Vertical stability in a current-carrying stellarator

G. Y. Fu[a)]
Princeton Plasma Physics Laboratory, Princeton University, P. O. Box 451, Princeton, New Jersey 08543

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An analytic stability criterion is derived for the vertical mode in a large aspect ratio stellarator with uniform current density profile. The effects of vacuum magnetic field generated by helical coils are shown to be stabilizing due to enhancement of field line bending energy. For a wall at infinite distance from the plasma, the amount of external poloidal flux needed for stabilization is given by $f=(\kappa^2-\kappa)/(\kappa^2+1)$, where κ is the axisymmetric elongation and f is the ratio of vacuum rotational transform to the total transform. © *2000 American Institute of Physics.* [S1070-664X(00)02504-0]

It is known that tokamak plasmas suffer from vertical instability when the plasma shaping is sufficiently elongated. On the other hand, the tokamak beta limit tends to increase with elongation as implied by the well-known Troyon limit.[1] Thus, advanced tokamak operations require feedback stabilization of the vertical mode in order to achieve high beta.

Recent numerical calculations have shown that the vertical mode is robustly stable in a current-carrying quasiaxisymmetric stellarator,[2,3] whereas an equivalent tokamak is unstable. In this work, we show analytically that the vertical mode is much more stable in a current-carrying stellarator than in an equivalent tokamak. The stabilization comes from vacuum magnetic field generated externally by helical coils. The external poloidal magnetic field enhances the field line energy relative to the current-driven term associated with a vertical motion. In the following, we will derive an analytic stability criterion of the vertical mode in a current-carrying stellarator plasma.

We start from the energy principle.[4] The perturbed plasma energy is a sum of plasma potential energy δW_p and vacuum magnetic energy δW_v,

$$\delta W_p=\frac{1}{2}\int_p dv[\mathbf{B_1}^2+\mathbf{J}\cdot(\xi\times\mathbf{B_1})], \tag{1}$$

$$\delta W_v=\frac{1}{2}\int_v dv\,\mathbf{B_1}^2, \tag{2}$$

where $\mathbf{B_1}$ is the perturbed magnetic field, $\mathbf{J}$ is the equilibrium plasma current, and ξ is the plasma displacement. We have also assumed that the perturbation is incompressible.

For simplicity, we consider a large aspect ratio, low beta stellarator plasma. The plasma shape can then be approximated by a cylinder with cross-section shape varied along the axial direction due to helical coils. Using the stellarator expansion[5] via averaging along the axial direction, the equilibrium and stability problem is reduced to a two-dimensional one. Then, the equilibrium and perturbed magnetic field are reduced to

$$\mathbf{B}=\mathbf{z}\times\nabla\Psi+B_z\mathbf{z}, \tag{3}$$

$$\mathbf{B_1}=\mathbf{z}\times\nabla\Psi_1, \tag{4}$$

where we have used Cartesian coordinates (x,y,z) with z the coordinate along the axial direction, and $\mathbf{z}$ the unit vector. Here, Ψ and Ψ_1 are the equilibrium and perturbed poloidal magnetic flux, respectively. The equilibrium flux $\Psi=\Psi_c+\Psi_v$ is a sum of internally generated flux Ψ_c due to current and externally generated flux Ψ_v due to helical coils. To make further analytic progress, we assume uniform current density and uniform vacuum rotational transform; the total equilibrium flux can then be written as

$$\Psi=(\Psi_{v0}+\Psi_{c0})\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right), \tag{5}$$

for an elliptical shape with $\kappa=b/a$ being the ellipticity. Here, Ψ_{v0} and Ψ_{c0} are the flux values at the plasma edge due to helical coils and plasma current, respectively. The corresponding equilibrium current is $\mathbf{J}=J_0\mathbf{z}$, with

$$J_0=2\Psi_{c0}\left(\frac{1}{a^2}+\frac{1}{b^2}\right). \tag{6}$$

We consider the vertical perturbation as a rigid shift along the y direction (i.e., the direction along the elongation). Then, $\xi=\xi_y\mathbf{y}$, and $\Psi_1=-2\xi_y\Psi_0 y/b^2$ where $\Psi_0=\Psi_{v0}+\Psi_{c0}$. The potential energy is reduced to

$$\delta W_p=\frac{1}{2}\int_p dv\left[|\nabla\Psi_1|^2+J_0\xi_y\frac{\partial\Psi_1}{\partial y}\right] \tag{7}$$

$$=2V\xi_y^2\left[\frac{1}{b^4}\Psi_0^2-\frac{1}{b^4}(\kappa^2+1)\Psi_{c0}\Psi_0\right], \tag{8}$$

where V is the plasma volume. The vacuum energy is reduced to

$$\delta W_v=\frac{1}{2}\int_v dv|\nabla\Psi_1|^2, \tag{9}$$

where Ψ_1 satisfies

$$\nabla^2\Psi_1=0, \tag{10}$$

in the vacuum. Equation (10) can be solved conveniently using confocal coordinates (θ,μ) as

$$x=\sqrt{b^2-a^2}\sinh(\mu)\cos(\theta), \tag{11}$$

$$y=\sqrt{b^2-a^2}\cosh(\mu)\sin(\theta). \tag{12}$$

The solution is then given by

$$\Psi_1=-2\xi_y\frac{\Psi_0}{b}\sin(\theta)e^{-(\mu-\mu_0)}, \tag{13}$$

in absence of a conducting wall. Here, $\mu=\mu_0$ defines the plasma boundary shape with $\tanh(\mu_0)=a/b$. The integral in Eq. (9) can be evaluated straightforwardly and gives

$$\delta W_v=\frac{2V\xi_y^2}{ab^3}\Psi_0^2. \tag{14}$$

Then, the total perturbed plasma energy is given by

$$\delta W=\frac{2V\xi_y^2}{b^4}[(1+\kappa)\Psi_0^2-(\kappa^2+1)\Psi_{c0}\Psi_0]. \tag{15}$$

Physically, the first term in the bracket is the sum of the field line bending energy and the vacuum magnetic energy, and the second term is the destabilizing term driven by current. The externally generated poloidal flux is stabilizing because it enhances the field line bending energy and the vacuum energy by a factor of $(\Psi_0/\Psi_{c0})^2$, whereas the current driven term is only enhanced by a factor of Ψ_0/Ψ_{c0}. This is true when the external poloidal flux adds to the internal flux. In the case where external flux subtracts the internal flux, the external poloidal flux can be destabilizing when $0<\Psi_0/\Psi_{c0}<1$. When $\Psi_0/\Psi_{c0}<0$, the plasma is always stable vertically, regardless of the value of Ψ_0/Ψ_{c0}.

Equation (15) gives the following stability criterion for the fraction of external rotational transform $f=\iota_{\text{ext}}/\iota$ needed for stabilization:

$$f=\frac{\kappa^2-\kappa}{\kappa^2+1}. \tag{16}$$

Note that $f=1-\Psi_{c0}/\Psi_0$. This result has been confirmed by the numerical calculations[2] using the 3D global stability code Terpsichore.[6] For $f=0$, this stability criterion (i.e., $\kappa=1$) reduces to that of a tokamak without conducting wall stabilization.[7]

We note that effects of stellarator field on positional stability of a current-carrying plasma had been investigated experimentally by Sakurai and Tanahashi.[8] It was found that the stellarator field produced a large negative vertical field index which made the plasma much more stable in the horizontal direction. It was also found that the plasma was vertically stable, although the vertical field index was negative. Thus, it was concluded[8] that the field index cannot be used as a stability criterion for the vertical mode in a stellarator. This work explains how a stellarator plasma can be more stable vertically than in an equivalent tokamak plasma.

The result of this work has important implications for design of current-carrying quasi-axisymmetric stellarators (QAS). It shows why current-carrying QAS can be robustly stable to the vertical mode. It implies that the beta limit of QAS can be raised by increasing the axisymmetric elongation. Indeed, this conjecture has been confirmed by initial numerical results.[9] Another application of this result could be stabilization of the vertical mode in a tokamak plasma in order to access high elongation without feedback stabilization. It may be possible to design a simple helical coil specifically to stabilize the vertical mode without affecting favorable features of axisymmetric tokamaks. The feasibility of this remains to be investigated.

In conclusion, we have derived an analytic stability criterion for the vertical mode in a current-carrying stellarator plasma. The vertical mode can be stabilized by the externally generated poloidal flux due to enhancement of field line bending energy.

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[a]Electronic mail: fu@pppl.gov

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