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Alfvén wave tomography for cold magnetohydrodynamic plasmas

I. Y. Dodin and N. J. Fisch
Princeton Plasma Physics Laboratory, P.O. Box 451, Princeton, New Jersey 08543

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Alfvén wave propagation in slightly nonuniform cold plasmas is studied by means of ideal magnetohydrodynamics (MHD) nonlinear equations. The evolution of the MHD spectrum is shown to be governed by a matrix linear differential equation with constant coefficients determined by the spectrum of quasistatic plasma density perturbations. The Alfvén waves are shown not to affect the plasma density inhomogeneities, as they scatter off of them. The application of the MHD spectrum evolution equation to the inverse scattering problem allows tomographic measurements of the plasma density profile by scanning the plasma volume with Alfvén radiation. [DOI: 10.1063/1.1448499]

I. INTRODUCTION

The problem of finding the eigenwaves of various plasmas is most easily considered under the strict assumption of homogeneous plasma properties. Such an approach allows significant simplification of dynamic equations leading to relatively simple dispersion relations for the waves propagating in uniform plasmas, further called the partial waves. However, the assumption of plasma homogeneity is often inapplicable to real systems.

Consider a plasma medium, which is adequately described in terms of ideal magnetohydrodynamics (MHD). One of the features of MHD-like plasmas consists of the fact that magnetoactive collisionless plasmas can sustain steady state localized structures, maintaining total (kinetic plus magnetic) pressure balance with the ambient media. Such structures are often called magnetic bubbles (for density depressions) or magnetic bottles (for enhanced density)[1] because of the disturbance of the magnetic field caused by the diamagnetic effect of their localized plasma density changes. The spatial distribution of magnetic bubbles (or bottles) represents a nonuniform plasma pressure profile, whose spatial harmonics can be treated as static waves described by the dispersion relation $\omega(\mathbf{k})=0$. From this point of view, the problem of finding the eigenmodes of ideal nonuniform MHD plasma can be considered in terms of MHD partial waves, scattering on plasma inhomogeneities (magnetic bubbles or bottles), as long as the partial waves remain well defined, i.e., the plasma pressure inhomogeneity remain small enough. Therefore, the lowest-order effect coming into play when the plasma density inhomogeneity is taken into account can be expected to be the coupling of conventional Alfvén waves, which represent the partial waves of MHD plasmas, on the quasistatic waves of plasma density perturbations described previously.

In this paper, we consider the evolution of Alfvén waves' spectrum specifically due to their scattering on quasistatic perturbations of plasma density in a cold slightly nonuniform ideal MHD plasma. (The term ''slightly nonuniform'' refers to the amplitude of plasma density perturbations, but does not limit the ratio of the wavelength and the characteristic spatial scale of the density perturbations considered, which is allowed to be of the order of unity.) This problem is a three-wave interaction problem, where one of the waves has zero frequency and nonzero wave vector and the two other waves are conventional Alfvén waves, whose dispersion relations are derived under the assumption of uniform plasma medium. Various nonlinear and three-wave interactions in magnetohydrodynamics have been intensively studied; for detailed review, see, e.g., Ref. 2, and references therein. In particular, the effect of Bragg scattering of MHD waves on spatial lattices of plasma structure with given wave numbers has been experimentally studied in the context of ionospheric irregularities.[3,4]

In the present paper, we find explicit solutions for the problem of partial waves scattering in cold slightly nonuniform plasma. On the other hand, inverting the problem, one can get the full information about the density spatial distribution out of the obtained scattering properties of given plasmas. In principle, these results allow realization of quasilinear Alfvén tomography, i.e., the procedure of obtaining the plasma density profile from measurements of the Alfvén spectrum transformation. We call the proposed tomography quasilinear because despite the nonlinearity of the MHD equations used, the final equation for the Alfvén spectrum is shown to be linear, which makes its solving procedure equivalent to the standard tomography problem solution.[5,6] Tomographic measurements are currently being used, for example, in magnetic fusion plasma devices with application of x-ray waves for measuring the electron temperature in hot core of tokamak plasma[7–9] and in ionospheric electron density measurements in radio-frequency range.[10–12] We demonstrate that in cold slightly nonuniform plasmas, Alfvén radiation can also be used for similar purposes broadening the spectral limitations of plasma tomography.

The paper is arranged as follows: In Sec. II, we introduce the nonlinear MHD equations and define the types of partial waves and ways of their interaction. In Sec. III, we derive the Hamiltonian evolution equation for Alfvén waves spectrum. We prove that in the cold plasma limit, the Hamiltonian remains constant (under the assumption of small waves amplitudes), and is determined by quasistatic plasma density perturbations not changing in time. The solution of the Alfvén tomography problem is discussed in Sec. IV. In Sec. V, we summarize the main results of our work.

II. MOTION EQUATIONS AND PARTIAL WAVES OF MHD PLASMA

Let us consider plasma immersed in a static uniform external magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, $\boldsymbol{\nabla} B_0 \equiv 0$, where $\hat{\mathbf{z}}$ is the unit vector along the z axis. Under the assumption of the MHD-like plasma motion, in the cold plasma limit, when the plasma kinetic pressure p is negligible compared with the pressure of the external magnetic field $B_0^2/8\pi$ ($\beta = 8\pi p/B_0^2 \ll 1$), the full set of plasma motion equations can be written in the following form:

$$\partial_t \mathbf{U} - V_\mathrm{A}^2(\nabla\times\mathbf{b})\times\hat{\mathbf{z}} = \mathbf{N_U}, \tag{1}$$

$$\partial_t \mathbf{b} - \nabla\times(\mathbf{U}\times\hat{\mathbf{z}}) = \mathbf{N_b}, \tag{2}$$

$$\partial_t \chi + \nabla\cdot\mathbf{U} = N_\chi. \tag{3}$$

Here $\mathbf{U}$ is the plasma flow velocity, $\mathbf{b} = \mathbf{B}_\sim/\mathbf{B}_0$ is the normalized magnetic field perturbation, $\chi = \rho_-/\rho_0$ is the density perturbation normalized on the average uniform plasma density ρ_0, and $V_\mathrm{A}^2 = B_0^2/4\pi\rho_0$ is the squared Alfvén velocity. Nonlinear "forces" written on the right-hand sides of the equations can be expressed as

$$\mathbf{N_U} = -\chi\partial_t\mathbf{U} - (1+\chi)(\mathbf{U}\cdot\nabla)\mathbf{U} + V_\mathrm{A}^2(\nabla\times\mathbf{b})\times\mathbf{b}, \tag{4}$$

$$\mathbf{N_b} = \nabla\times(\mathbf{U}\times\mathbf{b}), \tag{5}$$

$$N_\chi = -\nabla\cdot(\chi\mathbf{U}). \tag{6}$$

For our further purposes, it is more convenient to rewrite Eqs. (1) and (2), and (4) and (5) in a matrix form introducing a vector of the transverse flow velocity $\mathbf{U}_\perp = U_x\hat{\mathbf{x}} + U_y\hat{\mathbf{y}}$:

$$\mathbf{D}\cdot\mathbf{U}_\perp = \mathbf{N}_\perp, \tag{7}$$

where differential operator $\mathbf{D}$ is given by

$$\mathbf{D} = \begin{pmatrix} \partial_{tt}^2 - V_\mathrm{A}^2(\partial_{xx}^2 + \partial_{zz}^2) & -V_\mathrm{A}^2\partial_{xy}^2 \\ -V_\mathrm{A}^2\partial_{xy}^2 & \partial_{tt}^2 - V_\mathrm{A}^2(\partial_{yy}^2 + \partial_{zz}^2) \end{pmatrix} \tag{8}$$

and transverse nonlinear force $\mathbf{N}_\perp$ can be expressed as

$$\mathbf{N}_\perp = \partial_t\mathbf{N}_{\mathbf{U},\perp} + V_\mathrm{A}^2(\nabla\times\mathbf{N_b})\times\hat{\mathbf{z}}. \tag{9}$$

Consider now the case when plasma inhomogeneity is purely oscillatory in time and space. Then, in the linear approximation one can treat Alfvén velocity as constant and since $\mathbf{N}_\perp$ is negligible, the linearized equation (7) represents a wave equation for an anisotropic medium. Looking for the eigenstate of plasma motion governed by (7) in the form of a plane wave

$$\mathbf{U}_\perp = \mathbf{U}_\perp^{(0)}\exp(-i\omega t + i\mathbf{k}_\perp\cdot\mathbf{r}_\perp + ik_{||}z),\ \mathbf{U}_\perp^{(0)} = \mathrm{const} \tag{10}$$

(here $\mathbf{k}_\perp = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$, $\mathbf{r}_\perp = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$, $k_{||} = \mathbf{k}\cdot\hat{\mathbf{z}}$ and $\mathbf{k}$ is the three-dimensional wave vector) one immediately gets the well-known dispersion relations for compressional Alfvén waves (further called the CA modes)

$$\omega_\mathrm{CA}^2(\mathbf{k}) = k^2V_\mathrm{A}^2 \tag{11}$$

and the shear Alfvén waves (SA modes) correspondingly:

$$\omega_\mathrm{SA}^2(\mathbf{k}) = k_{||}^2V_\mathrm{A}^2. \tag{12}$$

The plane wave representation (10) of MHD plasma eigenmodes breaks down as soon as Alfvén velocity becomes location dependent. As long as the inhomogeneity remains smooth $(kL \gg 1, L = \rho_0/|\nabla\rho_0|)$, the Wentzel–Kramers–Brillouin (WKB) theory adequately describes the wave propagation process, and the representation (10) with the wave amplitude $\mathbf{U}_\perp^{(0)}$ slowly changing in space stays a good approximation for the eigenmodes of MHD plasma. No wave scattering is taking place in this case. On the other hand, as soon as the wavelength $2\pi k^{-1}$ becomes comparable with the spatial scale of density inhomogeneity ($kL \sim 1$), the geometrical optics (or WKB approximation) breaks down and different approach becomes needed for describing the wave propagation in nonuniform plasmas.

For the purpose of considering arbitrary values of kL, we note that the modes (11) and (12) do not form a complete set of linear solutions for all MHD plasma perturbations, and static density perturbations should be treated separately from Alfvén waves. Indeed, as soon as one introduces a nonzero plasma temperature (which is still allowed to be infinitely small to satisfy the condition of negligible plasma β), MHD plasma becomes capable of containing steady state localized structures, maintaining the total pressure balance with the ambient medium described by the equilibrium condition

$$\nabla\left(p + \frac{B^2}{8\pi}\right) = \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi}. \tag{13}$$

Such structures are often called magnetic bubbles[1] (or magnetic bottles, depending on whether plasma density is depressed or enhanced inside a structure). Magnetic bubbles are often observed in space,[13–15] created artificially in the Earth ionosphere[3,4] and laboratory conditions[16] and generally represent a certain scientific interest for astrophysics and plasma science.

From (13), it follows that as long as the plasma kinetic pressure remains isotropic, no equilibrium pressure variation along the field lines can be maintained self-consistently, which means that in isotropic plasmas, magnetic bubbles can only be two-dimensional (2D) structures axial symmetric along the static uniform magnetic field B_0. Pressure anisotropy is often present in real plasmas[14] but since its level is usually low, bubbles tend to elongate in the direction of external field,[3,14] so that their longitudinal size is significantly larger than the transverse one. Thus, for simplicity we will assume the bubbles to be completely two-dimensional, so that for small variations of plasma kinetic pressure the equilibrium condition (13) is equivalent to

$$\mathbf{b} = -\hat{\mathbf{z}}\frac{\gamma\beta}{2}\chi, \tag{14}$$

where $\gamma = O(1)$ is introduced through the plasma equation of state $p = p_0(\rho/\rho_0)^\gamma$. For our purposes, it is convenient to think of such 2-D small-amplitude static plasma density perturbations as of static waves (S modes), which dispersion relation is given by

$$\omega_S(\mathbf{k}) = 0, \quad k_{||} = 0. \tag{15}$$

Low-amplitude Alfvén waves (11) and (12) and S modes (15) represent the *partial* waves of MHD plasmas meaning that each of these waves is an eigenmode of a *linear* MHD system only where mode coupling is negligible, but, as soon as the nonlinear drive $\mathbf{N}_\perp$ is taken into account, these waves generally cannot exist independently. Consider a "pumping" Alfvén wave with frequency ω_p and wave vector $\mathbf{k}_p$ propagating in a region of spatially modulated plasma density and let the modulation be purely sinusoidal with the wave vector $\mathbf{k}_S$. The presence of quadratic nonlinearity in the expression for $\mathbf{N}_\perp$ shows that the energy of the pumping wave will be transferred into a scattered wave with the frequency and the wave vector given by

$$\omega_{\mathrm{sc}} = \omega_p, \quad \mathbf{k}_{\mathrm{sc}} = \mathbf{k}_p + \mathbf{k}_\mathbf{S}. \tag{16}$$

Equation (16) represents the conditions of resonant three-wave interaction, or well-known Bragg scattering of pumping Alfvén wave on a spatial plasma structure. The scattered wave, in turn, is also scattered by the lattice with the wave vector $\mathbf{k}_S$ producing a third wave and amplifying the pumping one, etc., so that the amplitudes of the waves will evolve in time. In the following sections, we show the explicit way of finding the scattering properties of arbitrarily modulated plasma density in the case when the amplitudes of these modulations remain small.

The conditions for resonant interaction (16) indicate the possible pairs of Alfvén linear modes, which can couple to the inhomogeneities of plasma density. For scattering of one SA mode into another SA mode, the necessary conditions of coupling

$$k_{||,\mathrm{SA},1} = k_{||,\mathrm{SA},2}, \quad \mathbf{k}_{\perp,\mathrm{SA},1} = \mathbf{k}_{\perp,\mathrm{SA},2} + \mathbf{k}_S \tag{17}$$

show that every shear Alfvén wave is potentially unstable to transformation into another similar wave with the same parallel and different transverse wave numbers. On the contrary, in CA↔SA scattering process, condition (16) combined with the dispersion relations (11) and (12) require the equalities

$$k^2_{||,\mathrm{CA}} + k^2_{\perp,\mathrm{CA}} = k^2_{||,\mathrm{SA}}, \quad k_{||,\mathrm{CA}} = k_{||,\mathrm{SA}}, \quad \mathbf{k}_{\perp,\mathrm{CA}} = \mathbf{k}_{\perp,\mathrm{SA}} + \mathbf{k}_S, \tag{18}$$

which lead to the additional conditions on wave numbers:

$$\mathbf{k}_{\perp,\mathrm{CA}} = 0, \quad \mathbf{k}_{\perp,\mathrm{SA}} = -\mathbf{k}_S. \tag{19}$$

In other words, CA wave with nonzero transverse wave number cannot interact resonantly with SA waves via Bragg scattering on static density perturbations. The third type of wave transformation, namely CA↔CA process, simultaneously requires

$$k_{||,\mathrm{CA},1} = k_{||,\mathrm{CA},2}, \quad \mathbf{k}_{\perp,\mathrm{CA},1} = \mathbf{k}_{\perp,\mathrm{CA},2} + \mathbf{k}_S, \tag{20}$$

and, similarly to SA↔SA process, for arbitrary $\mathbf{k}_S$ (unless $k_S > 2k_{\perp,\mathrm{CA},1}$), there always exists a scattered Alfvén wave meaning that the pumping CA wave is always unstable to Bragg scattering on plasma density perturbations.

III. EVOLUTION OF MHD SPECTRUM IN BRAGG SCATTERING PROCESS

In order to derive the equation for the evolution of Alfvén spectrum in Bragg scattering process, let us represent the transverse plasma flow velocity in the form

$$\mathbf{U} = \int d^3\mathbf{k}[\mathbf{U}_{\mathrm{CA},\,\mathbf{k}}(\mathbf{r},t)e^{-i\omega_{\mathrm{CA}}(\mathbf{k})\,t} + \mathbf{U}_{\mathrm{SA},\,\mathbf{k}}(\mathbf{r},t)e^{-i\omega_{\mathrm{SA}}(\mathbf{k})\,t} + \mathbf{U}_{S,\,\mathbf{k}}(\mathbf{r},t)]e^{i\mathbf{k}\cdot\mathbf{r}} + \mathrm{c.c.}, \tag{21}$$

where the partial modes spectra $\mathbf{U}_{\mathrm{CA}}$, $\mathbf{U}_{\mathrm{SA}}$, and $\mathbf{U}_S$ are generally slow functions of time and space compared with the characteristic frequency of Alfvén waves and the largest wavelength being considered; c.c. stands for the complex conjugate term. Let us then introduce 2-D "polarization" vectors $\boldsymbol{\xi}$ and scalar wave amplitudes $U_\mathbf{k}$ for each partial wave α according to $\mathbf{U}_{\perp,\alpha,\mathbf{k}} = \boldsymbol{\xi}_\alpha U_{\alpha,\mathbf{k}}$, where for compressional Alfvén waves polarization is defined as $\boldsymbol{\xi}_{\mathrm{CA}} = \mathbf{k}_\perp / k_\perp$ and for shear Alfvén waves as $\boldsymbol{\xi}_{\mathrm{SA}} = \hat{\mathbf{z}} \times \mathbf{k}_\perp / k_\perp$. Considering the Fourier representation of (7) in high-frequency (Alfvén) part of flow velocity spectrum and multiplying it by polarization vector one gets the equation for the amplitude of each of CA and SA spectra:

$$\frac{dU_\mathbf{k}}{dt} = i\,\frac{\boldsymbol{\xi}\cdot\mathbf{N}_{\perp\,\omega,\mathbf{k}}}{2\,\omega(\mathbf{k})}. \tag{22}$$

Here $d/dt = \partial/\partial t + \mathbf{V}_{\mathrm{gr}}\cdot\nabla$ is the convective time derivative along wave package trajectory, $\mathbf{V}_{\mathrm{gr}}$ is the group velocity, $\mathbf{N}_{\perp\,\omega,\mathbf{k}}$ is the Fourier-transformed right-hand side of (7); frequency $\omega(\mathbf{k})$ is calculated according to one of the dispersion relations (11) and (12) depending on the wave considered. Equation (22) is obtained under the assumption that $U_\mathbf{k}$ is changing slowly compared to $e^{-i\omega t}$, so that $\partial^2(U_\mathbf{k}e^{-i\omega t})/\partial t^2 \approx -(\omega^2 U_\mathbf{k} + 2i\omega\,\partial U_\mathbf{k}/\partial t)e^{-i\omega t}$, and similarly—for spatial derivatives of $U_\mathbf{k}e^{-i\mathbf{k}\cdot\mathbf{r}}$. Since ω is connected with $\mathbf{k}$ via dispersion relation, the terms proportional to $U_\mathbf{k}$ cancel out when $\mathbf{D}\cdot\mathbf{U}_\mathbf{k}$ is multiplied with polarization vector $\boldsymbol{\xi}$ ($\boldsymbol{\xi}\cdot\mathbf{D}(\omega(\mathbf{k}),\mathbf{k})\cdot\boldsymbol{\xi} = 0$). Therefore, one is left with only the first-order derivatives on the left-hand side of the equation, which lead to the final result (22).

Denote the order of velocity perturbations for CA and SA modes with a small parameter ϵ_A, and the order of small density perturbations for the S modes with ϵ_S. Require that the perturbations of the magnetic field b_S caused by the S modes are small compared to those caused by the Alfvén waves b_A:

$$\mathbf{b}_S \sim \epsilon_S \beta \ll \mathbf{b}_\mathrm{A}. \tag{23}$$

Thus, although such S modes disturb the plasma density, they do not change the magnetic field in cold plasmas.

Relation (23) allows significant simplification of the nonlinear force $\mathbf{N}_\perp$ because in this case the only nonlinear drive comes from the term $\chi\,\partial_t\mathbf{U}$. Under the assumption of infinitely small β, one can then rewrite (22) as an integral equation

$$\frac{dU_{\mathbf{k}}}{dt}=\frac{i\omega(\mathbf{k})}{2}\int d^2\mathbf{k}'_\perp(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')\chi_S(\mathbf{k}_\perp-\mathbf{k}'_\perp)U_{\mathbf{k}'}, \tag{24}$$

where $\mathbf{k}'=\mathbf{k}'_\perp+k_\parallel\hat{\mathbf{x}}$, $\chi_S(\mathbf{k}_\perp)$ is the static density perturbations spectrum, and the density perturbations caused by Alfvén waves are assumed small compared to the static density perturbations of the S modes. Unless the Alfvén spectrum is represented by shear waves only, one needs to require that $\epsilon_A\ll\epsilon_S$ to satisfy the latter assumption.

The equivalent representation of (24) has a form

$$i\frac{dU_{\mathbf{k}}}{dt}=\mathbf{H}U_{\mathbf{k}}, \tag{25}$$

where the operator $\mathbf{H}$, defined over the set of eigenfunctions $\{\varphi_n(\mathbf{k}_\perp),\, n=0,1,...\}$, has matrix elements

$$H_{nm}=-\frac{1}{2}\int d^2\mathbf{k}_{\perp,1}d^2\mathbf{k}_{\perp,2}(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')\omega(\mathbf{k}_1)\varphi_n^*(\mathbf{k}_{\perp,1})$$
$$\times\chi_S(\mathbf{k}_{\perp,2}-\mathbf{k}_{\perp,1})\varphi_m(\mathbf{k}_{\perp,1}). \tag{26}$$

Consider now the limitations of the proposed approach. Equations (24) and (25) are derived under the assumption of resonant coupling, meaning that only those waves, whose frequencies and wave vectors strictly satisfy (16), are considered as interacting with each other. Certainly, this assumption is only satisfied for sufficiently large systems compared with the characteristic Alfvén wavelength $2\pi k^{-1}$ and on large time scales of interaction compared with the characteristic oscillation period $2\pi\omega^{-1}$. The former condition follows from the requirement of spatial resonance, which can now be rederived directly from (24). Indeed, the amplitude of Alfvén waves with a wave vector $\mathbf{k}$ is changed by the interaction with a wave having vector $\mathbf{k}'$ only if the spectral amplitude of static density perturbations χ_S is nonzero at $\mathbf{k}-\mathbf{k}'$, meaning that there exists a spatial lattice satisfying the spatial resonant interaction condition (16). Note that in (24) and (26), the integration in $\mathbf{k}$ space is taken only over the subspace of transverse vectors $\mathbf{k}_\perp$, since static density perturbations are assumed purely two-dimensional and the longitudinal wave number does not change in Bragg scattering process.

The temporal resonance condition (16) requiring the lower limitations on the interaction time in the proposed approach, however, cannot be derived from the obtained equations, since it has already been taken into account in the derivation, and thus should be considered separately. For SA$\leftrightarrow$SA scattering, this condition is satisfied automatically as soon as the space resonance requirement is fulfilled. Indeed, the latter assumes conservation of the parallel wave number of scattering wave ($k_{S,||}\equiv0\Rightarrow k_{||,\mathrm{SA},1}=k_{||,\mathrm{SA},2}$), which according to (12) is equivalent to the temporal resonance condition. For CA$\leftrightarrow$CA interaction, however, the situation is different because frequencies of compressional waves are determined by the transverse wave numbers as well. On the other hand, if one assumes the interaction of CA waves with *given* $k_\perp$, Eqs. (24) and (26) become applicable, too. Such a situation occurs, for example, in the scattering of a plane compressional wave with a given wave vector. In this case, the modes that are born in the Bragg scattering process will automatically get the same $k_\perp$ though the direction of their wave vectors will be rotated in $\mathbf{k}_\perp$ subspace depending on the plasma density profile.

Consider now the evolution of Alfvén spectrum in the frame moving with a group velocity with respect to the laboratory set of coordinates. For each spatial harmonic $\psi_{\mathbf{k}}$ of Alfvén spectrum in new frame of reference $U_{\mathbf{k}}(\mathbf{r},t)=\psi_{\mathbf{k}}(\mathbf{r}-\mathbf{V}_{\mathrm{gr}}t,t)$, one then gets the well-known Schrödinger equation

$$i\frac{\partial\psi_{\mathbf{k}}}{\partial t}=\mathbf{H}\psi_{\mathbf{k}}, \tag{27}$$

where the eigenvalues of the Hamiltonian $\mathbf{H}$ represent nothing else as Doppler-shifted frequencies of *spectrum* oscillations. Similar to the original Hamiltonian of quantum mechanics, $\mathbf{H}$ is Hermitian as can be seen directly from its definition (26). Since its eigenvalues are real, there follows the absolute stability of the transformed spectrum under the adopted approximation. Also, like the corresponding conservation law for the quantum mechanics ψ function, the Alfvén spectrum conserves the "normalization" $\int|\psi_{\mathbf{k}}|^2d^3\mathbf{k}=\mathrm{const}$.

Equations (24)–(27) needs to be solved together with the equation describing the time evolution of the density fluctuations spectrum $\chi_S(\mathbf{k})$. The latter equation can be obtained by Fourier transformation of (3) where only the resonant terms governing low-frequency drive must be kept. Alfvén wave density perturbations spectrum $\chi_{\mathbf{k}}$ can then be related to the flow velocity spectrum $\mathbf{U}_{\mathbf{k}}$ through $\chi_{\mathbf{k}}=\mathbf{k}\cdot\mathbf{U}_{\mathbf{k}}/\omega$, which gives for quasistatic density perturbations spectrum

$$\partial_t\chi_{S,\mathbf{k}_S}=-i\Bigg\{\mathbf{k}_S\cdot\mathbf{U}_{S,\mathbf{k}_S}+\int\frac{d^3\mathbf{k}}{\omega(\mathbf{k})}[(\mathbf{k}_S\cdot\mathbf{U}_{\mathrm{CA};\mathbf{k}_S-\mathbf{k}_\perp,-k_{||}})$$
$$\times(\mathbf{k}\cdot\mathbf{U}_{\mathrm{Ca},\mathbf{k}})+\mathrm{c.c.}]\Bigg\}, \tag{28}$$

where c.c. stands for the complex conjugate term. The right-hand side of (28) can be evaluated through Fourier representation of (2). In order to evaluate the terms that contain harmonics of the high-frequency magnetic field, note that for an Alfvén wave with given ω and $\mathbf{k}$, the corresponding perturbation of the field is given by

$$\mathbf{b}_{\mathbf{k}}=\hat{\mathbf{z}}\frac{\mathbf{k}\cdot\mathbf{U}_{\mathbf{k}}}{\omega}-\frac{k_{||}\mathbf{U}_{\mathbf{k}}}{\omega}, \tag{29}$$

as follows from linear form of (2). Substituting (29) into nonlinear equation (2) and performing the integration over all $\mathbf{k}$ with resonant conditions (16) taken into account, one can show that at low-frequency, magnetic field spectrum change is then determined by the evolution of quasistatic plasma density profile:

$$\partial_t\mathbf{b}_{S,\mathbf{k}}\approx\partial_t\chi_{S,\mathbf{k}}. \tag{30}$$

Since $\mathbf{b}_S \sim \epsilon_S \beta$ and $\partial_t \sim \epsilon_S$ for spectral quantities, the change of the field produced by evolution of S modes can be neglected, so the quasistatic plasma density profile remains unchanged $(\partial_t \chi_{S,\mathbf{k}} = 0)$. Thus,

$$\mathbf{H} = \text{const}, \tag{31}$$

making Eqs. (24)–(27) self-consistent and linear.

IV. APPLICATIONS OF MHD SPECTRUM EVOLUTION EQUATION: ALFVÉN WAVE TOMOGRAPHY

The fact that the matrix equation (24) and (25) is linear opens up the possibility of tomographic applications. Consider, first, harmonic spatial modulation of the plasma density:

$$\chi_S(\mathbf{k}_\perp) = \epsilon_S(\delta(\mathbf{k}_\perp - \mathbf{k}_S) + \delta(\mathbf{k}_\perp + \mathbf{k}_S)) \tag{32}$$

leading to the following form of Eq. (24):

$$\dot{U}_\mathbf{k} = \epsilon_S \frac{i\omega(\mathbf{k})}{2} (U_{\mathbf{k}-\mathbf{k}_S}\alpha_- + U_{\mathbf{k}+\mathbf{k}_S}\alpha_+), \tag{33}$$

where $\alpha_\pm = \boldsymbol{\xi}_\mathbf{k} \cdot \boldsymbol{\xi}_{\mathbf{k}\pm\mathbf{k}_S} = O(1)$ are polarization factors. Equation (33) can only be solved together with similar equations for $U_{\mathbf{k}\pm\mathbf{k}_S}$, which, in turn, require solving the equations for $U_{\mathbf{k}\pm n\mathbf{k}_S}$ with higher n. Therefore, the complete set of equations (33) is infinite $(n = 0,1,...,\infty)$ and, thus, hard to analyze for arbitrary initial conditions. However, an approximate solution of (33) can be found for limited-time scattering of a plane wave with a given wave vector $\mathbf{k}$. The first harmonics that will be generated during the interaction process will be shifted in $\mathbf{k}$ space only on single value of $\mathbf{k}_S$ from $\mathbf{k}$. These harmonics will later produce the waves having $\mathbf{k}$ vectors shifted on $\pm 2\mathbf{k}_S$, which then give rise to the waves with $\mathbf{k} \pm n\mathbf{k}_S$, $n = 2,3,...$, etc. Hence, in the beginning of the scattering process, harmonics with high n do not have sufficient time for being pumped and thus can be neglected under the approximation of limited scattering time. In the first (linear) stage of interaction, the harmonics with $n = 0, \pm 1$ are enough for an adequate description of the scattering process. Then, solving Eq. (33) one gets for the amplitudes of these waves:

$$U_\mathbf{k} \approx 1 - \frac{\alpha_+^2 + \alpha_-^2}{8}\epsilon_S \omega t, \quad U_{\mathbf{k}\pm\mathbf{k}_S} \approx i\epsilon_S \frac{\alpha_\pm \omega}{2} t \tag{34}$$

for "pump" wave with initial amplitude $U_\mathbf{k} = 1$. Formula (34) is valid until the dynamics of $U_{\mathbf{k}\pm\mathbf{k}_S}$ is entirely determined by the value of $U_\mathbf{k}$ and the second harmonics $U_{\mathbf{k}\pm 2\mathbf{k}_S}$ remain small, namely on time scales $t \ll (\epsilon_S \omega)^{-1}$. A more careful treatment requires taking a larger number of higher harmonics into account.

Until now, we have been solving the direct scattering problem obtaining the amplitudes of scattered Alfvén radiation from a known plasma density profile. However, the inverse scattering problem might also be of certain interest especially because of its certain possible practical applications. Indeed, if one knows the spectra of Alfvén radiation before and after scattering, in principle, one can reconstruct the plasma density profile from these data. The solution of the problem consists of inverting Eqs. (24) and (25), in order to obtain the matrix $\mathbf{H}$, from which the static density perturbation spectrum $\chi_S(\mathbf{k})$ can be derived:

$$\chi_S(\mathbf{k}_{2,\perp} - \mathbf{k}_{1,\perp}) = -\frac{2}{\omega(\mathbf{k}_1)} (\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2)^{-1} \varphi_n(\mathbf{k}_{1,\perp}) H_{nm} \varphi_m^*(\mathbf{k}_{2,\perp}). \tag{35}$$

In order to obtain the matrix elements H_{nm} from the results of Alfvén waves scattering, one can employ a known Alfvén wave source, comprised of wave packets characterized by their initial spectrum $U_\mathbf{k}^{(i)}$. Formal integration of (25) on a time interval $t \in (0,\tau)$ gives the expression for the final spectral function $U_\mathbf{k}^{(f)}$, which can be written in the following form:

$$U_\mathbf{k}^{(f)} = \mathbf{M} U_\mathbf{k}^{(i)}, \quad \mathbf{M} = \exp(-i\mathbf{H}\tau). \tag{36}$$

Scanning the plasma with N wave packets with different initial conditions $U_\mathbf{k}^{(i)}$, where N is the number of basis functions $\varphi_n(\mathbf{k}_\perp)$ used for spectrum representation, one gets enough independent equations for obtaining the elements of the matrix $\mathbf{M}$. (In order to get the exact density profile, one needs $N \to \infty$.) Taking the matrix logarithm of $\mathbf{M}$, one then gets the Hamiltonian $\mathbf{H}$, and, therefore, obtains the density spectrum (35).

The proposed procedure solves the general inverse scattering or tomography problem,[5,6] i.e., the multiple scanning measurements allow reconstruction of the static density perturbations in a cold plasma. Other tomographic methods actively used in plasmas include the application of x-ray waves for measuring the electron temperature in hot core of a tokamak plasma[7–9] and in ionospheric electron density measurements in the radio-frequency range.[10–12] The Alfvén radiation tomography discussed in the present work provides, in particular, the opportunity for studying density profiles in cold plasmas.

Practical difficulties that might be encountered in the present tomography problem are likely similar to the difficulties encountered in other tomographic applications. Since the complex logarithmic function is not defined uniquely, an inevitable uncertainty in tomographic reconstruction arises as one tries to obtain the matrix $\mathbf{H}$ from the matrix $\mathbf{M}$. However, using the linear stage of spectrum transformation discussed previously, or scanning the plasma volume with Alfvén waves of different frequencies, one can, in principle, further constrain possible reconstructions. The additional problems of Alfvén tomography that must be solved before the technique can be applied to real systems includes taking into account both thermal corrections and other nonlinear MHD effects.[2]

V. SUMMARY

In this article, we investigated the nonlinear coupling of Alfvén waves to inhomogeneities of cold collisionless plasmas. We demonstrated that in the limit of negligible plasma pressure $(\beta \to 0)$, the plasma inhomogeneities do not evolve in the interaction, so that the Alfvén wave coupling can be considered as Bragg scattering on fixed spatial lattices of plasma. In this case, the Hamiltonian (26) governing the evolution of the MHD spectrum remains constant, which makes the corresponding evolution equation (24)–(27) linear. Representing plasma quantities by their Fourier spectra leads to simple solution of the direct scattering problem.

In particular, we obtained the scattering properties of a given system including the case of a static sinusoidal plasma density perturbation. Knowing the spectrum of the incident and scattered Alfvén waves, we derived an expression for the Hamiltonian, from which the static density perturbations spectrum can be found easily. The solution of this inverse scattering problem indicates how tomographic reconstruction of plasma density perturbations might be achieved by means of imposed Alfvén radiation.

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