AS551 General Plasma Physics 1
November 17, 1997

Problem Set #7 (due Nov. 24)

For a plasma in a stationary equilibrium ($d/dt = 0$), with an isotropic pressure, the MHD equilibrium relation is:

$$\nabla p = \frac{\vec{j} \times \vec{B}}{c}$$

In the following problems, you will explore some of the properties of equilibria implied by this equation.

1. General cylindrical pinch equilibria. Consider a plasma with cylindrical symmetry ($\partial/\partial z = 0$, $\partial/\partial\theta = 0$), with specified radial profiles for the pressure $p(r)$, azimuthal field $B_\theta(r)$, and axial field $B_z(r)$. From the MHD equation for equilibria, derive the equation:

$$p(r) + \frac{B^2(r)}{8\pi} = p(0) + \frac{B^2(0)}{8\pi} - \frac{1}{4\pi}\int_0^r dr \frac{B_\theta^2}{r}$$

Assume the plasma extends to $r = a$, and is surrounded by a vacuum (with a magnetic field) outside that radius. For the case of uniform current distribution j_z = constant for $r < a$, and $j_z = 0$ for $r > a$, show that the plasma is paramagnetic, i.e. $B_z(0) > B_z(a)$, if $p(0) < B_\theta^2(a)/(4\pi)$. Is this result dependent, or not, on the current and pressure distributions in the plasma?

2. Bennet pinch condition. Consider a standard z-pinch equilibrium configuration (current in the z-direction, magnetic field only in the θ direction). The plasma is uniform in z and θ, and is confined to $r < a$ by the magnetic field. Calculate the volume integrated plasma energy (per length in z):

$$W = \int_0^a dr 2\pi r \frac{3}{2} p$$

Integrate this by parts (assuming $p = 0$ at $r = a$) to express this in terms of $\partial p/\partial r$, and use the MHD force balance equation to show that W can be expressed in terms of the total current carried by the plasma. (This is known as the Bennett pinch condition.)

3. Complete set of MHD waves. (This problem will require the most algebra.) In the notes entitled "MHD Waves and Instabilities", dated 11/22/96, I derived the dispersion relation for compressional Alfvén waves. Half way down the 4th mini-page of the notes is the general wave equation in MHD without making any compressional assumptions, which with a minor modification can be written as:

$$-\rho_0\omega^2\vec{\xi} = -\Gamma_0 p_0 \vec{k}\vec{k}\cdot\vec{\xi} - \frac{B_0^2}{4\pi}\hat{z}\times[\vec{k}\times(\vec{k}\times(\hat{z}\times\vec{\xi}))]$$

This can be written in the form

$$[-\rho_o\omega^2\overleftrightarrow{1} + \overleftrightarrow{M}]\cdot\vec{\xi} = \overleftrightarrow{A}\cdot\vec{\xi} = 0.$$

Setting the determinant of the matrix $\vec{\vec{A}}$ to zero will yield a cubic Eq. in ω^2. Find the 3 roots for ω^2 and identify each branch: the "shear Alfvén wave" (which is incompressible), the "compressional Alfvén wave", and the "sound wave" (whose propagation speed depends only on c_s and not on v_A). (To identify the separate branches, sometimes it is useful to consider the limit of low $\beta = p/(B^2/8\pi)$.) Also find the corresponding eigenvector $\vec{\xi}$ for each of the 3 eigenvalues for ω^2, and comment on important properties of the "polarization" $\vec{\xi}$ for each mode.

Order-of-Magnitude Quickies (Only rough order-of-magnitude estimates are needed in this section, don't try to solve any differential equations precisely in these problems.)

5. Quasineutrality and $E \times B$ drifts. Assume the electric field is large enough that the $E \times B$ velocity is comparable to the ion thermal speed $v_{ti} = \sqrt{T_i/m_i}$. Again simplify Poisson's Eq. using $\nabla \cdot \vec{E} \sim E/L$. Find the order of magnitude of the relative charge imbalance $\Delta n/n_0 = (n_E - n_i)/n_0$ needed to produce this electric field. Express your result in terms of the fundamental parameters of the Debye length λ_D and the Alfvén speed v_A.

6. Express the ratio of the Alfvén speed to the ion thermal velocity in terms of β, the ratio of the plasma pressure to magnetic pressure.

7. Why magnetic fields don't decay on a ν_{ei} time scale. Assume the initial electron distribution function $f(v)$ is such that the average electron velocity in any direction is zero. A given current density $j \sim n_e e u_e$ is then induced by accelerating all electrons equally from their initial velocity v to $v+u_e$ in a particular direction. What is the increase in their kinetic energy density ΔW_e? Calculate the dimensionless ratio $\Delta W_e/(B^2/(8\pi))$, using the rough order-of-magnitude estimate $\nabla \times \vec{B} \sim B/L \sim 4\pi j/c$, where B and L are characteristic magnetic and length scales. Express your final answer in terms of the collisionless skin depth c/ω_{pe}. [Implication: You should find that this ratio is tiny for typical parameters. Thus, even though collisional drag would seem to want to wipe out the electron current in a quick ν_{ei} time scale, only a tiny fraction of the magnetic energy is needed during each collision time to maintain the current.]

Express the magnetic diffusion coefficient in terms of ν_{ei} and c/ω_{pe}. How is this related to your calculation of $\Delta W_e/(B^2/(8\pi)$.

8. Calculate the ratio of the collisionless skin depth to the ion gyroradius and express the result in terms of β.

9. Express the ratio of the collisionless skin depth to the Debye length in terms of the electron temperature and the electron rest mass.

10. Table of "typical" plasma parameters. Consider three types of plasmas: (1) a "typical" laboratory fusion plasma with $n \sim 10^{14}/\text{cm}^3$, $T \sim 10^4$ eV, $B \sim 5$ Tesla, and $L \sim 100$ cm, (2) a "typical" plasma used for semiconductor etching or materials processing, with $n \sim 10^{12}/\text{cm}^3$, $T \sim 1$ eV, $B \sim 0.1$ Tesla, and $L \sim 10$ cm, and (3) the solar wind plasma with $n \sim 5/\text{cm}^3$, $T \sim 100$ eV, $B \sim 5$ nTesla, and $L \sim 10^5$ km (based on the radius of the earth's magnetopause, which is roughly $10R_E$, where the earth radius $R_E = 6375$ km. For these three plasmas calculate the following quantities, listing the results in a table for comparison:

the plasma to magnetic pressure ratio β, the Debye length λ_D, the collisionless skin depth c/ω_{pe}, the ion gyroradius ρ_i, the mean-free-path between collisions λ_{mfp}, (and the ratio between these four lengths and L), the Alfvén-to-light speed ratio v_A/c, and the number of particles in a Debye sphere $n\lambda_D^3$.

End of order-of-magnitude section.

11. A puzzler requiring very careful thinking.

In this problem we will assume that everything is uniform in the y and z directions, and vary only in the x direction. The magnetic field points only in the z direction. Assume the equations of ideal MHD hold, with a scalar pressure, and the plasma is confined to the region from $x = -a$ to $x = +a$ by the magnetic field. The plasma pressure is uniform for $|x| < a$. There are 2 ideal conducting walls at $|x| = a + L$, and there is a vacuum (with a magnetic field) in the region between the plasma and the walls.

Now suppose that the plasma is heated for a finite time (perhaps by absorbing photons from a nearby supernova explosion). As the pressure increases, the plasma will expand until a new force balance is reached with the external magnetic field. (Ignoring the launching of Alfvén or sound waves which might happen if the heating is too rapid.) If so, what is this new equilibrium state, and how does it scale with the amount of heating? What happens in the limits where the initial β is very small, or where the initial β (internal plasma pressure divided by external magnetic field pressure) is 1?

Now redo all of this using a CGL tensor pressure instead of a scalar pressure, assuming that the parallel temperature is fixed and that the heating occurs only in $T_\perp$ (perhaps because the heating process uses waves which resonate with the cyclotron frequency of the particles). Assume that during any expansion process that $p_\perp/(nB)$ is constant (ignoring heating). [Stated another way including heating and dynamics simultaneously, assume that $3d(p_\perp/(nB))/dt = P/(nB)$ where P is the heating power density.] What is a possible physical argument for assuming this? Show that this assumption is equivalent to taking $\Gamma = 2$ in the "equation of state", consistent with $\Gamma = (2 + m)/m$, where m is the "number of degrees of freedom" which can participate in the relevant component of the pressure.

How do your answers to the above questions change in the limit as the walls are moved to infinity ($L \to \infty$).