Diagnostic applications of transient synchrotron radiation in tokamak plasmas*

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Transient radiation, resulting from a brief, deliberate perturbation of the velocity distribution of superthermal tokamak electrons, can be more informative than the steady background radiation that is present in the absence of the perturbation. It is possible to define a number of interesting inverse problems that exploit the two-dimensional frequency-time data of the transient radiation signal.

I. INTRODUCTION

The radiation emanating from the interior of a hot tokamak plasma is a major source of information concerning the details of the hot interior. One can relate, for example, the synchrotron radiation intensity at different frequencies to the interior temperature at different magnetic field strengths. One might also envision that transient radiation patterns that accompany relaxation of the plasma after a brief, deliberate perturbation might be even more informative. After all, the *incremental* radiation signal available, even from only one viewing angle, is a two-dimensional pattern $R(\omega,t)$ in frequency-time space (rather than merely a one-dimensional function of frequency only), where t is the time elapsed following the perturbation.

This work outlines the diagnostic potential of these transient signals, particularly for synchrotron emission data, where microsecond resolution is available. The deliberate perturbation is necessary, because, although any frequent measurement of the radiation records a great deal of data, unless something is liable to change on that time scale, that data is redundant. It would not be informative, for example, to measure temperature (which governs the background radiation) every 50 μsec, were the temperature already known to change only on the time scale of a second. By producing a transient signal, however, the time measurements are endowed with informative potential.

The spontaneous emission per unit volume of plasma at frequency ω and into angle θ with respect to the magnetic field is denoted by $R_{\rm tot}(\omega,t;\theta)$. Contributing to this radiation is $R_{\rm back}(\omega;\theta)$, the relatively constant emission of thermal background electrons. However, of interest here is the incremental or transient signal, $R(\omega,t;\theta) = R_{\rm tot}(\omega,t;\theta) - R_{\rm back}(\omega;\theta)$, that results from the invasive, brief heating that we refer to as the probe heating. To be specific, separate the electron distribution function f into $f = f_{\rm M}(1+\phi_B+\phi)$, where $f_{\rm M}$ is a Maxwellian distribution, ϕ_B describes the relatively constant deviation from Maxwellian of the background distribution, and ϕ describes the time-dependent distribution specifically associated with the probe heating. The radiation of interest is then

$$R(\omega,t;\theta) = \int d^3p\, f_{\rm M}\phi(\mathbf{p},t)\eta(\omega,\mathbf{p};\theta), \tag{1}$$

where η is the radiation power at frequency ω, due to a single electron at momentum $\mathbf{p}$, which is radiated into angle θ. (The tokamak is observed in the vertical plane that includes the tangent to the magnetic field $\mathbf{B}$, so the strength of $\mathbf{B}$ may be assumed constant and known, and θ measures the angular deviation from purely perpendicular observation of the magnetic field.) The radiation η can be either synchrotron emission at the ordinary or extraordinary wave polarization or bremsstrahlung emission.

What makes this an interesting integral equation is that the dynamics of the fast electrons are thought to be well founded, so that the function $\phi(\mathbf{p},t)$, formally an arbitrary function of two velocity-space variables and time, is very much further constrained by the dynamic equations that evolve ϕ. Moreover, the parameters that affect these dynamics are relatively few in number; for example, the dynamics of the fast electrons is entirely insensitive to the temperatures of the background species and to rapid fluctuations of the background densities. The challenge then is to use the data R to learn about ϕ or the parameters governing the evolution of ϕ or η.

Suppose the incremental radiation R is detected directly, as might be the case for an optically thin plasma. A number of interesting inverse problems can then be defined.

(1) Invert Eq. (1) to find the two-dimensional initial perturbation $\phi(\mathbf{p},t=0)$ as a function of the two-dimensional radiation response $R(\omega,t;\theta)$. The radiation function $\eta(\omega,\mathbf{p};\theta)$ is, of course, a known kernel if its parametric dependencies are known. Here, considered as known through other measurements, are all plasma parameters that govern either the radiation response (such as the direction of the magnetic field $\mathbf{B}$), or the parameters that govern the dynamic response (such as the effective ion charge state $Z_{\rm eff}$).

(2) Consider as unknown a subset of the parametric dependencies of either the radiation function $\eta(\omega,\mathbf{p};\theta)$ (such as the magnetic field $\mathbf{B}$), or the parameters that govern the radiation response (such as $Z_{\rm eff}$), or the parameters that describe the initial perturbation $\phi(\mathbf{p},t=0)$. Then, determine these parametric dependencies by examining those which most closely give the two-dimensional radiation re-

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sponse $R(\omega,t;\theta)$. Assumed here, as in problem (1), is that the dynamics of the plasma are not in question.

(3) Here, it remains open whether the data are, in fact, consistent with the assumed dynamics; for example, one might question the transport mechanism of the fast electrons. Consider as unknown not only some subset of the plasma parameters, but also the precise dependencies of certain effects such as transport.

Formally, problem (1) demands the most complete information, a full two-dimensional function. Interestingly, for perpendicular observation ($\theta = 0$), there is an analytic solution available to this inverse problem. However, in the presence of noise, the description of $\phi(\mathbf{p},t=0)$ on the basis of $R(\omega,t;\theta)$ will be coarse.

Problem (2) is of great interest because certain plasma parameters, difficult to measure otherwise, might be deduced. Of particular interest is the dc parallel electric field E, something entirely unavailable otherwise. Typically less than a volt/meter in a tokamak, this field is far too small to be inferred through atomic phenomena, and cannot be measured directly by probes because the plasma is too hot. Its effect is manifest, however, in the dynamics of superthermal electrons, exactly those that synchrotron radiate most profusely.

In problems (2) and (3) a direct inversion of the data is not contemplated; rather one employs the two-dimensional frequency-time data, possibly an array of 10^3 observations, to inform on only a handful of unknown parameters. That only several quantities need be determined makes the problem approachable in terms of the adequacy of the data. The challenge is to find extremely efficient algorithms for processing the data.

Common to each of the inverse problems is the exploitation of a fortuitous separation of time scales $1/\omega \ll \tau_{\rm det} \ll \tau_c \ll \tau_{\rm par}$. From the first inequality we have that the radiation frequency ω (~ 100 GHz) is sufficiently characterized on the instrumental detection time scale of $\tau_{\rm det}$, which can be 50 μsec. The time history of the radiation response is well characterized since there are many detector observations in a superthermal electron slowing down time, τ_c (typically 10–100 msec), yet the parameters themselves change on the longer time scale, $\tau_{\rm par}$ (typically 1 sec), so that their values may be treated as constant during the transient analysis. Of course, there is the opportunity, by repeating the probe, to average the results of several transient analyses.

The use of synchrotron emission to deduce plasma properties is, of course, an established diagnostic for the electron temperature and recently there have been attempts to uncover further details of the electron momentum distribution function f.[1–7] A one-dimensional f was deduced elegantly in a relativistic electron ring geometry.[8] In these studies, the deduction of details of the electron distribution function was based on the synchrotron emission from the entire distribution of electrons; consequently, only one-dimensional data (in frequency) could be used to constrain f. Other studies have recognized some utility in transient radiation.[9,10]

In this work, the possibilities in diagnosing the tokamak plasma using the deliberately produced transient emissions[11–14] are reviewed. In Sec. II, the derivation of the Green's function solution for the radiation response is outlined, and helpful scale-invariant properties of the solution are noted. In Sec. III, certain practical difficulties of observing the radiation are considered, particularly when the plasma is not quite optically thin. Noteworthy is that the transient absorption properties of the plasma can be calculated using the same very efficient algorithms that make tractable the calculation of the emission properties. In Sec. IV, a framework for extracting information from the data is formulated and results of previous work in the first two inverse problems defined above are reviewed; the third type is not considered here. Two themes emerge here: first, that the transient response indeed informs on plasma parameters, and, moreover, that the parameters of interest can be deduced almost *orthogonally*, i.e., ignorance or even misinformation concerning some parameters does not impair significantly the inference of other parameters; and second, that fast algorithms are available for processing the data quickly, which is an important requirement in searching a large dimensional space.

II. RADIATION RESPONSE

Constraining the radiation pattern $R(\omega,\tau;\theta)$ is that the perturbation ϕ is governed by the linearized Fokker–Planck equation

$$f_{\rm M}\frac{\partial\phi}{\partial t} + q\mathbf{E}\cdot\nabla_{\mathbf{p}} f_{\rm M}\phi - C(\phi) = 0, \tag{2}$$

where C is a collision term. It is assumed here that the perturbation ϕ is small in terms of its contribution to the collision integral, so that the Fokker–Planck equation may be linearized. This is an excellent approximation, since the perturbation involves only a small number of electrons, even if they are energetic enough to dominate the radiation. The initial condition on ϕ, which is the result of the probe heating, is taken to be $f_{\rm M}\phi = -Q(\mathbf{p}/mc)/(mc)^3$, where m is the electron mass, c, the speed of light, is introduced for later normalization, and Q is the normalized initial deviation from background caused by the probe heating. For example, were the probe to consist of an impulse of a narrow spectrum of high-phase-velocity lower-hybrid waves, then $Q(\mathbf{p}/mc)$ would be finite in a narrow range of superthermal $\mathbf{p}$. Using normalized momentum, $\mathbf{u} = \mathbf{p}/mc$, and normalized time, $\tau = \nu_c t$, with collision frequency $\nu_c = nq^4 \log \Lambda/4\pi m^2\epsilon_0^2 c^3$, one can write the incremental radiation associated with the initial condition on $f_{\rm M}\phi$ as

$$R(\omega,\tau;\theta) = \int d^3p\, f_{\rm M}\phi(\mathbf{p},\tau)\eta(\omega,\mathbf{p};\theta)$$
$$= \int d^3u\, \psi(\omega,\mathbf{u},\tau;\theta)Q(\mathbf{u}), \tag{3}$$

where the second equality above recognizes that a large savings in effort is possible by defining a Green's function ψ for the radiation response.

Suppose the perturbation ϕ is concentrated at high velocities. A property of electrons on the tail of the distribution function, superthermal but not runaways, is that energy diffusion by collisions is ignorable compared to energy loss.

This makes possible enormous analytic progress in solving the relativistic Fokker–Planck adjoint equation for ψ, the Green's function for the radiation response. In the high-velocity limit, and in terms of the normalized variables $\tau = \nu_c t$ and $\mathscr{E} = qE/mc\nu_c$, the adjoint equation is[15]

$$\frac{\partial \psi}{\partial \tau} - \mathscr{E}\frac{\partial \psi}{\partial u_\parallel} + \frac{1}{u^3}\left(\gamma^2 u \frac{\partial \psi}{\partial u} - \gamma \frac{1+Z_{\rm eff}}{2} \times \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu}\psi\right) = 0, \quad (4)$$

where $\gamma^2 \equiv u^2 + 1$, $Z_{\rm eff}$ is the effective ion charge state, and Eq. (4) is to be solved with the initial condition $\psi(\omega,\mathbf{u};\theta,\tau = 0) = \eta(\omega,\mathbf{u};\theta)$.

To solve Eq. (4), separate ψ and the initial conditions into Legendre harmonics and expand in the electric field, $\psi_k(u,\tau) = \psi_k^{(0)} + \mathscr{E}\psi_k^1 + \mathscr{E}^2\psi_k^{(2)} + \cdots$. The equation for $\psi_k^{(0)}$ can then be integrated along characteristics to obtain the analytic solution[11]

$$\psi_k^{(0)} = \eta_k(x)\frac{\{[1+\gamma(u)]/u\}^{\alpha_k}}{\{[1+\gamma(x)]/x\}^{\alpha_k}}, \quad (5)$$

where the characteristic function $x(\tau,u)$ can be written as $x = g^{-1}[g(u) - \tau]$, with $g(u) \equiv u - \tan^{-1} u$; the inverse function, g^{-1}, is defined such that $g^{-1}[g(u)] = 1$. The equation governing $\psi_k^{(1)}$, to be solved with homogeneous initial conditions, is driven by the kth Legendre harmonic of $\partial\psi^{(0)}/\partial u_\parallel$. It turns out that this inhomogeneous term can be simplified enormously so that $\psi_k^{(1)}$ can be put into an efficient closed form.[14]

The radiation patterns will differ somewhat depending upon the polarization observed. The radiation intensity, for ordinary polarization (i.e., with the **E** vector parallel to the magnetic field), may be written as[16]

$$\eta^O(\omega,\theta,\mathbf{u}) = \sum_n \frac{e^2\omega^2}{2\pi c\lambda^2}\left(\frac{\sin\theta - u\mu/\gamma}{\cos\theta}\right)^2 \times J_n^2\left(n\frac{u}{\gamma}(1-\mu^2)^{1/2}\frac{\cos\theta}{\lambda}\right) \times \delta\left(\omega - \frac{n\omega_c}{\gamma\lambda}\right), \quad (6)$$

where n is the cyclotron harmonic, J_n is the nth Bessel function of the first kind, $\omega_c = eB/mc$ is the cyclotron frequency of nonrelativistic electrons, and $\lambda = 1 - u\mu\sin\theta/\gamma$ is the extent of the Doppler shift through viewing the radiation at angle θ. The radiation intensity at the extraordinary polarization may be written as

$$\eta^X(\omega,\theta,\mathbf{u}) = \sum_n \frac{e^2\omega^2}{2\pi c\lambda^2}\left(\frac{u}{\gamma}\right)^2(1-\mu^2) \times J_n'^2\left(n\frac{u}{\gamma}(1-\mu^2)^{1/2}\frac{\cos\theta}{\lambda}\right) \times \delta\left(\omega - \frac{n\omega_c}{\gamma\lambda}\right), \quad (7)$$

where J_n' is the derivative of the nth Bessel function of the first kind. Note that while in both instances of polarization the radiation vanishes for electrons with only parallel energy ($\mu^2 = 1$), the radiation at the extraordinary polarization is maximized when the electron has purely perpendicular motion ($\mu = 0$), whereas for the ordinary polarization, the radiation intensity is maximized for intermediate pitch angle ($0 < \mu^2 < 1$).

The comparison of many possible radiation responses to data is facilitated by fast algorithms. The Green's function makes efficient the simultaneous consideration of many initial perturbations $Q(\mathbf{p})$. Moreover, Eqs. (2) and (3) admit several scale-invariant transformations of the radiation response $R(\omega,t)$. For synchrotron radiation, there are three such transformations: having solved for $R(\omega,t;\Theta)$, where Θ is a set of parametric dependencies that includes the magnetic field amplitude B, electric field E, the density n, and the perturbation amplitude A, we also have for any constants a_1, a_2, and a_3,

$$R(\omega,t;a_1B,a_2A,a_3n,E) = a_1a_2R\left(\frac{\omega}{a_1},\frac{t}{a_3};B,A,n,\frac{E}{a_3}\right). \quad (8)$$

Further simplification of Eq. (8) is made possible by choosing to heat those electrons for which it is permissible to linearize $R = R_0 + ER_1$. These would be tail electrons, but they are not nearly so fast as to be runaway electrons that are strongly affected by the dc field.

A similar Green's function can be found for bremsstrahlung emission, which also enjoys certain scale-invariant properties. However, the time resolution for detecting bremsstrahlung is significantly greater in tokamak applications than for detecting synchrotron emission.

III. OBSERVING THE RADIATION

A number of practical issues must be resolved before we can associate the incremental radiation observed with the incremental radiation produced. First, the plasma must be optically thin to the produced radiation; otherwise, this radiation is trapped inside the tokamak and is useless for the purposes of informing on the tokamak interior. Second, it is necessary to take into account any small changes in the emitted radiation as it leaves the tokamak, occurring either because the ray path is affected by the plasma or because there is mild damping or growth of the wave amplitude. Third, the radiation must be intense enough to be observed, yet not so intense that the radiation complicates the electron dynamics. These conditions give a parameter range for the utility of this diagnostic, at least as it might be most easily practiced.

In traversing the plasma, the intensity of radiation, $I(\omega,s,t)$, evolves spatially by[17]

$$n_r^2(s)\frac{\partial}{\partial s}\frac{I}{n_r^2(s)} = R_{\rm tot}(\omega,t,s) - \alpha(\omega,t,s)I(\omega,t,s), \quad (9)$$

where $n_r(s)$ is the ray refractive index at distance s along the ray path,

$$R_{\rm tot} = \int d^3p\,\eta(\omega,\mathbf{p})f(\mathbf{p},t,s) \quad (10)$$

is the total spontaneous emission, and

$$\alpha = -\frac{8\pi^3 c^2}{n_r^2 \omega^2} \int d^3p\, \eta(\omega,\mathbf{p}) Df \quad (11)$$

is the absorption coefficient with

$$Df \equiv \left[\frac{\epsilon}{c^2 p_\perp} \frac{\partial f}{\partial p_\perp} - n(\theta)\cos(\theta)\left(\frac{p_\parallel}{c p_\perp} \frac{\partial f}{\partial p_\perp} - \frac{1}{c} \frac{\partial f}{\partial p_\parallel} \right) \right]. \quad (12)$$

Now suppose that, in addition to the background distribution function, we introduce a perturbation, i.e., $f \to f_B + \tilde{f}(t)$; we then also have $I \to I_B + \tilde{I}(t)$, $R_{\rm tot} \to R_{\rm back} + R(t)$, and $\alpha \to \alpha_B + \tilde{\alpha}(t)$. Here t enters as a parameter in the decay of the transient signal. The background signal is given by

$$n_r^2 \frac{\partial}{\partial s} \frac{I_B}{n_r^2} = \beta_B(\omega,s) - \alpha_B(\omega,s) I_B(\omega,s), \quad (13)$$

and the perturbed or incremental signal is obtained by subtracting Eq. (13) from Eq. (9) to obtain

$$\begin{aligned} n_r^2 \frac{\partial}{\partial s} \frac{\tilde{I}(\omega,t,s)}{n_r^2} &= R(\omega,t,s) - \alpha_B(\omega,s)\tilde{I}(\omega,t,s) \\ &\quad - \tilde{\alpha}(\omega,t,s) I_B(\omega,s) \\ &\quad - \tilde{\alpha}(\omega,t,s)\tilde{I}(\omega,t,s). \quad (14) \end{aligned}$$

Having calculated efficiently the incremental spontaneous emission $R(\omega,t,s)$ in Sec. II, the question is whether the same can be done for the radiation I that is, in fact, observed at the detector. By analogy to Eq. (3), the incremental absorption at each magnetic surface can be written as

$$\begin{aligned} \tilde{\alpha}(\omega,t) &= -\frac{8\pi^3 c^2}{n_r^2 \omega^2} \int d^3p\, \eta(\omega,\mathbf{p}) D\tilde{f} \\ &= -\frac{8\pi^3 c^2}{n_r^2 \omega^2} \int d^3p\, \psi(\omega,\mathbf{p},t) DQ(\mathbf{p}), \quad (15) \end{aligned}$$

and so can be calculated easily using the same Green's function ψ that facilitates the calculation of the incremental emission $R(\omega,t)$. Moreover, the incremental absorption $\tilde{\alpha}(\omega,t)$ obeys similar scale-invariant transformations. Define $\tilde{\xi} \equiv n_r^2 \tilde{\alpha}$. If we then have solved for

$$\begin{aligned} \tilde{\xi}(\omega,t;B,A,n,E) &= \tilde{\xi}_0(\omega,t;B,A,n) \\ &\quad + E\tilde{\xi}_1(\omega,t;B,A,n), \quad (16) \end{aligned}$$

then we also have for any constants a_1, a_2, a_3, and a_4,

$$\begin{aligned} &\tilde{\xi}(\omega,t;a_1 B,a_2 A,a_3 n,a_4 E) \\ &= \frac{a_2}{a_1} \left[\tilde{\xi}_0\left(\frac{\omega}{a_1}, \frac{t}{a_3}; B,A,n\right) \right. \\ &\quad \left. + \frac{a_4}{a_3} E\tilde{\xi}_1\left(\frac{\omega}{a_1}, \frac{t}{a_3}; B,A,n\right) \right]. \quad (17) \end{aligned}$$

Similar transformations exist for the refractive index, which, in principle, would simplify obtaining $\tilde{\alpha}$ from $\tilde{\xi}$, but these transformations are computationally far less critical, and, in any event, it is envisioned that $n_r \approx 1$ for the frequencies of interest here.

Thus both the incremental damping and the incremental absorption are susceptible to very efficient calculation. The background absorption and background radiation need only be calculated once, and hence, all terms on the right-hand side of Eq. (14) are readily available, so that the incremental radiation $\tilde{I}(t)$ observable at the detector can be calculated efficiently. Care must be taken, however, to evaluate the ray paths for each frequency. Note that while Eq. (14) contains terms nonlinear in the perturbation $\tilde{f}$, it is, in fact, a linear equation for the perturbed radiation $\tilde{I}$, which can be written directly as

$$\begin{aligned} \tilde{I}(\omega,t,s) &= n_r^2(s) e^{-H(s)} \int_0^s ds'\, n_r^{-2}(s') \\ &\quad \times e^{H(s')} [R(\omega,t,s') - \tilde{\alpha}(\omega,t,s') I_B(\omega,s')], \quad (18) \end{aligned}$$

where

$$H(\omega,t,s) \equiv \int_0^s ds' [\alpha_B(\omega,s') + \tilde{\alpha}(\omega,t,s')]. \quad (19)$$

The parameter regime of interest is where $H(\omega,t,L) \ll 1$, where L is the path length traversed by the radiation to the detector, typically about the minor radius in a tokamak. When this condition is not satisfied, the transient radiation does not carry information about conditions in the plasma interior. In practice, in present-day tokamaks, the extraordinary mode might be employed. This is the more highly emitted and more highly absorbed radiation, but, in reactors, only radiation of the ordinary polarization, observed between cyclotron harmonics, escapes freely from the plasma interior.

The incremental radiation need not exceed the background radiation; it need only be distinguishable from it. Nonetheless, the signal-to-noise ratio is clearly enhanced for large transient signals. There are two limitations to the signal amplitude: for practical purposes, the power to create the signal is limited, and, for simplicity, the present analysis assumes that the dynamics of the perturbed distribution is independent of the radiation.

A measure of the radiation recoil is the ratio of instantaneous power loss of a single fast electron owing to spontaneous synchrotron emission, $P_{\rm syn}$, to the instantaneous power loss resulting from collisions, $P_{\rm col}$, which is approximately[14]

$$\frac{P_{\rm syn}}{P_{\rm col}} = \frac{2 \ln \Lambda}{3\gamma} \frac{\omega_c^2}{\omega_p^2} (1-\mu^2)\left(\frac{p}{mc}\right)^3. \quad (20)$$

The Green's function ψ derived here is valid only for $P_{\rm syn}/P_{\rm col}$ small, where collisions, rather than radiation recoil, dominate the electron dynamics. For electrons at about 600 keV, corresponding to electrons in the tail of the distribution function in a 25 keV reactor plasma, $P_{\rm syn}/P_{\rm col} \sim 0.1$. If desired, a more accurate Green's function can be derived by accounting for the recoil through an expansion in $P_{\rm syn}/P_{\rm col}$. While $P_{\rm syn}/P_{\rm col}$ vanishingly small may be mathematically expedient, in practice, a larger value is desirable. After all, this ratio is also a measure of efficiency, giving the fraction of the absorbed probe power that is reflected back as incremental synchrotron radiation data, and a high probe power is more costly.

If the recoil effect of spontaneous emission is small, then one can bound as well the effect of stimulated emission or absorption of radiation. Let $\tilde{I}_{\rm BB}(\omega,t) = R(\omega,t)/\tilde{\alpha}(\omega,t)$ be the blackbody radiation associated with the incremental distribution, were that distribution in thermal equilibrium.

This radiation is presumably far greater in the frequency regime of interest than is the blackbody radiation associated with the background electrons, at least where the perturbation exists. Then, so long as the radiation traversing the plasma obeys the relatively mild restriction $I \ll \tilde{I}_{BB}$, the effect of electrons absorbing the radiation is no greater than the recoil from electrons emitting radiation.

IV. INVERSE PROBLEMS

In the presence of data, the relative probability of certain sets of plasma parameters becomes significantly enhanced; the application of the constraints imposed by the data is the second type of inverse problem proposed in the Introduction. Consider, for example, an optically thin plasma for which experimental measurements are of the form $R_x(\omega,t) = R(\omega,t) + \widetilde{R}(\omega,t)$, where the extraneous signal $\widetilde{R}(\omega,\tau)$ is Gaussian noise, uncorrelated both in frequency and time, with $\langle \widetilde{R} \rangle = 0$ and $\langle \widetilde{R}^2 \rangle = \sigma^2$. Given a set of plasma parameters $\{\Theta\}$ and a noise level σ, the probability, $P(R_x|\Theta;\sigma)$, of any particular data set R_x can then be calculated. Through Bayes's theorem, $P(\Theta|R_x;\sigma) = P(R_x|\Theta;\sigma)P(\Theta)/P(R_x)$, the data can be used to refine the probability of any parameter set $\{\Theta\}$ over the *a priori* distribution $P(\Theta)$ for that parameter set.

What is of interest here is the probability distribution of the plasma parameter set $\{\Theta\}$, given that the data were obtained in the presence of noise σ and generated with the specific plasma parameter set $\{\Theta_p\}$, which can be written as

$$P(\Theta|\Theta_p;\sigma) = \sum_{\{R_x\}} P(\Theta|R_x;\sigma)P(R_x|\Theta_p;\sigma)$$
$$= \lim_{N_R\to\infty} \frac{1}{N_R} \sum_{j=i}^{N_R} P(\Theta|R_x^{(j)};\sigma), \qquad (21)$$

where, in the first equality, the summation over all possible data sets $\{R_x\}$ is both infeasible and, in practice, unnecessary; the second equality obtains, since, by construction, $P(\Theta|R_x;\sigma)$ is sampled with probability $P(R_x|\Theta_p;\sigma)$. Generally $N_R \sim 80$ suffices to approximate $P(\Theta|\Theta_p;\sigma)$. Of course, the fast algorithms for generating $R(\omega,t)$ are indispensable since R must be obtained for each competitive data set.

The peakedness of the probability distribution $P(\Theta|\Theta_p;\sigma)$ is an indication of the sensitivity of data to the plasma parameters and the worth of the constraints that the data imposes. This distribution has been examined numerically, and it has been found[12–14] that, in fact, the *a priori* probabilities $P(\Theta)$ can be very much improved upon, even in the presence of considerable noise and when several parameters are simultaneously unknown.

Remarkable in this analysis is the number of competing data sets that can be easily numerically simulated and considered. In one example,[14] radiation emanating from the core and periphery of a tokamak, viewed by one detector, is simulated with over 1.4×10^6 competing parameter sets. Here, in a coarse model, the two radiating regimes have, respectively, known densities n_c and n_p, the same known ion charge state, but unknown electric fields $\mathscr{E}_c$ and $\mathscr{E}_p$, that are to be determined. Treated as unknown here is how large a perturbation, A, is created at each of the two points. Likewise treated as unknown are the current profile and, hence, the viewing angle with respect to the magnetic field at each of these points. It is further supposed that the location in velocity space of the absorbed probing radiation is also known and the same at each point, possibly because of a resonance condition. Thus the detector sums

$$R(\omega,t) = R(\omega,t\,|n_c,Z_{\rm eff};\mathscr{E}_c,A_c,\theta_c)$$
$$+ R(\omega,t\,|n_p,Z_{\rm eff};\mathscr{E}_p,A_p,\theta_p), \qquad (22)$$

where c labels parameters at the plasma center and p labels parameters at a peripheral point.

The challenge is to find the probability distribution over all 1.4×10^6 competing sets of parameters in the six-dimensional space $(\mathscr{E}_c,A_c,\theta_c,\mathscr{E}_p,A_p,\theta_p)$, given a very crude *a priori* probability distribution and the data $R(\omega,t)$. This example might be relevant in diagnosing current drive experiments in which a loop voltage on axis is not yet relaxed via magnetic diffusion. The problem, nontrivial at first sight, turns out to require minimal numerical effort, in view of the fast algorithms for calculating R and the scale-invariant transformations available in Eq. (8); in fact, only 14 different radiation patterns (at different θ) were actually calculated directly.

Of course, here, were $n_c = n_p$, there would be no distinguishing the radiation source. However, even a 10% variation in density turns out to be exploitable. In practice, purely experimental noise can be kept much lower and a larger differential in density makes this discrimination much easier.

The detailed results, given elsewhere,[14] exhibit a peakedness to the joint probability distribution sufficient to characterize the parameters of interest. Moreover, the sharpness of marginal joint probability distributions (say, for the electric fields) is not very much affected by knowledge of the other parameters (such as the viewing angles), indicating a certain orthogonal dependence of the radiation pattern on different plasma parameters. These findings are consistent with more extensive studies[12] that examined the relative orthogonality of parameters such as $Z_{\rm eff}$, θ, and parameters characterizing the probe spectrum.

There is an interesting inverse problem of the first type that can be posed for the special case of purely perpendicular observation ($\theta = 0$, i.e., observation along a line of sight parallel to the tokamak major axis) and vanishing loop voltage. Suppose that all the relevant plasma parameters are known, but that the velocity-space details of the incremental perturbation are to be found. Generally, this would be a two-dimensional inverse problem in which $Q(\mathbf{u})$ is found by inverting Eq. (3), assuming as known the four-dimensional kernel ψ, but in this case, $Q(\mathbf{u})$ can be found by solving a set of only one-dimensional inversions.

Denoting the Legendre components of $Q(\mathbf{u})$ by $Q_k(u)$, Eq. (3) can be put in the form

$$R(\omega,\tau;\theta) = \sum_k \frac{4\pi}{2k+1} \int_0^\infty u^2\,du\,\psi_k(\omega,u,\tau;\theta)Q_k(u). \quad (23)$$

For $\theta = 0$, the sign of the initial parallel electron velocity cannot be resolved by measuring $R(\omega,\tau)$ (since the Doppler frequency shift is absent when observing perpendicularly), but compensating for this drawback is a fortuitous circumstance: electrons initially at the same energy do not subsequently differ in energy; such electrons remain on the same energy shell and, regardless of their distribution along that shell or the energy of the shell, all radiate at the same frequency, *both initially and subsequently*.

Consequently, to deduce the $Q_k(u_0)$, constraints from nth harmonic radiation at a later time can be applied without ambiguity to electrons initially at momentum u_0. At time $\tau = \tau_0 = 0$, only the measurement of the frequency $\omega = n\omega_c/\gamma(u_0)$ constrains the $Q_k(u_0)$. In the absence of a dc electric field, superthermal electrons slow down in energy, but diffuse only in pitch angle. Hence, at some later time, say $\tau = \tau_1$, electrons initially at momentum u_0 will all be found on the same energy surface, say $u = u_1$, and radiating at the same frequency $\omega = n\omega_c/\gamma(u_1)$. Therefore, to deduce the $Q_k(u_0)$, all available information is found in the projection $R(\omega,\tau) \to r(\tau) \equiv R[\omega(\tau),\tau]$, where $\omega(\tau) = n\omega_c/\gamma(x)$ tracks electrons slowing down on the trajectory $x(\tau,u_0)$. The result is that the $Q_k(u)$ are determined from only one-dimensional inverse problems of the form

$$r(\tau) \equiv R\,[\omega(\tau),\tau] = \sum_k W_k(\tau,u)Q_k(u), \qquad (24)$$

where u enters only as a parameter.[11]

V. SUMMARY

In summary, the relatively modest diagnostic system that we propose includes both the brief, probing rf signal that leads to the incremental synchrotron signal, and an array of frequency detectors with submillisecond time resolution. In this manner, a great deal of data is focused on but a few choice parameters, including the otherwise unmeasurable dc loop voltage. Powerful analytic tools make feasible a numerical analysis of data that would otherwise be unthinkable, allowing a very large parameter space to be scanned efficiently.

The type of diagnostic considered here is somewhat unusual in that what is recognized is that the data obtained may be extremely informative, but do not necessarily measure only one particular parameter. Rather, the data, which may number 10^2–10^4 separate observations, constrain the values that certain parameters may jointly take. Hence the solution space is a large dimensional space. Fortunately, only a few parameters affect the data, and, actually, do so somewhat orthogonally, so the solution space, large as it is, is not untractable.

Extensions of the work reported here include the third type of inverse problem, in which certain laws regarding the electron dynamics may be questioned in much the same spirit as are the parameters that govern the dynamics. Also, application of the technique to specific toroidal geometries is being pursued, and it is hoped that information that is both novel and useful can soon become available in present-day tokamaks.

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