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INVERSE PROBLEM FOR INCREMENTAL SYNCHROTRON RADIATION

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Abstract—Significantly more information is available from synchrotron emission from a plasma when the plasma is purposefully disturbed. An inverse problem, to deduce properties of the disturbance given time-dependent radiation data, is proposed. The fast time response of radiation detectors is fully exploited by this approach. A special case of interest, perpendicular observation of a steady-state plasma, lends itself to an analytic inversion.

1. INTRODUCTION

A MAGNETICALLY confined plasma is a copious emitter of synchrotron radiation. In fact, the radiated power is a matter of concern in the design of high temperature nuclear fusion reactors. Here, however, we concern ourselves with the possibility of diagnosing, by means of this radiation, important momentum space features of high energy electrons in a Tokamak plasma.

One way in which this radiation might be used relies on the measurement of the two-dimensional pattern $R(\omega, \theta)$ of radiation emitted at frequency ω into angle θ, where θ measures the angular deviation from purely perpendicular observation of the magnetic field. (The Tokamak is observed in the vertical plane that includes the tangent to the magnetic field **B**, so the strength of **B** may be assumed constant and known.) The 2-D pattern $R(\omega, \theta)$ might then be used to infer the two-dimensional electron distribution function $f(p_\parallel, p_\perp)$, where $p_\parallel$ and $p_\perp$ refer to the electron momentum, respectively, parallel and perpendicular to the magnetic field. This approach, however, suffers because it requires the deployment of any array of microwave detectors to resolve the θ-dimension. In practice, possibly only one detector is available, and, while it has been used to place useful constraints on $f(p_\parallel, p_\perp)$ (see in the Tokamak literature, e.g. CELATA and BOYD, 1977; TAMOR, 1979; BORNATICI *et al.*, 1983; CELATA, 1985; HUCHINSON and KATO, 1986; KATO and HUCHINSON, 1986) or to deduce elegantly in a relativistic electron ring geometry a one-dimensional f (MAHAJAN *et al.*, 1974), a full inversion is not possible.

A second way to obtain information might be to employ r.f. or other power to induce in the plasma a momentum-space flux $\mathbf{\Gamma}(\mathbf{p}, t)$, and to measure the incremental radiation emitted by the perturbed distribution, i.e. the additional radiation produced as a result of the probing r.f. power. One can then pose the following problem: to deduce the momentum space details of the source function, $\mathbf{\Gamma}(\mathbf{p}, t)$, given this incremental radiation. Details of this source function give the velocity space details of power absorption, which may be of immediate interest in r.f. heating or current drive experiments. Also, the details of the absorption informs on the electron distribution function itself, since, basically, where (in velocity space) power is absorbed, there must be electrons.

To isolate the incremental radiation, we are at liberty to modulate or otherwise to control the time-dependence of the source; so suppose we impose an impulse $\mathbf{\Gamma} = \mathbf{S}(\mathbf{p})\delta(t)$. The incremental radiation now decays in time as the electrons suffer collisions, obeying laws we think we know, so that this time decay reveals the details of the original impulse. For example, incremental radiation associated with fluxes of fast electrons would decay slowly; for nonrelativistic electrons the decay time goes as $1/p^3$. If there are N time points collected during the electron slowing-down time, then the 2-D impulse response, $R(\omega, t; \theta)$, produces N times the constraints on $\mathbf{S}(\mathbf{p})$ than does the immediately available spectra $R(\omega, t = 0; \theta)$. Thus, using only one observation angle, we hope to deduce the 2-D wave-induced flux $\mathbf{S}(\mathbf{p})$ from the 2-D radiation data, $R(\omega, t; \theta)$, θ fixed. Of course, if more detectors were available, the multiplication in information would be the same, and the additional information might be used to uncover spatial dependencies.

The utility of information provided by radiation during collisional relaxation has been recognized before: for example, ALIKAEV *et al.* (1976) observe radiation decay subsequent to intense cyclotron heating in the TM-3 Tokamak and GIRUZZI *et al.* (1986) observe numerically the transient radiation pattern associated with cyclotron heating in the presence of a d.c. electric field. The goal of the present work is to provide the mathematical framework to make precise the extent of information so obtained. Here, the analysis is limited to steady-state plasmas. We assume that the plasma is optically thin to the observed radiation and that the plasma density, effective ion charge state, and equilibrium magnetic field are known on the flux surface on which the electron velocity distribution is perturbed and from which the incremental radiation is observed. Moreover, with only one detection angle, we must assume that these known quantities may be treated as relatively constant over the region to which the incremental radiation is attributed. For example, we hope to treat the case of heating by lower-hybrid waves launched along the Tokamak horizontal plane, thereby creating a superthermal perturbation, while the incremental synchrotron emission is observed in a vertical plane and at, typically, the third harmonic, to which the plasma is likely to be optically thin. The solutions obtained here are valid only for induced fluxes of superthermal electrons. On the other hand, slower electrons radiate less, often at a distinguishably different frequency, and are thus unlikely to be an important source of confusion.

Note that while frequency and time are formally conjugate variables, a huge separation in time scales permits them to be treated as independent variables; the frequencies to be measured are typically hundreds of gigahertz, while, for example, 400 keV electrons in a plasma with density 10^{13} cm^{-3} slow down in about a hundred milliseconds. Radiation detectors with microsecond response time are readily available, so that $\sim 10^5$ time points are available in principle in a decay time. Thus, the potential exists for a huge multiplication in the amount of processible information.

Relying on theories of wave-damping and quasilinear diffusion, one might use $\mathbf{S}(\mathbf{p})$ to infer $f(\mathbf{p})$, at least where $\mathbf{S}(\mathbf{p})$ is finite. However, details of $\mathbf{S}(\mathbf{p})$ are often of immediate interest, because we then know where in electron momentum space wave power is absorbed. This locus determines, for example, the efficiency of current-drive by lower-hybrid waves (FISCH, 1987). Suppose the plasma current were maintained by these waves. By delivering through the waveguides an incremental impulse of power, under otherwise steady conditions, the distribution of resonant electrons may be available from the incremental radiation, possibly providing useful feedback for waveguide-phasing. Moreover, in such a manner one might resolve the outstanding question of the so-called spectral gap observed in recent experiments.

The paper is organized as follows: in Section 2, we find the Green's function for the radiation response. This relates the radiation pattern $R(\omega, t; \theta)$ to the perturbed electron flux $\mathbf{S}(\mathbf{p})$. This is the basic equation that we pose for inversion, i.e. to deduce the flux $\mathbf{S}(\mathbf{p})$. In this work, we do not venture beyond solving for $\mathbf{S}(\mathbf{p})$; for example, we do not solve for the background electron velocity distribution, although, as we noted, possibly the background could be related through other theories to $\mathbf{S}(\mathbf{p})$, particularly in those regions where $\mathbf{S}(\mathbf{p})$ is finite. These other theories, in turn, depend on the nature of the wave that induces the flux $\mathbf{S}(\mathbf{p})$, but our analysis is entirely independent of the precise wave that induces $\mathbf{S}(\mathbf{p})$.

In Section 3, we show that for the special case of perpendicular observation ($\theta = 0$), there is a series inversion that may be arrived at by recursion. This special case succumbs to an analytic solution because perturbations in $\mathbf{S}(\mathbf{p})$ on different energy shells may be treated independently. The dependence of each energy shell on the velocity pitch-angle can be inferred essentially because higher harmonics in the velocity pitch-angle decay faster. In Section 4, we relate the $\theta = 0$ solution to the ill-posed heat equation. The connection to this more familiar equation facilitates a discussion of the robustness to noise of the solution. The introduction of noise limits the number of harmonics that can be reliably inverted. In Section 5 we summarize our conclusions and present some ideas on the relaxation of some of the more stringent assumptions that were required here.

2. FORMULATION OF THE PROBLEM

To relate the incremental radiation $R(\omega, t; \theta)$ to an externally imposed impulsive momentum-space flux $\mathbf{\Gamma}(\mathbf{p}, t)$, we write the distribution function f as $f = f_M(1+\phi_B+\phi)$, where f_M is a Maxwellian distribution, ϕ_B describes the deviation from Maxwellian of the background distribution, and ϕ describes the distribution specifically associated with the source $\mathbf{\Gamma}$. Assume that f obeys the linearized Fokker–Planck equation, which for the term we are interested in may be written as

$$f_M \partial\phi/\partial t - C(\phi) = -\nabla_p \cdot \mathbf{\Gamma}(\mathbf{p}, t) \equiv -\delta(t)\nabla_p \cdot \mathbf{S}(\mathbf{p}), \tag{1}$$

where C is a collision term and steady-state (no d.c. electric field) has been assumed. Then the incremental radiation due to $\mathbf{\Gamma}$ may be written for an optically thin plasma as

$$R(\omega, \tau; \theta) = \int \mathrm{d}^3p\, f_M \phi(\mathbf{p}, \tau) I(\omega, \mathbf{u}; \theta) = \int \mathrm{d}^3u\, \psi(\omega, \mathbf{u}, \tau; \theta) Q(\mathbf{u}), \tag{2}$$

where we employ normalized momentum, $\mathbf{u} = \mathbf{p}/mc$, and normalized time, $\tau = \nu_c t$, with collision frequency $\nu_c = nq^4 \log \Lambda / 4\pi m^2 \varepsilon_0^2 c^3$, and we define $Q(\mathbf{u}) \equiv -(mc)^3 \nabla_p \cdot \mathbf{S}(\mathbf{p})$. Here I is the radiated power into angle θ and frequency ω of a single electron at momentum $\mathbf{p}$, and ψ is the Green's function for the radiation response, i.e. ψ solves the relativistic Fokker–Planck adjoint equation, written for superthermal excitation in the high-velocity limit as

$$\frac{\partial \psi}{\partial \tau}+\frac{1}{u^3}\left(\gamma^2 u \frac{\partial \psi}{\partial u}-\gamma \frac{1+Z}{2} \frac{\partial}{\partial \mu}(1-\mu^2) \frac{\partial}{\partial \mu} \psi\right)=0, \tag{3}$$

with the initial condition $\psi(\omega, \mathbf{u}; \theta, \tau=0)=I(\omega, \mathbf{u}; \theta)$, where Z is the ion charge state and we defined $\gamma^2(u) \equiv 1+u^2$ and $\mu \equiv p_{\|}/p$. A detailed derivation of these equations, although not with the precise boundary conditions used here, may be found in the review by FISCH (1987). The Green's function $\psi(\omega, \mathbf{u}, \tau; \theta)$ may be thought of as a moment of the Green's function that solves equation (1) for ϕ; in particular, it is that moment that gives the radiation response. The physical interpretation of ψ is apparent from the equalities in equation (2); suppose a cloud of electrons is given an initial normalized momentum $\mathbf{u}$, then $\psi(\omega, \mathbf{u}, \tau; \theta)$ gives the probability at time τ of radiation into angle θ at frequency ω of electrons originating in that cloud. The physical interpretation of the quantity $Q(\mathbf{u})$ is the detailed distribution of these electron clouds, produced by the brief, initial, perturbing r.f. pulse.

Equation (3) can be solved in closed form. For notational convenience in writing ψ and R, we suppress now the implied θ dependence, since, in any event, only one observation angle is contemplated. Separate ψ and the initial conditions into Legendre harmonics, ψ_k and I_k, and the resulting equations for the ψ_k may be integrated along characteristics to obtain

$$\psi_k(\omega, u, \tau)=I_k(\omega, \rho)\left[\frac{\rho}{u}\left(\frac{\gamma(u)+1}{\gamma(\rho)+1}\right)\right]^{\alpha_k}, \tag{4}$$

where the characteristic $\rho(u, \tau)$ solves $G(\rho)=G(u)-\tau$, where $G(u)=\int_0^u (x^2/(1+x^2))\,\mathrm{d}x=u-\tan^{-1} u$, and where $\alpha_k \equiv (1+Z)k(k+1)/2$. For u nonrelativistic, $\rho \rightarrow (u^3-3\tau)^{1/3}$, indicating that after a time $\tau=u^3/3$, electrons initially with speed u have slowed down to $u=0$, at which point they no longer radiate. Although equation (3) is derived rigorously for superthermal u only, it may be applied here universally, since radiation from slowed-down less energetic electrons is small compared to initially, and the terms neglected in equation (3), energy diffusion, merely act to thermalize, so that, in any event, the incremental radiation vanishes.

Denoting the Legendre components of $Q(\mathbf{u})$ by $Q_k(u)$, we can write equation (2) in the form

$$R(\omega, \tau)=\sum_k \frac{4\pi}{2k+1} \int_0^{\infty} u^2\,\mathrm{d}u\, \psi_k(\omega, u, \tau) Q_k(u). \tag{5}$$

This compact form, namely equation (5), in which we have been able to write the radiation pattern, is a major result of this work. The task here is to invert equation (5), i.e. solve for the Q_k given R. Note that equation (5) is a Fredholm equation of the first kind, with a 4-D kernel. Neither the existence nor the uniqueness of the solution is guaranteed. In fact, of particular interest is the null space and range of the kernel, which maps Q into R. The null space, spanned by those $Q_k(u)$ which are mapped into $R = 0$, represents the irreducible ambiguity in deducing Q from R. The range of ψ is spanned by those $R(\omega, \tau)$ for which solutions Q exist. Experimental data falling outside the range cast doubt either on the validity of the data or on the applicability of the assumptions that form the basis of the theory.

The numerical inversion of equation (5) is simplified by exploiting the properties of the kernel ψ; in particular, note that equation (5) is of the form

$$R(\omega, \tau) = \sum_k \int_0^\infty \mathrm{d}\xi\, \Psi_k(\omega, \xi - \tau) q_k(\xi), \tag{6}$$

where $\xi = G(u)$. Let $\tilde{\Psi}_k(\omega, s) \leftrightarrow \Psi_k(\omega, \xi)$, $\tilde{R}(\omega, s) \leftrightarrow R(\omega, \tau)$, and $\tilde{q}_k(s) \leftrightarrow q_k(\xi)$ be Fourier transform pairs. Then equation (6) can be put in the form

$$\tilde{R}(\omega, s) = \sum_k \tilde{\Psi}_k(\omega, -s) \tilde{q}_k(s), \tag{7}$$

which simplifies equation (6) to a set, parameterized by s, of Fredholm summations of the first kind, but with 2-D, rather than 4-D, kernels.

3. THE SPECIAL CASE; $\theta = 0$

An important special case is purely perpendicular observation ($\theta = 0$), i.e. observation along a line of sight parallel to the Tokamak major axis. The radiation intensity, say, for ordinary polarization (i.e. with **E** vector parallel to the magnetic field) may then be written as

$$I(\omega, \theta = 0, \mathbf{u}) = \sum_n \frac{e^2 \omega^2}{2\pi c} \frac{u^2 \mu^2}{\gamma^2} J_n^2\left(n \frac{u}{\gamma} (1 - \mu^2)^{1/2}\right) \delta(\omega - n\omega_c/\gamma), \tag{8}$$

where n is the cyclotron harmonic, J_n is the nth Bessel function of the first kind, and $\omega_c = eB/mc$ is the cyclotron frequency of nonrelativistic electrons. Note that the sign of the parallel electron velocity is not resolvable by this measurement, since the Doppler frequency shift is absent when observing perpendicularly. Compensating for this drawback is a fortuitous circumstance: electrons initially at the same energy do not subsequently differ in energy; such electrons remain on the same energy shell and, regardless of their distribution along that shell or the energy of the shell, all radiate at the same frequency, *both initially and subsequently*.

Consequently, we can add up constraints to deduce from nth harmonic radiation the flux of electrons initially at momentum u_0, i.e. to deduce the $Q_k(u_0)$. At time $\tau = \tau_0 = 0$, only the measurement of the frequency $\omega = n\omega_c/\gamma(u_0)$ constrains the $Q_k(u_0)$. In the absence of a d.c. electric field, superthermal electrons slow down in energy, but diffuse only in pitch angle. Hence, at some later time, say $\tau = \tau_1$, electrons initially at momentum u_0 will all be found on the same energy surface, say $u = u_1$, and radiating at the same frequency $\omega = n\omega_c/\gamma(u_1)$. Therefore, to deduce the $Q_k(u_0)$, all available information is found in the projection $R(\omega, \tau) \to r(\tau) \equiv R(\omega(\tau), \tau)$, where, $\omega(\tau) = n\omega_c/\gamma(\rho)$ tracks electrons slowing down on the trajectory $\rho(u_0, \tau)$.

It is often the case that the inversion is further simplified, because the incremental excitation is imposed over a region of energy space narrow enough that the *incremental* radiation at one harmonic is dominant. If this were not true, it would be necessary to account for the possibility that radiation from one energy shell of electrons could be confusable at some instant of time with radiation at a different harmonic from a different shell. However, it turns out that this accounting is often not necessary. For example, for excitations over the relatively wide range 250–500 keV the incremental radiation at the third harmonic can be distinguished, both initially and at later times, from radiation at other harmonics.

Thus, rewriting $\delta(\omega - n\omega_c/\gamma(\rho)) = \delta(u - u(\omega,\tau))\rho\gamma^2(u)/(\omega u^2)$, we substitute equation (4) and equation (8) into equation (5). We take the projection $\omega \to \omega(u,\tau)$ and integrate over the δ-function, which now selects only one u, which enters as a parameter. We obtain

$$r(\tau) \equiv R(\omega(\tau),\tau) = \frac{e^2\omega}{c}\sum_k \gamma^2(u)Q_k(u)\left(\frac{\gamma(u)+1}{u}\right)^{\alpha_k}\left(\frac{\rho}{\gamma(\rho)+1}\right)^{\alpha_k}\frac{\rho^3}{\gamma^2(\rho)}H_{nk}(\rho), \tag{9}$$

where $\rho = \rho(u,\tau)$ and $\omega = \omega(u,\tau)$, as discussed, and

$$H_{nk}(\rho) = \int_{-1}^{1} P_k(\mu)\mu^2 J_n^2\left(n\frac{\rho}{\gamma(\rho)}(1-\mu^2)^{1/2}\right)\mathrm{d}\mu, \tag{10}$$

where P_k is the kth Legendre harmonic. Note that $H_{nk} = 0$ for k odd.

Now equation (9) is a relatively easy inverse problem; it is 1-D with u entering as a parameter. While this equation is readily approached numerically, interestingly, there is a trick available here to invert it analytically. We deduce the Q_k from the behavior of r as $\rho \to 0$, or, equivalently, $\tau \to G(u)$. The exploitable property of equation (9) is that the coefficients of the higher order Q_k have higher order zeros. Since we are interested in the behavior particularly as $\rho \to 0$, we can Taylor expand the Bessel function to put equation (9) into the form

$$\Phi(\rho) = \sum_k \hat{Q}_k F_k(\rho), \tag{11}$$

where, we rewrote $\tau = G(u) - G(\rho)$, and we defined

$$\begin{aligned}
\Phi(\rho) &= r(\tau)/[(e^2\omega\rho/c)(\rho/\gamma(\rho))^{2n+2}],\\
\hat{Q}_k &= \gamma^2(u)((\gamma(u)+1)/u)^{\alpha_k}Q_k(u),\\
F_k(\rho) &= \left(\frac{\rho}{\gamma(\rho)+1}\right)^{\alpha_k}\left(a_{nk}^{(0)} - \frac{n^2}{2(n+1)}\left(\frac{\rho}{\gamma(\rho)}\right)^2 a_{nk}^{(1)} + \cdots\right),
\end{aligned} \tag{12}$$

where the $a_{nk}^{(i)}$ are constants that arise from the small argument expansion of the Bessel function, e.g.

$$a_{nk}^{(0)} = (n^n/2^n n!)^2 \int_{-1}^{1} P_k(\mu)\mu^2(1-\mu^2)^n \, d\mu. \tag{13}$$

Note that as $\rho \to 0$, $F_k(\rho) \sim \rho^{\alpha_k}$. Now suppose that $\Phi(\rho)$ is analytic—one approach to assure this in practice is to smooth the experimental data before processing it. (The question of lost information because of the smoothing is discussed in Section 4.) Then, if we differentiate equation (11) α_k times and evaluate at $\rho = 0$, terms higher than k vanish, and we find (for k even)

$$\hat{Q}_k = \frac{2^{\alpha_k}}{\alpha_k!\, a_{nk}^{(0)}} \frac{\partial^{\alpha_k}}{\partial \rho^{\alpha_k}} \left[\Phi(\rho) - \sum_{j=0}^{k-1} \hat{Q}_j F_j(\rho) \right]_{\rho=0}, \tag{14}$$

which solves for the $\hat{Q}_k$ (and hence for the Q_k) by recursion. (Assumed here is that Z and hence α_k is an integer—a modest generalization of the method handles noninteger Z.)

It is possible that the excitation $\mathbf{S}(\mathbf{p})$ is known to be unidirectional, i.e. $Q(\mu < 0) = 0$. This might be the case, for example, in a unidirectionally launched lower-hybrid wave spectrum for the purpose of current-drive. This auxiliary knowledge may be exploited to particular advantage. Separate $Q(\mu)$ into $Q_e + Q_o$ which are, respectively, its even and odd components. Now $Q(\mu < 0) = 0$ implies that $Q_e = Q_o$, and, hence, for $\mu > 0$, we have $Q = 2Q_e$. Thus, our solution for the even Legendre harmonics, which gives Q_e, represents, in fact, a complete and unambiguous inversion when unidirectional excitation can be assumed. Without this additional constraint, the null space of ψ consists of all odd harmonics; with the undirectional assumption, the null space vanishes. The range of ψ, however, may be shown to be a notably small subset of R-space.

Although we have solved the inverse problem given emissions of the ordinary wave polarization, an entirely analogous procedure may be used to deduce $Q(\mathbf{p})$ from emissions of the extraordinary wave polarization. This gives rise to several interesting questions, including the puzzle of how to approach the formally overspecified inverse problem when, happily, emission data of both polarizations are available.

4. RELATION TO THE ILL-POSED HEAT EQUATION

A question of major importance is the sensitivity of the data inversion to noise. The data inversion here is technically *ill-posed*, and while there are a number of techniques for performing an inversion (see e.g. TIKHONOV and ARSENIN, 1977), in the presence of noise the higher harmonics are expected to be essentially irrecoverable and only a partial reconstruction of Q may be possible. It is possible to make a connection to more standard equations of the ill-posed type, in particular, to a certain posing of the heat equation. To do so, it is helpful to consider an alternative derivation of 1-D projection equation, equation (9). Assume a distribution of electrons $f(\mu, t)$ that corresponds to a shell of electrons initially at some momentum u_0 and deduce $f(\mu, t)$ from the data. The angular distribution relaxes according to an equation of the form

$$\frac{\partial f(\mu, t)}{\partial t} = D(u)\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu}f(\mu, t), \tag{15}$$

where the diffusion coefficient D goes, e.g. as $1/u^3$ in the nonrelativistic limit. As discussed above, as this shell distribution relaxes in μ, it also slows down in energy, so that it radiates at different frequencies. The radiation of a function of time may be put into the form

$$r(t) = \int f(\mu, t)I(\mu, \omega, u)\,\mathrm{d}\mu, \tag{16}$$

where the known weighting function I is the radiated power of a single electron, and its arguments, $u = u(t)$ and $\omega = \omega(t)$ track the speed of the electron shell and the frequency with which they emit radiation. Given the initial condition $f(\mu, t = 0) = f_0(\mu)$, equation (16) may be solved by summing over Legendre harmonics

$$f(\mu, t) = \sum_k a_k P_k(\mu) \exp[-k(k+1)\chi(t)], \tag{17}$$

where $\chi(t) = \int_0^t \mathrm{d}\tau\, D(\tau)$, and where the a_k are the harmonics of the initial condition. Substituting now for $f(\mu, t)$ in equation (3), we can put $r(t)$ into the form

$$r(t) = \sum_k a_k \tilde{I}_k(t) \exp[-k(k+1)\chi(t)], \tag{18}$$

where $\tilde{I}_k(t)$ is the kth Legendre harmonic of the radiation function I, and equation (18) is equivalent to equation (9).

To deduce the a_k from the data $r(t)$ is an ill-posed inversion, in that the higher harmonics will certainly be sensitive to small amounts of noise. A comparison of equation (18) with equation (17) is revealing. Equation (17) represents the solution to a heat equation. One backward posed heat equation would be to deduce, e.g., an initial temperature distribution from later measurements at one point in space for all time; or, equivalently, in equation (17), to deduce from $f(\mu = \mu_0, t)$, the a_k, something that would succeed essentially for only the low k terms. The form of equation (18) would be exactly the same if $\tilde{I}_k(t)$ were independent of time. However, the main features should be the same for the case at hand, where $\tilde{I}_k(t)$ is merely a milder function of time than is the exponential function, $\exp[-(k(k+1)\chi(t))]$. In any event, the difficulty is in deducing the higher harmonics.

Suppose that the data $r(t)$, polluted by noise, are given at some finite number of time points, say M. Then one might expect on the basis of scaling arguments that harmonics such that $n^2 > M/l$ would not be deducible, where l, dependent on the noise and the desired accuracy, is the number of time points that would accurately recover a single harmonic. For example, if $M = 10^4$ and $l = 10$, then $n \sim 30$ is about the largest recoverable harmonic. A numerical experiment on inverting equation (9) was reported recently (FISCH *et al.*, 1987) in which ten harmonics were recovered accurately in the presence of a relatively large amount of simulated noise.

5. SUMMARY

In summary, we have proposed an invasive method for multiplying by orders of magnitude the information obtainable from synchrotron radiation. While certainly there are a number of practical matters to sort out, the key has been to use knowledge of the dynamics of fast electrons. These are the electrons that emit the most radiation and about which we are generally most curious. The inverse problem for an interesting special case has been solved exactly. A more general case may be approached numerically, and further generalization is available by considering more complicated plasma dynamics. Perturbative solutions about our analytical result, say for small d.c. electric field or orbservation slightly off from $\theta = 0$, might be useful extensions.

If the dynamics of fast electrons are not known precisely, then the methods here might be employed to pick between competing models of their dynamics. Here, it is worth bearing in mind that while some of the assumptions made here appear quite stringent, the amount of information demanded is actually relatively modest compared to the information content of our solution. For example, although we require knowledge of the ion charge state to the extent that that parameter affects the time-dependence of the radiation response, this demand is only one number, whereas, a function with two arguments, the divergence of the initial perturbative flux, is deduced. In the event that *a priori* partial information were available with respect to the initial flux, it might be fruitful to speculate on a reverse problem, where inferences might be drawn concerning macroscopic plasma parameters, such as the ion charge state or the current profile, which are difficult to deduce through other means.

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