac{e\phi(r)}{k_B T_e}. \quad (16)$$

These Gaussian density profiles, already identified and observed in plasma centrifuge experiments, must not be considered as dynamical constraints, but as definitions of the parameters ω_e^* and ω_i^* characterizing the concavities of electron and ion densities near the discharge axis as depicted in Fig. 1. With these definitions, the ions vorticity equation (14) becomes the classical second order algebraic equation: $\Omega_i^2 + \Omega_i \omega_i = -\omega_i^{*2}$. This relation is similar to the one occurring in the classical Brillouin problem, where $2\omega_i^*$ plays the role of ω_{pi}^2 the nonneutral plasma frequency; in both cases, these pulsations provide a measure of the free energy content responsible for the induced or spontaneous plasma rotation. Here, this free energy available for spontaneous rotation is internal: thermal and electrostatic (ambipolar field), and in the Brillouin problem, it is purely electrostatic and applied.

The occurrence of the exponent $m\omega^{*2} r^2/2 \pm e\phi$ in place of $\pm e\phi$ in the generalized Boltzmann equilibrium Eq. (15) is not surprising as it is just the classical Lagrangian coupling $L(v_\theta, r) = mv_\theta^2/2 \pm ev_\theta A_\theta(r) \mp e\phi(r)$ between particles and fields, where $v_\theta = \Omega r$ is the azimuthal velocity and $A_\theta(r) = Br/2$ the azimuthal vector potential so that $L = m\Omega^2 r^2/2 \pm m\Omega\omega r^2/2 \mp e\phi = -m\omega^{*2} r^2/2 \pm e\phi$.

The two solutions of the collisionless rigid body rotation problem are given by the classical formulas already identified and analyzed within the framework of nonneutral and quasineutral collisionless plasmas studies

$$\frac{\Omega_{\pm i}}{\omega_i} = -\frac{1}{2} \mp \frac{1}{2} \sqrt{1 - 4\frac{\omega_i^{*2}}{\omega_i^2}}, \quad (17)$$

$$\frac{\Omega_{\pm e}}{\omega_e} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4\frac{\omega_e^{*2}}{\omega_e^2}}. \quad (18)$$

The Brillouin limit,[12] which will disappear when collisions are taken into account, can be written as a set of two conditions: $4\omega_e^{*2}/\omega_e^2 < 1$ and $4\omega_i^{*2}/\omega_i^2 < 1$. These constraints can be interpreted as follows: the free energy available for radial expansion must not overcome the magnetic field energy

aimed at confining the plasma otherwise no rigid body rotation equilibrium is possible. The slow and fast rotation modes are depicted in Fig. 2 (we have restricted the figure to the electron case, the ion one is similar up to a minus sign) where we have normalized the angular velocity Ω to the cyclotron frequency ω and normalized the measure of the free energy ω^* also to ω.

Among these two modes, only the slow branches Ω_{-e} and Ω_{-i} are to be considered for the spontaneous rotation problem. The fast branch needs an initial energy input far above the discharge thermal free energy content and cannot be considered as spontaneous but as initially strongly driven. That the slow mode is indeed the observable mode of spontaneous rotation can be proved as follows. Consider the limit of small rotation, that is, to say, of small chemical potential gradient ($\omega_e^* \ll \omega_e$ and $\omega_i^* \ll \omega_i$). Under this assumption, we can Taylor expand Eq. (18)

$$\Omega_{-e} = \frac{\omega_e^{*2}}{\omega_e} + \frac{\omega_e^{*4}}{\omega_e^3} + \cdots. \quad (19)$$

Then, we can identify the meaning of these first two terms of Eq. (19) within the framework of classical drift theory. In a cold magnetized discharge, three adiabatic flows are to be considered: (i) the electric E cross B drift $\mathbf{V}_E$, (ii) the diamagnetic flow $\mathbf{V}_M$, and (iii) the centrifugal inertial drift $\mathbf{V}_C$, whose classical expressions are

$$\mathbf{V}_E = -\frac{\nabla\phi}{B} \times \mathbf{b}, \quad (20)$$

$$\mathbf{V}_M = -\frac{\nabla P \times \mathbf{B}}{neB^2} = -k_B T \frac{\nabla n}{eBn} \times \mathbf{b}, \quad (21)$$

$$\mathbf{V}_C = \frac{m[(\mathbf{V}_E + \mathbf{V}_M) \cdot \mathbf{e}_\theta]^2}{r} \frac{\mathbf{e}_r \times \mathbf{b}}{eB}. \quad (22)$$
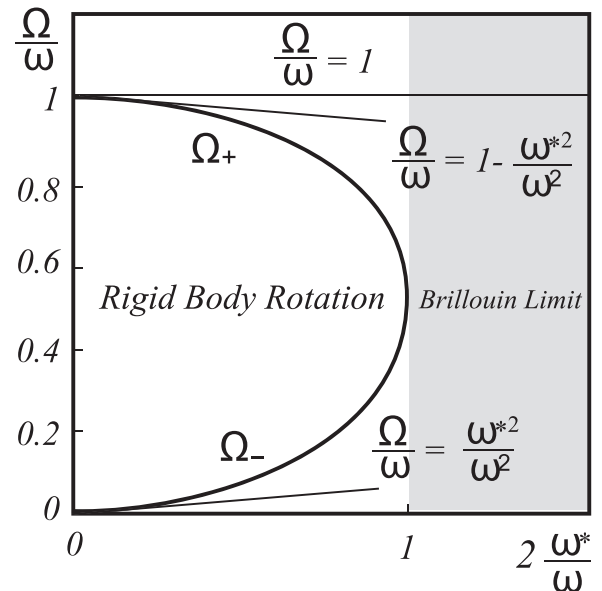

FIG. 2. Angular velocity of the slow and fast modes as a function of the free energy parameter ω^*/ω.

Consider the electron population near the center of the discharge, as a result of the cylindrical symmetry, the density and potential display parabolic profiles because the density gradient $\partial n / \partial r \sim r$ and the electric field $\partial \phi / \partial r \sim r$, and, as the pulsations ω^* are defined as combinations of the concavity plus the slope of the electric field, we can easily check that the first term of the expansions equations (19) is nothing but the cross fields plus the diamagnetic azimuthal flow and that the second term is nothing but the centrifugal drift flow

$$(\mathbf{V}_E + \mathbf{V}_M) \cdot \mathbf{e}_\theta = \frac{\omega_e^{*2}}{\omega_e} r, \qquad (23)$$

$$\mathbf{V}_C \cdot \mathbf{e}_\theta = \frac{\omega_e^{*4}}{\omega_e^3} r. \qquad (24)$$

Higher order terms in Eq. (19) can be analyzed as coupling between drift flows and higher order drifts and the previous expansions shows that, to the lowest order, the slow mode is just the sum of the E cross B drift and diamagnetic azimuthal flows; this confirms that this mode is the spontaneous one. A definite characterization of the difference between the two modes will be assessed at the end of this section on the basis of the energy content compared to the thermal energy content and will confirm this conclusion.

Several additional hypotheses can be considered to explore the possibility of equilibrium spontaneous rotations in plasma discharges. For example, as the difference between the ions and electrons rotations might provide a free energy source for instabilities, we can study the consequences of an isorotation hypothesis, $\Omega_{-e} = \Omega_{-i}$. This hypothesis severely constrains the density and potential profiles, as the electron rotation is far larger than the ion one. In fact, if the free energy content is dominated by the ambipolar potential, the dominant term is the E cross B azimuthal drift, which naturally provides isorotation.

IV. COLLISIONAL SLOW MODE DECELERATION

The classical model of one dimensional plasma discharges is a well posed problem;[39,40] five unknown functions, the electrons and ions densities, $n_e(x)$ and $n_i(x)$, the electrons and ions fluid velocities, $v_e(x)$ and $v_i(x)$, and the ambipolar electric potential, $\phi(x)$, are solutions of five equations, electrons and ions mass (charge) balances, electrons and ions momentum balances, and the constraint on local quasi-neutrality. Thus, for given neutral density and plasma temperature, the various densities, velocities, and potential profiles are determined up to an unknown boundary value, the central density $n(x=0)$, which can be separately predicted on the basis of the global power balance. Then, to complete the description, this quasineutral solution is to be matched with the sheath solution to fulfill the constraint of global steady state neutrality.

The classical model of two dimensional (the radial coordinate r and the axial coordinate z) axisymmetric magnetized cylindrical discharges also provides a well posed problem; nine unknown functions, the electrons and ions densities, $n_e(r,z)$ and $n_i(r,z)$, the electrons and ions radial fluid velocities, $v_{re}(r,z)$ and $v_{ri}(r,z)$, the electrons and ions axial fluid velocities, $v_{ze}(r,z)$ and $v_{zi}(r,z)$, the ambipolar electric potential, $\phi(r,z)$, and the electrons and ions vorticities, $\Omega_e(r,z)$ and $\Omega_i(r,z)$, are solutions of nine equations: electrons and ions mass (charge) conservations, electrons and ions momentum balances along, across, and around the magnetic field, and the constraint on local quasineutrality. However, because the nature of this latter problem is different from the former one, partial differential equations versus ordinary differential equations, few analytical results are available to understand the full dynamics of magnetized discharges and simple scaling between the various equilibrium parameters are not available.

The aim of the present study is to understand part of this dynamics and to relate the electrons and ions angular rotation velocities to the radial density and potential gradients. These quantities can be predicted on the basis of the particle balances, but this requires the full account of the axial and radial dynamics and the evaluation of the ratio of the radial plasma losses toward the wall to the axial plasma losses toward the top and bottom end plates. We will not consider this full problem here and will make no use of the electrons and ions particles conservations equations; thus, keeping only four unknown, Γ_{re} and Γ_{ri}, the radial particles fluxes, as well as $2\Omega_e$ and $2\Omega_i$ the vorticities, and considering only the four momentum balance equations in the plane perpendicular to the axial magnetic field, we will end up with a set of relations between the density, the potential, the vorticities, and the radial fluxes. This set of relation will be sufficient to analyze the impact of collisions on both the rigid body rotation modes and the radial classical collisional transport.

As collisionless rigid body rotations have been observed in many plasma experiments, we will focus on this class of solutions and show that such a type of solutions can be found for collisional plasmas, thus extending the previous results of collisionless discharges and providing a simple relation to correlate rotations, radial fluxes, density, magnetic field, and collisionality in magnetized discharges. The two main differences with respect to the previous collisionless case will be the occurrence of a fourth order algebraic equation in the collisional case as opposed to a second order one in the collisionless case and in the appearance of radial transport.

Helmholtz representation theorem, for an arbitrary continuously differentiable velocity field in a plane, provides a canonical decomposition of $\mathbf{v}$ as the sum of a curl plus a gradient, the curl being expressed in terms of a stream function Ψ and the gradient in term of a velocity potential Φ

$$\mathbf{v} = \nabla \times \Psi \mathbf{b} - \nabla \Phi = \nabla \Psi \times \mathbf{b} - \nabla \Phi. \qquad (25)$$

This classical decomposition Eq. (25) as a sum of an irrotational part plus a solenoidal part is meaningful as the stream function Ψ can be interpreted as the specific density of angular momentum $\mathbf{b} \cdot \int \int \int (\mathbf{r} \times m\mathbf{v}) d\mathbf{r} = \int \int \int 2m\Psi d\mathbf{r}$ around the discharge axis, and the $\Delta\Phi$ as the specific density of radial momentum flux $-\int \int m\mathbf{v} \cdot d\mathbf{s} = \int \int \int m\Delta\Phi d\mathbf{r}$. We will demonstrate that the stream function Ψ and the velocity potential Φ provides the right framework to analyze the rotational dynamics of a collisional discharge. We will find that they are related to the discharge free energy content through the general relations

$$\frac{\Psi_e}{\alpha_e} = \Phi_e = \frac{\nu_e}{m} \frac{\frac{m}{2}v_e^2 - e\phi + k_B T_e \ln\frac{n_e}{n_0}}{\nu_e^2 + (\omega_e - 2\Omega_e)^2}, \qquad (26)$$

$$\frac{\Psi_i}{\alpha_i} = \Phi_i = \frac{\nu_i}{M} \frac{\frac{M}{2}v_i^2 + e\phi + k_B T_i \ln\frac{n_i}{n_0}}{\nu_i^2 + (\omega_i + 2\Omega_i)^2}. \qquad (27)$$

This result clearly displays the fact that the velocity potential Φ vanishes when the collision frequency vanishes as the magnetic confinement becomes perfect and no radial outward flux can be observed. Electron and ion momentum balances in steady state discharges require that the sum of the inertial, Coulomb, Laplace, pressure, and friction forces is equal to zero. This requirement is described by equations (6) and (7) where we assume that the neutral velocity $\mathbf{v}_g = 0$ as the density of this population is several orders of magnitude larger than the charged particles density, which means that internal gas viscosity provides a strong coupling with the wall and that the global inertia tensor of the neutral gas rotor is far larger than the electrons and ions ones, both effects, inertia and viscosity, resulting in a strong tendency to remain at rest. Arranging the unknown electrons and ions velocities on the left hand side and the equilibrium free energy drive on the right side, we end up with

$$\mathbf{v}_e + \frac{\omega_e - 2\Omega_e}{\nu_e} \mathbf{v}_e \times \mathbf{b} = -\frac{1}{m\nu_e}\nabla\left(-e\phi + \frac{mv_e^2}{2} + k_B T_e \ln\frac{n_e}{n_0}\right), \qquad (28)$$

$$\mathbf{v}_i - \frac{\omega_i + 2\Omega_i}{\nu_i} \mathbf{v}_i \times \mathbf{b} = -\frac{1}{M\nu_i}\nabla\left(e\phi + \frac{Mv_i^2}{2} + k_B T_i \ln\frac{n_i}{n_0}\right). \qquad (29)$$

For simplicity, we will momentarily drop the *i* and *e* indexes as Equations (28) and (29) can be reduced to the same form of the type

$$\mathbf{v} + \alpha \mathbf{v} \times \mathbf{b} = -\nabla\psi. \qquad (30)$$

It can be checked that the general solution of this type of linear equation can be expressed as the sum of the radial ($\nabla\psi$) and azimuthal ($\mathbf{b} \times \nabla\psi$) components

$$\mathbf{v} = \frac{\alpha}{1 + \alpha^2}\nabla\psi \times \mathbf{b} - \frac{1}{1 + \alpha^2}\nabla\psi. \qquad (31)$$

We recognize Helmholtz representation of the velocity field Eq. (25) so that the stream function and the velocity potential for solid body rotation in collisional magnetized plasma are given by Equations (26) and (27). Taking the curl and the divergence of Eq. (31), we can express the angular and radial velocities as

$$(\nabla \times \mathbf{v}) \cdot \mathbf{b} = -\Delta\Psi = -\frac{\alpha}{1 + \alpha^2}\Delta\psi = \frac{1}{r}\frac{\partial}{\partial r}rv_\theta = 2\Omega, \qquad (32)$$

$$\nabla \cdot \mathbf{v} = -\Delta\Phi = -\frac{1}{1 + \alpha^2}\Delta\psi = \frac{1}{r}\frac{\partial}{\partial r}rv_r = 2\frac{\Omega}{\alpha}. \qquad (33)$$

The first new result with respect to the collisionless case is the occurrence of a radial velocity v_r in addition to the azimuthal one v_θ. Integrating the previous relations Equations (32) and (33) with respect to the radius gives the ions and electrons azimuthal v_θ and radial v_r velocities: $v_{\theta e} = \Omega_e r$, $v_{\theta i} = \Omega_i r$ and $v_{re} = \Omega_e r/\alpha_e$, $v_{ri} = \Omega_i r/\alpha_i$. To express the angular velocities Ω_i and Ω_e as a function of the plasma parameters, we consider Eq. (32) and restrict the analysis to the ion case as the electron case is similar up to a minus sign

$$2\Omega_i(1 + \alpha_i^2) = -\frac{\alpha_i}{M\nu_i}\Delta\left(\frac{M}{2}v_i^2 + e\phi + k_B T_i \ln\frac{n_i}{n_0}\right). \qquad (34)$$

We can either look for the rotation pulsation Ω_i as the unknown or for the generalized Hall parameter α_i as the unknown; we will consider this latter case. The term associated with the inertial force free energy, Δv^2, can be expressed as $\Delta v_i^2 = \Delta(v_{\theta i}^2 + v_{ri}^2) = 4\Omega_i^2(1 + \alpha_i^2)/\alpha_i^2$, thus

$$\left(\alpha_i^2 - \frac{\omega_i^2}{\nu_i^2}\right)(1 + \alpha_i^2) - \frac{2\alpha_i^2}{M\nu_i^2}\Delta\left(e\phi + k_B T_i \ln\frac{n_i}{n_0}\right) = 0. \qquad (35)$$

Assuming density profiles of the type Eqs. (15) and (16), we introduce ω^* the Laplacian of the chemical potential as in Section III: $\omega^{*2} = -\Delta[\pm e\phi + k_B T \ln(n/n_0)]/2M$ so that we end up with the quadratic equation

$$\alpha^4 + \left(1 - \frac{\omega^2}{\nu^2} + 4\frac{\omega^{*2}}{\nu^2}\right)\alpha^2 - \frac{\omega^2}{\nu^2} = 0. \qquad (36)$$

Only one positive roots for α^2 is to be considered, and this leads us to the slow and fast mode expression for the ions and electrons

$$\frac{\Omega_{\pm e}}{\omega_e} = \frac{1}{2} \pm \frac{1}{2\sqrt{2}}$$
$$\times \sqrt{1 - 4\frac{\omega_e^{*2}}{\omega_e^2} - \frac{\nu_e^2}{\omega_e^2} + \sqrt{\left(1 - 4\frac{\omega_e^{*2}}{\omega_e^2} - \frac{\nu_e^2}{\omega_e^2}\right)^2 + 4\frac{\nu_e^2}{\omega_e^2}}}, \qquad (37)$$

$$\frac{\Omega_{\pm i}}{\omega_i} = -\frac{1}{2} \mp \frac{1}{2\sqrt{2}}$$
$$\times \sqrt{1 - 4\frac{\omega_i^{*2}}{\omega_i^2} - \frac{\nu_i^2}{\omega_i^2} + \sqrt{\left(1 - 4\frac{\omega_i^{*2}}{\omega_i^2} - \frac{\nu_i^2}{\omega_i^2}\right)^2 + 4\frac{\nu_i^2}{\omega_i^2}}}. \qquad (38)$$

These relations are the collisional counterpart of the collisionless relations Equations (17) and (18), and they express the fluids solid body rotations as a function of the discharge parameters. If we consider the limit $\nu = 0$, we just recover Eqs. (17) and (18).

We will study the behavior of both the slow and fast collisional rotation as a function of the collision parameter $x \equiv \nu^2/\omega^2$ and as a function of the free energy parameter $y \equiv 4\omega^{*2}/\omega^2$. We have plotted the two branches of Eq. (38)
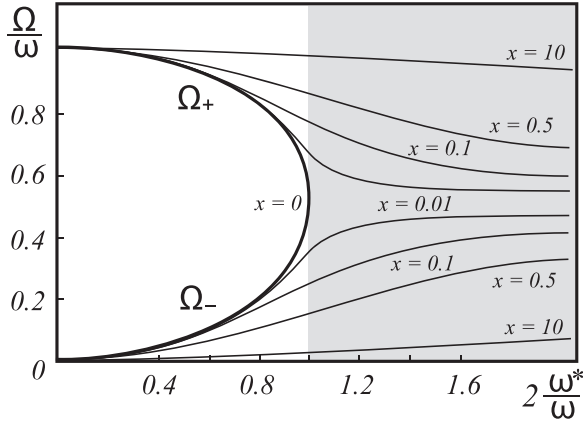
FIG. 3. Rigid rotor angular velocity as a function of the free energy parameter ω^*/ω for various x.

in Fig. 3 where the abscissa is $\sqrt{y}$, the ordinate $\Omega_\pm/\omega$ and the family of curves is labeled by the parameter x. Clearly, collisions speed up the fast mode, slow down the slow one, and breakdown the Brillouin limits, which no longer exist in this regime. That collisions speed up the fast mode and slow down the slow one is a direct consequence of the inequality:

$$1 - y - x + \sqrt{(1 - y - x)^2 + 4x} > 2(1 - y), \quad \text{if} \quad \omega^{*2} > \omega^2,$$

which is the case. For a given free energy parameter y, the behavior of the rotation Ω/ω can be analyzed as a function of the collision parameter $x = \nu^2/\omega^2$, and two cases are to be considered: (i) if $y < 1$, even in the absence of collisions, $x = 0$, there exist a finite rotation velocity, (ii) if $y > 1$, the collisionless Brillouin limit implies that at small ν^2/ω^2 the rotation velocity converge toward zero. The first case is depicted in Fig. 4 and the second one in Fig. 5. The transition between the two regime takes place at the collisionless Brillouin limit $y = 1$ independently of x.

The fact that the slow mode Ω_- is the one which arise spontaneously in discharges without initial energy input can be confirmed as follows. In a collisional discharge, the two major components of the fluid velocity perpendicular to the static magnetic field are the E cross B drift $\mathbf{V}_E$ modified by
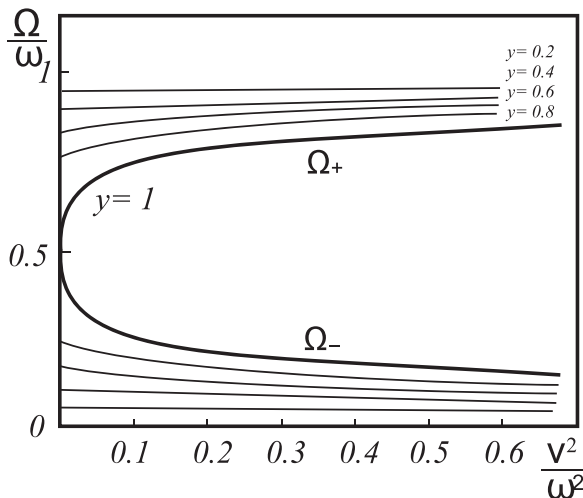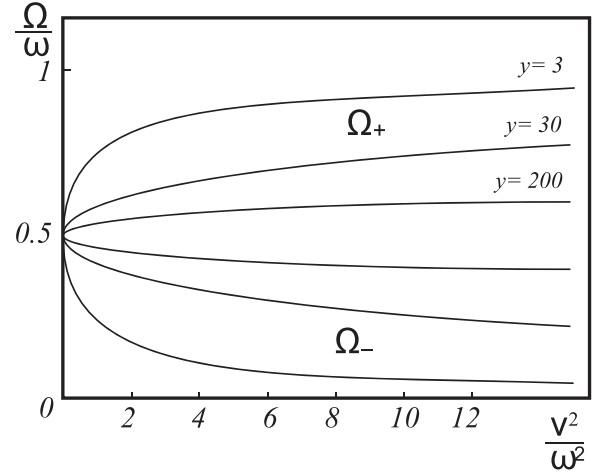

FIG. 4. Rigid rotor angular velocity as a function of the collision parameter ν/ω for various $y < 1$.

the collisions and the diamagnetic flow $\mathbf{V}_M$ modified by the collisions. The classical expressions of these two fluid velocities are given by

$$\mathbf{V}_E = -\frac{\omega^2}{\omega^2 + \nu^2} \frac{\nabla\phi \times \mathbf{B}}{B^2} \approx -\frac{\omega^2}{\nu^2} \frac{\nabla\phi \times \mathbf{B}}{B^2}, \quad (39)$$

$$\mathbf{V}_M = -\frac{\omega^2}{\omega^2 + \nu^2} \frac{\nabla P \times \mathbf{B}}{neB^2} \approx -\frac{\omega^2}{\nu^2} \frac{k_B T}{n} \frac{\nabla n \times \mathbf{B}}{eB^2}. \quad (40)$$

Expanding Eq. (37) for large ν and finite ω^*/ν, we obtain the following first order terms:

$$\Omega_{-e} = \frac{\omega_e^{*2}}{\nu_e^2} \omega_e + \cdots \quad (41)$$

for the electron population and a similar expression for the ion one. Near the axis of the discharge, the density gradients are linear with respect to the radial position, $\partial n/\partial r \sim r$, as well as the electric field profile, $\partial \phi/\partial r \sim r$, the former being Gaussian and the latter parabolic as a result of quasineutrality, we obtain

$$(\mathbf{V}_E + \mathbf{V}_M)_e \cdot \mathbf{e}_\theta = \omega_e^{*2} \frac{\omega_e}{\nu_e^2} r, \quad (42)$$

which confirms that the slow collisional rigid body rotation agree with classical drift theory. A collisional magnetized discharge is determined by the collision frequencies, ν_i and ν_e, the cyclotron frequencies, ω_i and ω_e, and the free energy parameters ω_i^{*2} and ω_e^{*2}, near the center of the discharge quasineutrality provides a relevant hypothesis, and this additional hypothesis allows to express a further relation between electron and ions densities. Following the analysis of Sec. III, plasma density and electrostatic potential turn out to be Gaussian and parabolic, and they can be interpreted in terms of characteristic lengths a and b. Based on this relation: $\omega_e^{*2} \sim v_{Te}^2/a^2$, so $\omega_e^{*2} \sim (\rho_{Le}/a)^2 \omega_e^2$ and for the electronic slow branch $\Omega_{-e} \approx (\rho_{Le}/a)^2 \omega_e^3/\nu_e^2 \ll \omega_e$ as expected.

Besides the vorticities, $2\Omega_e$ and $2\Omega_i$, the radial particles fluxes, Γ_{re} and Γ_{ri}, are to be analyzed in order to complete the picture of collisional rigid body rotation of magnetized

FIG. 5. Rigid rotor angular velocity as a function of the collision parameter ν/ω for various $y > 1$.

plasma discharges. In the remaining part of this article, we will restrict our analysis to the slow branch of rigid body plasma rotations.

V. ROTATION INDUCED TRANSPORT

Classical transport in a magnetized discharge is described by the linear combination of Ohm and Fick laws, which are modified to account for the confining influence of the magnetic field. If we introduce the classical unmagnetized mobility $\mu = e/m\nu$ and the classical unmagnetized diffusion coefficient $D = k_B T/m\nu$, the collisional particles flux $\Gamma_{b\perp}$ perpendicular to a static magnetic field is given by the Braginskii relation[35,37]

$$\frac{\Gamma_{b\perp}}{n} = -\frac{\mu}{1+\omega^2/\nu^2}\nabla_\perp\varphi - \frac{D}{1+\omega^2/\nu^2}\nabla_\perp\ln\frac{n}{n_0}, \quad (43)$$

where ω is the cyclotron frequency, n the particle density, and φ the sum of the electrical ϕ and mechanical potentials acting on the discharge. For a rotating discharge, we have to consider the influence of the centrifugal potential $\Omega r^2/2$ in addition to the ambipolar or applied electrostatic potential ϕ. The radial electron and ion classical fluxes, $\Gamma_{eb} = \Gamma_{eb\perp}\cdot\mathbf{e}_r$ and $\Gamma_{ib} = \Gamma_{ib\perp}\cdot\mathbf{e}_r$, are thus given by the relations

$$\Gamma_{eb} = n_e\frac{e}{m}\frac{\nu_e}{\nu_e^2+\omega_e^2}\left(\frac{m}{e}\frac{\partial}{\partial r}\frac{\Omega_e^2 r^2}{2}+\frac{\partial\phi}{\partial r}-\frac{k_B T_e}{e}\frac{\partial}{\partial r}\ln\frac{n_e}{n_0}\right), \quad (44)$$

$$\Gamma_{ib} = n_i\frac{e}{M}\frac{\nu_i}{\nu_i^2+\omega_i^2}\left(\frac{M}{e}\frac{\partial}{\partial r}\frac{\Omega_i^2 r^2}{2}-\frac{\partial\phi}{\partial r}-\frac{k_B T_i}{e}\frac{\partial}{\partial r}\ln\frac{n_i}{n_0}\right). \quad (45)$$

We can express these two fluxes as a function of the free energy parameter $2M\omega^{*2} = -\Delta[e\phi + k_B T\ln(n/n_0)]$ for both electrons and ions in order to get the final form of the Braginskii fluxes in a rigid body rotating discharge

$$\Gamma_{eb} = n_e\frac{\omega_e^{*2}+\Omega_e^2}{\nu_e^2+\omega_e^2}\nu_e r, \quad (46)$$

$$\Gamma_{ib} = n_i\frac{\omega_i^{*2}+\Omega_i^2}{\nu_i^2+\omega_i^2}\nu_i r, \quad (47)$$

where we have used the previous definition of the free energy parameter ω^{*2} integrated with respect to the radius: $-M\omega^{*2}r = \partial[e\phi + k_B T\ln(n/n_0)]/\partial r$. The relations equations (37) and (38) describe the impact of collisions on the azimuthal motion. The radial dynamics for collisional rigid body rotations can be described through the integration of relation equation (33) with respect to the radius. The radial flux of particles associated with collisional rigid body rotations for electrons Γ_{en} and ions Γ_{in} is

$$\Gamma_{en} = n_e\frac{\Omega_{-e}}{\alpha_{-e}}r = n_e\frac{\Omega_{-e}}{\omega_e-2\Omega_{-e}}\nu_e r, \quad (48)$$

$$\Gamma_{in} = n_i\frac{\Omega_{-i}}{\alpha_{-i}}r = -n_i\frac{\Omega_{-i}}{\omega_i+2\Omega_{-i}}\nu_i r. \quad (49)$$

The index n refers to a neoclassical flux because these dissipative fluxes are different from the classical Braginskii fluxes Eqs. (46) and (47), under the same electric, rotational, and pressure constraints. As we are considering only the slow branch, for ions, both Ω_i and α_i are negative, and for electrons, both Ω_e and α_e are positive, so that the radial fluxes are always directed from the center toward the edge. The origin of the difference between Γ_n and the classical Braginskii flux Γ_b is similar to the origin of the difference between the neoclassical and classical fluxes with a rotational transform in screw pinches. This difference must be traced back to the orbit difference between a simple cyclotron motion and the combination of a cyclotron motion and a rotation of the guiding center.

It is worth noting that, as shown in Appendix A, these results can be recovered using a single particle orbit analysis if assuming a particular choice of orbit. The formula for the rigid body rotation is then the same as the one found above with a fluid analysis, Equations (37) and (38), provided that we consider the free energy content, which is purely electrostatic here rather than thermal and electrostatic for the fluid model, on the same footing for both models. Moreover, the frictional damping, which induces a damping at the orbital level and a dissipative radial flux at the fluid level, can be interpreted as a radial flux, which turns out to display the same neoclassical expression than that discovered for the fluid model, Equations (48) and (49). This convergence of results provides a strong validation of the obtained results.

In order to quantify the impact of rigid body rotation on radial transport, we introduce the ratio of the neoclassical flux to the classical flux and restrict the analysis to the ion case as the electron case is qualitatively similar

$$\frac{\Gamma_{in}}{\Gamma_{ib}} = -\frac{1+\dfrac{\nu_i^2}{\omega_i^2}}{1+\dfrac{2\Omega_{-i}}{\omega_i}\dfrac{\omega_i^{*2}}{\omega_i^2}+\dfrac{\Omega_{-i}^2}{\omega_i^2}}\dfrac{\Omega_{-i}}{\omega}, \quad (50)$$

where Ω_{-i} is given by Eq. (38). For a given collision parameter $x = \nu^2/\omega^2$, the ratio of the neoclassical flux to the classical flux Γ_n/Γ_b can be analyzed as a function of the free energy parameters $y = 4\omega^{*2}/\omega^2$. We have displayed in Fig. 6 the set of curves covering the weak collisionality regime; the
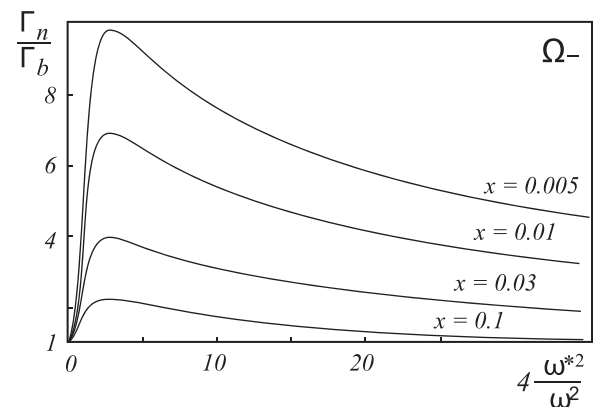

FIG. 6. Normalized radial flux for large values of the free energy parameters ω^*/ω and $x < 2$.
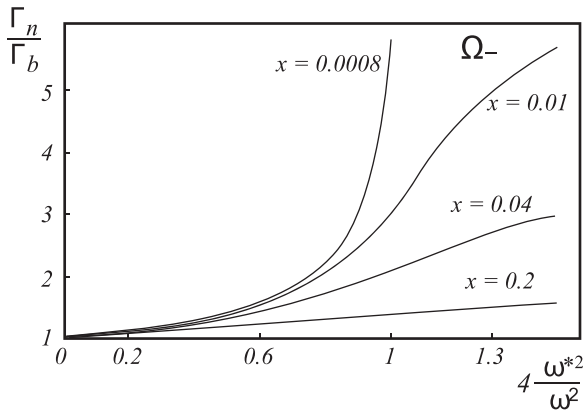
FIG. 7. Normalized radial flux for small values of the free energy parameters ω^*/ω.

ratio $\Gamma_n/\Gamma_b \sim 1$ is reached when $\omega^{*2}/\omega^2 \sim 0$ and $\omega^{*2}/\omega^2 \sim +\infty$ as expected. In the range $\omega^{*2}/\omega^2 \sim 1$ for weakly magnetized and weakly collisional plasma, the ratio Γ_n/Γ_b can reach significant values. This justifies a deeper analysis of this regime $\omega^{*2}/\omega^2 \sim 1$, which is displayed in Fig. 7. The fact that, at low collisionality, near the value $4\omega^{*2}/\omega^2 \sim 1$ we observe a steepening of the curve can be explained by the fact this value is precisely the Brillouin limit for collisionless discharges. We also find a change of behavior for $x \sim 2$, the flux ratio Γ_n/Γ_b switch from Fig. 6 behavior to a monotonous decreasing function from 1 to 0, still converging to 1 for large ν/ω or small ω^*/ω. This behavior is due to the occurrence of a maximum of the classical collisional factor $\nu/(\omega^2 + \nu^2)$ when $\omega \sim \nu$. The convergence toward $\Gamma_n/\Gamma_b \sim 1$ for large ν and finite ω^*/ν can be confirmed through a Taylor expansions of Equations (37) and (38): $\nu_e^2\Omega_{-e} \approx \omega_e\omega_e^{*2}$ and $\nu_i^2\Omega_{-i} \approx -\omega_i\omega_i^{*2}$, then Eqs. (48) and (49) give $\Gamma_{en} \approx n_e\Omega_{-e}\nu_e r/\omega_e$ and $\Gamma_{in} \approx -n_i\Omega_{-i}\nu_i r/\omega_i$, and finally, the first order terms of Eqs. (46) and (47) $\Gamma_{eb} \approx n_e\Omega_{-e}\nu_e r/\omega_e$ and $\Gamma_{ib} \approx -n_i\Omega_{-i}\nu_i r/\omega_i$. Thus, both classical and neoclassical fluxes matches for large ν and finite ω^*/ν.

VI. SUMMARY AND CONCLUSION

In this study, we have shown that the classical rigid body motion of a magnetized collisionless plasma column can be extended to collisional discharges and that this extension is physically meaningful. The energy content of the slow mode is far below the free energy content of the discharge so that it channels only a small fraction of this content. This clearly indicates that this rotation mode arises spontaneously, as the Taylor expansion of its vorticity exhibits the occurrence of electric drift and diamagnetic flow providing the coupling mechanism between the electrochemical potential $(\pm e\phi + k_BT\ln(n/n_0))$ drive and the mechanical rigid body vorticity 2Ω.

Two quantities are particularly useful to understand this coupling and the energy channeling from the electrochemical potential to the rotational degree of freedom: (i) the canonical vorticity $2\Omega \pm \omega$ and (ii) the generalized Hall parameter α. The results presented and analyzed here are relevant to the central part of a magnetized cylindrical plasma column, away from the lateral wall, where viscosity comes into play, and

far from the top and bottom plates, where the field line ends up and the plasma flow displays sheath. Near these plates, the dependency on the axial coordinate must be considered in order to account for the particles fluxes along the field line.

Under these assumptions, we have expressed the four unknowns, Γ_{re} and Γ_{ri}, the radial particles fluxes, and $2\Omega_e$ and $2\Omega_i$, the axial vorticities, as a function of the discharge parameters on the basis of the four momentum balance equations in the plane perpendicular to the axial magnetic field. These results are summarized by Eqs. (37) and (38) for the vorticities and (48) and (49) for the fluxes.

Similar to the crossover from electric coupling $e\phi$ to magnetic coupling $e\mathbf{v} \cdot \mathbf{A}$, we have identified the occurrence of a neoclassical regime of radial transport due to the difference between classical cyclotron orbits and rigid body orbits. Further insight to this problem can be gained if we consider the fraction of entropy production dS/dt which is diverted from the radial dissipative out flow to the rigid body rotation: $dS_\theta/dS_r = (dS_\theta/dt)/(dS_r/dt) = \nu v_\theta^2/\nu v_r^2 = \alpha^2$; thus, the generalized Hall parameter provides a direct measure of this entropy production channeling ratio.

The very simple picture provided by a single particle analysis has been shown to be coherent with a fluid model. Further studies on the stability of the slow collisional modes, on the kinetic theory of the neoclassical fluxes and on the matching to the viscous boundary layer remain to be done to build a complete picture of rigid rotor vorticity generation, sustainment, and transport in plasma discharges.

APPENDIX A: SINGLE PARTICLE ORBIT VERSUS FLUID MODEL

We will address the issue of collisional rigid body rotation within the framework of single particle orbit analysis. This type of model will necessarily miss the diamagnetic effect associated with the radial pressure gradient, but it will keep the electric field drift.

Consider a charged particle interacting with a magnetic field such that ω is the cyclotron frequency and ν the friction frequency. In addition to these conservative and dissipative forces, we take into account a linear electric field described by a quadratic potential similar to the quadratic electrostatic potential encountered in the fluid model. The equation for the particle orbit $\mathbf{r}(t)$ is given by

$$\frac{d^2\mathbf{r}}{dt^2} = \omega^{*2}\mathbf{r} + \omega\frac{d\mathbf{r}}{dt} \times \mathbf{e}_z - \nu\frac{d\mathbf{r}}{dt}, \qquad (A1)$$

where ω^* describe the concavity of the parabolic electrostatic potential. This equation is then projected on a Cartesian basis $(\mathbf{e}_x, \mathbf{e}_y)$ perpendicular to the magnetic field

directed by the unit vector $\mathbf{e}_z$. We introduce the complex variable $Z(t) \equiv x(t) + jy(t)$. The orbit perpendicular to the magnetic field is described by the complex ordinary differential equation

$$\frac{d^2Z}{dt^2} + [\nu + j\omega]\frac{dZ}{dt} - \omega^{*2}Z(t) = 0. \qquad (A2)$$

As this equation is linear, which was not the case for the fluid model, we look for solutions of the type $Z(t) = Z_0 \exp j\Omega t$, where Z_0 is a complex constant. The unknown pulsation Ω is a solution of the algebraic equation

$$\Omega^2 + (\omega - j\nu)\Omega + \omega^{*2} = 0 \qquad (A3)$$

which is easily solved as the sum of two rotations associated with the small and the large root

$$2\Omega = -\omega + j\nu \pm \sqrt{(\omega - j\nu)^2 - 4\omega^{*2}}. \qquad (A4)$$

It might look surprising that the fluid model gives a fourth order equation (36) and the particle model a second order equation (A3); this is a consequence of the fact that the fluid model deals with real variables and the particle one with complex one, so Eq. (A3) can be interpreted as two coupled second order real algebraic equation whose decoupling provides a fourth order equation. The real and imaginary part of Eq. (A4) can be expressed as

$$\text{Re}[\Omega] = \Omega_{\mp} = -\frac{\omega}{2} \pm \frac{1}{2\sqrt{2}}\sqrt{\omega^2 - 4\omega^{*2} - \nu^2 + \sqrt{(\omega^2 - 4\omega^{*2} - \nu^2)^2 + 4\nu^2\omega^2}}, \qquad (A5)$$

$$\text{Im}[\Omega] = \gamma_{\mp} = \frac{\nu}{2} \mp \frac{1}{2\sqrt{2}}\sqrt{-\omega^2 + 4\omega^{*2} + \nu^2 + \sqrt{(\omega^2 - 4\omega^{*2} - \nu^2)^2 + 4\nu^2\omega^2}}, \qquad (A6)$$

where Eq. (A5) is nothing but the fluid relation Eq. (38). The general single particle orbit is given by the linear combination $Z(t) = Z_+ \exp j(\Omega_+ + j\gamma_+)t + Z_- \exp j(\Omega_- + j\gamma_-)t$. In order to match the slow mode rigid body rotation, we have to restrict to $Z_+ = 0$.

The interpretation of the imaginary part γ_- requires a further analysis to match the fluid results. If we consider the solution $Z(t) = Z_0 \exp j(\Omega_- + j\gamma_-)t$, we end up with a damped or an escaping orbit. To get an understanding of the link between this orbit and the neoclassical fluid flux studied in Sec. V, we construct the ratio, Eq. (A7), of the radial particle velocity $v_r = dr/dt$ divided by the radial position $r(t) = \sqrt{x^2(t) + y^2(t)}$ and we restrict the analysis to the slow mode $Z(t) = Z_0 \exp j(\Omega_- + j\gamma_-)t$

$$\left.\frac{v_r}{r}\right|_{particle} = \frac{1}{r}\left.\frac{dr}{dt}\right|_{particle} = \text{Re}\left[\frac{Z(t)\frac{d}{dt}Z^*(t)}{Z(t)Z^*(t)}\right] = -\gamma_-. \qquad (A7)$$

Using Eqs. (33) and (10), the very same ratio of the radial velocity to the radial position can also be calculated within the fluid model, with

$$\left.\frac{v_r}{r}\right|_{fluid} = -\left(1 - \frac{\omega}{\omega + 2\Omega_-}\right)\frac{\nu}{2}. \qquad (A8)$$

Injecting Eqs. (A5) into (A8) yields the same result, that is, to say,

$$\left.\frac{v_r}{r}\right|_{fluid} = -\gamma_-. \qquad (A9)$$

This set of result might lead to the conclusion that a single particle model provides a shorter path to the fluid results concerning slow rigid body rotation and neoclassical fluxes. On the one hand, this is true in terms of formula, but, on the other hand, the identification of the radial neoclassical flux and the inclusion of the diamagnetic pressure effect in ω^* can only be achieved with a fluid picture. As usual in plasma physics, this is the interplay between fluid and particles pictures, which provides an understanding of the dynamics rather than a restricted fluid or particle model alone.

[1]B. Lehnert, Phys. Scr. **2**, 106 (1970).
[2]B. Lehnert, Phys. Scr. **7**, 102 (1973).
[3]M. Krishnan, M. Geva, and J. L. Hirshfield, Phys. Rev. Lett. **46**, 36 (1981).
[4]M. Krishnan, Phys. Fluids **26**, 2676 (1983).
[5]E. D. Bosco, S. W. Simpson, R. S. Dallaqua, and A. Montes, J. Phys. D: Appl. Phys. **24**, 2008 (1991).
[6]M. J. Hole and S. W. Simpson, Phys. Plasmas **4**, 3493 (1997).
[7]R. S. Dallaqua, S. W. Simpson, and E. D. Bosco, J. Phys. D: Appl. Phys. **30**, 2585 (1997).
[8]M. Hole and S. Simpson, IEEE Trans. Plasma Sci. **27**, 620 (1999).
[9]R. C. Davidson and N. A. Krall, Phys. Rev. Lett. **22**, 833 (1969).
[10]A. J. Theiss, R. A. Mahaffey, and A. W. Trivelpiece, Phys. Rev. Lett. **35**, 1436 (1975).
[11]I. Kotelnikov, M. Rom, and R. Pozzoli, Phys. Lett. A **372**, 1445 (2008).
[12]R. C. Davidson, *Physics of Nonneutral Plasmas* (Addision-Wesley, Redwood City, 1990).
[13]J. M. Wilcox, Rev. Mod. Phys. **31**, 1045 (1959).
[14]S. A. Diamond and D. L. Turcotte, Phys. Fluids **14**, 928 (1971).
[15]B. Lehnert, Nucl. Fusion **11**, 485 (1971).
[16]A. Bekhtenev, V. Volosov, V. Pal'chikov, M. Pekker, and Y. Yudin, Nucl. Fusion **20**, 579 (1980).
[17]C. Teodorescu, W. C. Young, G. W. S. Swan, R. F. Ellis, A. B. Hassam, and C. A. Romero-Talamas, Phys. Rev. Lett. **105**, 085003 (2010).
[18]A. J. Fetterman and N. J. Fisch, Phys. Plasmas **18**, 094503 (2011).
[19]A. J. Fetterman and N. J. Fisch, Phys. Plasmas **18**, 055704 (2011).
[20]I. V. Sokolov, Usp. Fiz. Nauk. **161**, 175 (1991).

[21]J. Bekenstein, Astrophys. J. **319**, 217 (1987).

[22]S. M. Mahajan and Z. Yoshida, Phys. Plasmas **18**, 055701 (2011).

[23]G. Shvets, N. J. Fisch, and J.-M. Rax, Phys. Rev. E **65**, 046403 (2002).

[24]I. Y. Kostyukov, G. Shvets, N. J. Fisch, and J. M. Rax, Phys. Plasmas **9**, 636 (2002).

[25]C. Thaury, E. Guillaume, S. Corde, R. Lehe, M. Le Bouteiller, K. Ta Phuoc, X. Davoine, J. M. Rax, A. Rousse, and V. Malka, Phys. Rev. Lett. **111**, 135002 (2013).

[26]J.-M. Rax, J. Robiche, and N. J. Fisch, Phys. Plasmas **14**, 043102 (2007).

[27]J.-M. Rax and J. Robiche, Phys. Plasmas **17**, 103112 (2010).

[28]T. Ohkawa and R. L. Miller, Phys. Plasmas **9**, 5116 (2002).

[29]S. Shinohara and S. Hori, Jpn. J. Appl. Phys., Part 1 **46**, 4276 (2007).

[30]R. Gueroult, J.-M. Rax, and N. J. Fisch, Phys. Plasmas **21**, 020701 (2014).

[31]N. J. Fisch, Y. Raitses, and A. Fruchtman, Plasma Phys. Controlled Fusion **53**, 124038 (2011).

[32]R. Gueroult, A. Fruchtman, and N. J. Fisch, Phys. Plasmas **20**, 073505 (2013).

[33]Y. Yue and S. W. Simpson, J. Phys. D: Appl. Phys. **29**, 2866 (1996).

[34]M. J. Hole and S. W. Simpson, J. Phys. D: Appl. Phys. **34**, 3028 (2001).

[35]R. R. Prasad and M. Krishnan, J. Appl. Phys. **61**, 113 (1987).

[36]R. R. Prasad and M. Krishnan, Phys. Fluids **30**, 3496 (1987).

[37]S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. Leontovitch (Consultants Bureau, New York, 1965), Vol. 1.

[38]W. B. Kunkel, W. R. Baker, A. Bratenahl, and K. Halbach, Phys. Fluids **6**, 699 (1963).

[39]M. A. Lieberman and A. J. Lichtenberg, *Principles of Plasma Discharges and Material Processing* (John Wiley and Sons, Hoboken, 2005).

[40]F. F. Chen and J. P. Chang, *Lecture Notes on Principles of Plasma Processing* (Kluwer Academic/Plenum Publishers, New York, 2003).

[41]A. Fruchtman, Plasma Sources Sci. Technol. **17**, 024016 (2008).

[42]A. Fruchtman, IEEE Trans. Plasma Sci. **36**, 403 (2008).

[43]A. Fruchtman and J.-M. Rax, Phys. Plasmas **17**, 043502 (2010).