Fluid models of phase mixing, Landau damping, and nonlinear gyrokinetic dynamics*

G. W. Hammett,† W. Dorland, and F. W. Perkins
Princeton Plasma Physics Laboratory, Princeton University, P.O. Box 451, Princeton, New Jersey 08543

(Received 25 November 1991; accepted 18 March 1992)

Fluidlike models have long been used to develop qualitative understanding of the drift-wave class of instabilities (such as the ion temperature gradient mode and various trapped-particle modes) which are prime candidates for explaining anomalous transport in plasmas. Here, the fluid approach is improved by developing fairly realistic models of kinetic effects, such as Landau damping and gyroradius orbit averaging, which strongly affect both the linear mode properties and the resulting nonlinear turbulence. Central to this work is a simple but effective fluid model [Phys. Rev. Lett. **64**, 3019 (1990)] of the collisionless phase mixing responsible for Landau damping (and inverse Landau damping). This model is based on a nonlocal damping term with a damping rate $\sim v_t|k_\parallel|$ in the closure approximation for the nth velocity space moment of the distribution function f, resulting in an n-pole approximation of the plasma dispersion function Z. Alternatively, this closure approximation is linearly exact (and therefore physically realizable) for a particular f_0 which is close to Maxwellian. "Gyrofluid" equations (conservation laws for the *guiding-center* density n, momentum $mnu_\parallel$, and parallel and perpendicular pressures $p_\parallel$ and $p_\perp$) are derived by taking moments of the gyrokinetic equation in guiding-center coordinates rather than particle coordinates. This naturally yields nonlinear gyroradius terms and an important gyroaveraging of the shear. The gyroradius effects in the Bessel functions are modeled with robust Padé-like approximations. These new fluid models of phase mixing and Landau damping are being applied by others to a broad range of applications outside of drift-wave turbulence, including strong Langmuir turbulence, laser–plasma interactions, and the α-driven toroidicity-induced Alfvén eigenmode (TAE) instability.

I. INTRODUCTION

Our general goal is to develop improved gyrofluid equations for nonlinear simulations of turbulence and transport in plasmas. "Gyrofluid" equations (or "gyro-Landau-fluid equations" as some have called them) describe the dynamics of a limited set of fluid moments of the gyrokinetic equations (e.g., the guiding-center density n, parallel flow $u_\parallel$, parallel pressure $p_\parallel$, and perpendicular pressure $p_\perp$). The gyrofluid equations also include approximate models of kinetic effects such as phase mixing, Landau damping (and its inverse), gyroaveraging, and the dominant nonlinearities.

Central to the improved accuracy of these new fluid equations is our new fluid model of phase mixing and Landau damping.[1] This model has wide applicability outside of our immediate interest in drift-wave-type plasma turbulence. In the present conference, variations of this model are being applied by Goldman and Newman[2] to the Zakharov equations of strong Langmuir turbulence to include self-consistent plasma heating effects, by Kaiser *et al.*[3] to laser light filamentation problems in laser–plasma interactions, and by Spong *et al.*[4] to numerical studies of the nonlinear dynamics of the α-driven toroidicity-induced Alfvén eigenmode (TAE) instability.

In this short paper we will focus on the phase-mixing paradigm to give intuitive insight into our fluid model. We will also provide a brief overview of the extension to include the effects of finite-gyroradius averaging and toroidal drifts, which will be given fuller treatment in subsequent papers.[5,6]

II. PREVIOUS FLUID EQUATIONS AND THE LANDAU DAMPING PROBLEM

Fluid equations provide a convenient reduced description for many problems, and are frequently more amenable to analytic insight, nonlinear analysis, or computational solution than the full kinetic equations. However, it has been widely thought (with a few exceptions) that fluid equations are inherently unable to model Landau damping or other resonance effects, and that if these effects play a role in the problem at hand, then one must abandon fluid equations in favor of a fully kinetic treatment. For example, Krall and Trivelpiece's standard text *Principles of Plasma Physics* states:

"A property of Langmuir waves that is predicted by the Vlasov theory but which is completely outside the scope of fluid theory is the collisionless damping of electrostatic potentials..."[7]

They relate this failure to the closure problem in the fluid moment equations hierarchy (in their words, they say that fluid theory requires an *ad hoc* assumption about the equation of state relating the pressure to the density, while "Vlasov theory reveals the correct equation of state"). Spitzer expresses a similar idea in his classic little textbook:

*Paper 8I3, Bull. Am. Phys. Soc. **36**, 2454 (1991).
†Invited speaker.

 0899-8213/92/072052-10$04.00 © 1992 American Institute of Physics

"When conditions change with time, determination of the way in which $p_\perp$ and $p_\parallel$ change is not possible in any simple way, since heat energy may flow along the lines of force. When collisions are infrequent, such a heat flow depends on the detailed nature of the velocity distribution function, and cannot be determined in any simple way from the macroscopic equations."[8]

Later in his book (p. 159) he goes on to say that, in these situations, "the macroscopic" (or fluid) "equations are not very useful, and the velocity distribution function $f(\mathbf{r},\mathbf{w},t)$ must be analyzed."

Another argument which is sometimes given for why the usual fluid equations fail to reproduce Landau damping was first given in an early paper by Oberman.[9] He showed that the kinetic theory reduces to the usual fluid results in the cold-plasma limit $kv_t/\omega \ll 1$ (where v_t is the average thermal speed of the plasma particles and ω/k is the phase velocity of the wave), and then says (with a slight change of notation):

"What has happened to the Landau damping? One cannot expect the Landau damping to manifest itself in such a procedure, a power series expansion in (kv_t/ω), for in the Landau problem, in this limit, the damping goes as

$$\mathrm{Im}(Z) \sim e^{-1/(kv_t/\omega)^2},$$

i.e., the damping goes to zero faster than any power of (kv_t/ω)."

The key to our success where earlier fluid attempts failed is the use of an improved closure approximation (e.g., a better approximation for the equation of state) which introduces a dissipative term with a damping rate $\sim |k_\parallel| v_t$. Physically, this $k_\parallel$-dependent damping term mimics the collisionless-phase-mixing process underlying Landau damping (Sec. III). Mathematically, our closure turns out to be equivalent to an n-pole Padé approximation of the Z function, which works even though the exponential term has no formal Taylor-series expansion. [An example of a "Padé" approximation is $\exp(-1/\epsilon 2) \approx 1/(1+\epsilon-2+\cdots)$. Also, Padé approximations are frequently more robust with faster convergence than Taylor-series approximations.]

Although many have held that fluid equations must be abandoned if Landau damping is important, there have been others who have suggested ways to model Landau damping by adding various dissipative terms to fluid equations. In the discussion after their Eq. (3.43), Kadomtsev and Pogutse[10] note in passing that the collisional dispersion relation for the current-convective instability (derived from Braginskii's collisional equations) becomes similar to the collisionless dispersion relation (which involves Landau damping) when the mean-free path $\lambda \approx 1/k_\parallel$. In this limit, the damping rate associated with parallel viscosity is $\mu k_\parallel^2 \sim v_t \lambda k_\parallel^2 \sim |k_\parallel| v_t$. They elaborated on this in a later paper[11] on self-consistent magnetic fluctuations where they employed a resistivity proportional to $|k_\parallel| v_t$. That paper presented a set of equations that are similar to our two-moment set below, and contained the essential idea of adding a damping term with a damping rate $\approx |k_\parallel| v_t$. However, there are still some differences between our equations and theirs (our equations provide a smooth n-pole approximation valid in adiabatic, cold-plasma, and intermediate limits, and also retain additional nonlinearities). Furthermore, their proposed extension to three-moment equations [see their Eq. (3.9)] involves a parallel heat flow proportional to $E_\parallel$, while our expression for the parallel heat flow is proportional to $\nabla_\parallel T_\parallel$. We show below that phase mixing occurs even if $E_\parallel = 0$. Physically, this enables us to phase mix perturbations which were generated nonlinearly (such as in ion Compton scattering), in addition to phase mixing those perturbations which were linearly proportional to $E_\parallel$.

Other examples of proposed Landau damping models include Lee and Diamond's[12] choice of a parallel momentum viscosity $\mu_\parallel \approx v_{ti}^2/|\omega|$ and a parallel heat conductivity $\chi_\parallel = 0$, and Waltz's choice[13] $\mu_\parallel = \chi_\parallel = \min(2^{1/2} v_{ti}/|k_\parallel|, 2v_{ti}^2/|\omega_r|)$. All of these previous models employ a nonzero $\mu_\parallel$, while we have shown[1] that in order to conserve energy and to properly reproduce several features of the ion temperature gradient (ITG) instability, one must use a three-moment set of fluid equations with $\mu_\parallel = 0$ and rely only on $\chi_\parallel$ to model Landau damping. If, however, one can ignore the temperature evolution equation (perhaps when looking at drift waves with $\nabla T = 0$), one can use a simpler two-moment set with a nonzero $\mu_\parallel$ model. The recent work of Chang and Callen is discussed below in Sec. III G.

III. THE PHASE-MIXING PARADIGM

In this section we will present a very simple phase-mixing paradigm to provide insight into the essential features of our fluid model of Landau damping. [Although phase mixing itself is well known, it is a very useful framework for understanding the problem at hand. It was Dr. Oberman who first suggested that I (GWH) should perhaps consider this perspective.]

A. Exact kinetic phase mixing

Consider a one-dimensional kinetic equation for a noninteracting neutral gas ($E_\parallel = 0$ for now),

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} = \delta(t) f_0(z,v), \tag{1}$$

where $f(z,v,t)$ is the particle distribution as a function of position z, velocity v, and time t; $f_0(z,v)$ provides the initial conditions at $t=0$. This equation involves the usual linear propagator, and the solution of this equation is the Green's function which can then be used to solve more complicated problems by convolving with additional source terms on the right-hand side. For example, linear Landau damping adds the source term $-(e/m)E_\parallel(\mathbf{x},t)\partial f_M(v)/\partial v$. Thus, an accurate fluid model of Eq. (1) should also be applicable to more complicated problems.

The exact solution of Eq. (1) is just $f(z,v,t) = f_0(z - vt,v)H(t)$. Consider an initial f_0 that is Maxwellian in velocity but has a small density perturbation with a single Fourier harmonic in the z direction,

$$f_0(z,v)=(n_0+n_1e^{ikz})f_M(v). \tag{2}$$

The solution at future times is just

$$f(z,v,t)=(n_0+n_1e^{ik(z-vt)})(1/\sqrt{2\pi v_t^2})e^{-v^2/(2v_t^2)}.$$

Note that f just oscillates in time with a frequency $\omega=kv$ and there is no damping. However, all v moments of f will exponentially decay in time. For example, consider the density as a function of time:

$$n(z,t)=\int dv\, f=n_0+n_1\frac{e^{ikz}}{\sqrt{2\pi v_t^2}}\underbrace{\int dv}_{\text{mixing}}\underbrace{e^{-ikvt}}_{\text{phase}}e^{-v^2/(2v_t^2)}.$$

Evaluating this integral we find that any initial density perturbation will decay away with a Gaussian time dependence:

$$n_1(t)=n_1(0)e^{-k^2v_t^2t^2/2}. \tag{3}$$

B. The limits of the usual fluid equations

We first present the usual fluid equations and show how they fail to reproduce this phase mixing. For later use, we will start with a kinetic equation that includes the electric field E:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial z}+\frac{e}{m}E\frac{\partial f}{\partial v}=0, \tag{4}$$

where e/m is the charge-to-mass ratio of the particles. As usual, fluid equations are derived by taking moments (operating with $\int dv\, v^l$), which leads to the following set of equations for the particle density $n=\int dv\, f$, the momentum density $mnu=m\int dv\, fv$, and pressure $p=m\int dv\, f(v-u)^2$:

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial z}(un)=0, \tag{5}$$

$$\frac{\partial}{\partial t}(mnu)+\frac{\partial}{\partial z}(umnu)=-\frac{\partial p}{\partial z}+enE, \tag{6}$$

$$\frac{\partial p}{\partial t}+\frac{\partial}{\partial z}(up)=-2p\frac{\partial u}{\partial z}-\frac{\partial q}{\partial z}, \tag{7}$$

where $q=m\int dv\, f(v-u)^3$ is the heat flux. Each of these equations is an exact integral of the kinetic equation, and expresses an exact nonlinear conservation law for the conservation of particles, momentum, energy, etc. Although exact, each of these equations is coupled together in an infinite hierarchy: to find the evolution of n requires knowledge of u, to find u requires p, p requires q, etc. The problem is in the $v\partial f/\partial z$ term of Eq. (4) which always introduces the v^{l+1} moment into the evolution equation for the v^l moment.

The usual treatment of this closure problem has been to ignore one of the higher moments, such as in the cold-plasma approximation $p=0$. Sometimes the dynamical equation for p is kept but $q=0$ is assumed. Sometimes a higher moment is approximated in terms of lower moments, for example assuming an equation of state such as $p=p_0(n/n_0)^\Gamma$. A $q=0$ approximation in Eq. (7) is equiv-

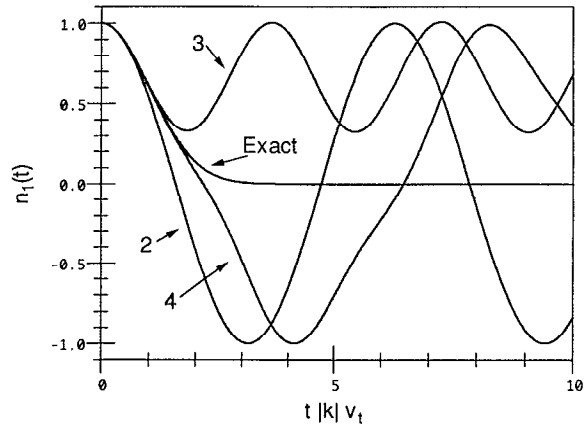

FIG. 1. The evolution of a density perturbation from the exact kinetic theory, illustrating phase mixing, and from two-, three-, and four-moment fluid models (all without dissipation) which fail to reproduce phase mixing.

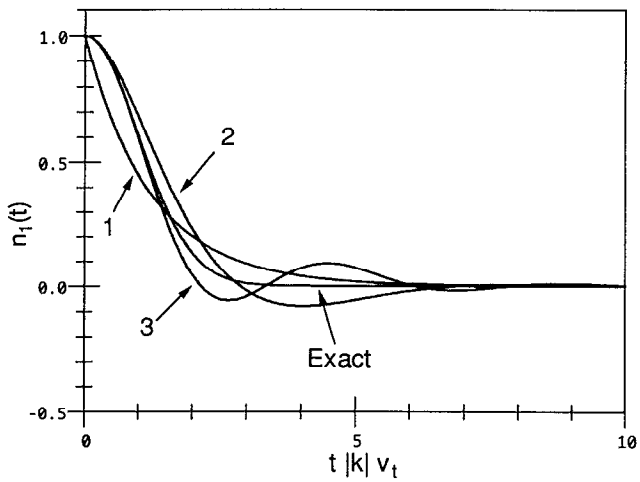

FIG. 2. The evolution of a density perturbation from the exact kinetic theory, illustrating phase mixing, and from one-, two-, and three-moment fluid models (including dissipation terms to model the phase mixing).

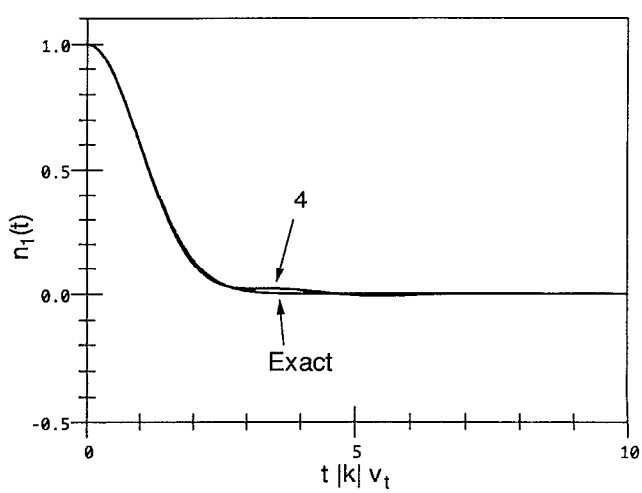

FIG. 3. The same as for Fig. 2, but with a four-moment fluid model (including dissipation) that provides an excellent fit to the exact kinetic theory.

alent to using this equation of state with the ratio of specific heats $\Gamma=3$ [consistent with $\Gamma=(2+d)/d$, where the number of degrees of freedom $d=1$ in this one-dimensional gas]. A $\Gamma=1$ approximation is sometimes used for the isothermal limit where the heat flux q is assumed to be so large (an opposite extreme from the $q=0$ approximation) that it immediately wipes out all temperature gradients.

It is reasonable to try the simplest approximations first, and in fact a great deal of progress has been made in plasma physics using such approximations, probably because they satisfy important conservation laws while still being analytically tractable. However, none of these simple closures is able to reproduce phase mixing. Linearizing and Fourier transforming the first two fluid equations, Eqs. (5) and (6) with $E=0$ for now, and using the approximation $p_1 = T_0\Gamma n_1$, one obtains:

$$-i\omega n_1+ikn_0u_1=0, \tag{8}$$

$$-i\omega mn_0u_1+ik\Gamma T_0n_1=0. \tag{9}$$

The solution is composed of a sum of modes with frequencies $\omega = \pm\sqrt{\Gamma}|k|v_t$. Selecting $\Gamma=1$ [to match the isothermal initial condition of Eq. (2)], and matching the initial conditions $n_1(t) = n_1(0)$ and $u_1(t) = 0$, yields the solution $n_1(t) = n_1(0)\cos(|k|v_tt)$. Figure 1 shows that this initially agrees with the exact kinetic phase-mixing result of Eq. (3), but it eventually diverges from the exact decaying result since it is oscillatory. Also shown in Fig. 1 is the linearized solution of Eqs. (5)–(7) with the $q=0$ approximation, showing a better fit for a slightly longer time, although it eventually diverges because it too is oscillatory.

C. Attempted extension to higher moments

One might hypothesize that the fluid equations could be improved by keeping more fluid moments before making a closure approximation. For example, rather than setting $q=0$, one could keep the fluid equation for q:

$$\frac{\partial q}{\partial t}+\frac{\partial}{\partial z}(uq)=-3q\frac{\partial u}{\partial z}+3\frac{p}{mn}\frac{\partial p}{\partial z}-\frac{\partial r}{\partial z}, \tag{10}$$

although this will require a closure approximation for the next higher moment $r\equiv m\int dv\, f(v-u)^4\equiv nm\langle(v-u)^4\rangle$. A reasonable choice might seem to be $r \approx nm\alpha\langle(v-u)^2\rangle^2=\alpha p^2/(nm)$, where the factor $\alpha=3$ for a Maxwellian f. However, the results of linearizing Eqs. (5)–(10) and numerically solving them for the phase-mixing test case are also shown in Fig. 1. Again, for a slightly longer time the four-moment solution fits better than the two- and three-moment solutions, but its long time behavior is oscillatory.

It is possible to show that a wide class of linear closure approximations fail to reproduce phase mixing. The general expression for the evolution of the lth moment, $W_l = \int dv\, f(v-u)^l\equiv n\langle(v-u)^l\rangle$ can be shown to be

$$\frac{\partial W_l}{\partial t}+\frac{\partial}{\partial z}(uW_l)=-lW_l\frac{\partial u}{\partial z}+l\frac{W_{l-1}}{mn}\frac{\partial p}{\partial z}-\frac{\partial W_{l+1}}{\partial z}. \tag{11}$$

[Since $W_1\equiv 0$, these equations are supplemented by Eq. (6) for u.] Linearizing $W_l(z,t) = W_{l,0} + \delta W_l(z,t)$, the general linear closure approximation for the unknown highest moment is $\delta W_L = \Sigma_{l=0,L-1}C_l\delta W_l$. We now have L equations in L unknowns. Linearizing and Fourier transforming them will lead to a matrix with purely real coefficients, because when Eq. (11) is linearized and Fourier transformed, each term involves either a $\partial/\partial t\to -i\omega$ or a $\partial/\partial z\to ik$ so that all of the i's cancel. (This is true only if the closure coefficients C_l are real, which we will assume for now.) Setting the determinant of this matrix to zero will result in an Lth-order polynomial in ω, whose roots are the eigenfrequencies of the system. All of the coefficients of that polynomial are real, so that the roots must come in complex-conjugate pairs: for any damped root that might model phase mixing, there must be also be a growing root, which would be unphysical. Presumably the closure coefficients could be chosen in such a way that all of the eigenfrequencies are real, which would be more physical but which would still fail to reproduce phase mixing.

D. The simplest possible fluid model of phase mixing

In this subsection we present a one-moment fluid model of phase mixing. It may seem overly simplistic, but it does illustrate the basic idea behind the model. In the next subsection we show how the model can be made more accurate by keeping additional fluid moments before damping is introduced.

It is perhaps obvious that one could try to model phase mixing by adding some type of damping to the fluid equations, but the key is to find the proper form for that damping. Mathematically, the addition of damping is equivalent to allowing the closure coefficients C_l of the previous subsection to be complex. Consider just the first fluid equation, Eq. (5), for the conservation of particles. We will approximate the higher moment un in terms of the lower moment with an expression of the form $un\approx -D\,\partial n/\partial z$, i.e., a Fick's law for diffusion, which will certainly introduce damping into our equations. The solution of this one-moment model is simply $n_1(t) = n_1(0)\exp(-Dk^2t)$. This exponential decay has a time scale $\nu\sim Dk^2$, while the exact kinetic result in Eq. (3) was a Gaussian decay with a time scale $\nu\sim|k|v_t$. Although these two results have different forms, we can ensure that they have the same time-averaged values (as illustrated in Fig. 2) by setting

$$D=\sqrt{(2/\pi)}(v_t/|k|).$$

Since $D_k\propto 1/|k|$ in wave-number space, D itself is actually an integral operator in real space, as shown in Ref. 1. Defining the particle flux $\Gamma_k=-D_k\,\partial n/\partial z=-D_kikn_k$ (where this particle flux Γ is not to be confused with the earlier use of Γ as the ratio of specific heats), one can evaluate the inverse Fourier transform to find that the real-space representation of the particle flux Γ is

$$\Gamma(z)=\frac{1}{(2\pi)^{1/2}}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dk\, e^{ik}e^{-|k|\epsilon}\Gamma_k$$
$$=-\frac{2^{1/2}v_t}{\pi^{3/2}}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dz'\,\frac{z'n(z+z')}{z'^2+\epsilon^2}$$
$$=-\frac{2^{1/2}v_t}{\pi^{3/2}}\int_0^{\infty} dz'\,\frac{n(z+z')-n(z-z')}{z'}, \qquad (12)$$

where we have used the convolution theorem, and the factor of $\exp(-|k|\epsilon)$ was added to define infinite integrals. Equation (12) shows that the particle flux Γ is driven by an average nonlocal density difference. Many numerical codes use a spectral representation in the magnetic field direction and so can use the simple Fourier representation for Γ_k rather than the convolution form for $\Gamma(z)$.

E. Extension of phase-mixing models to higher moments

As illustrated in Fig. 2, the accuracy of these fluid models of phase mixing can be improved by keeping more fluid moments before introducing a closure approximation. For example, the two-moment model shown in Fig. 2 is based on keeping both the density and momentum conservation equations, Eqs. (5) and (6), and approximating p in terms of lower moments as $p \approx nT_0 - mn\mu\, \partial u/\partial z$. The damping is provided by the viscositylike μ which is assumed to have the form $\mu = \mu_1 v_t/|k|$, in analogy with the scaling found for D in the one-moment case. After linearization and Fourier transforming, the density perturbation is found to have the frequency components $\omega/|k|v_t = \pm\sqrt{1-\mu_1^2/4} - i\mu_1/2$. We choose $\mu_1=\sqrt{\pi/2}$ to give the same time-averaged response as the exact kinetic result.

The two-moment model does not conserve energy since Eq. (7) is ignored and a constant temperature T_0 is assumed. However, momentum and particles are conserved since the only approximation made is in p, which appears only inside a derivative as $\partial p/\partial z$ and so represents a flux of momentum from one place to another while conserving total momentum. The two-moment model does reproduce the main qualitative features of the kinetic result and may be sufficiently accurate for some problems where the temperature is fairly constant, such as $\nabla T=0$ drift waves or the α-driven TAE instability (as is being done in Ref. 4).

Although the one-moment model was not even first-order accurate in t for small t, the two-moment model is now second-order accurate in t, independent of the choice of μ. Higher moment fluid models become increasingly accurate as a Taylor series in t, while the damping which is added to the highest moment ensures that the proper long-time behavior is recovered as well. Figure 2 shows the results from the three-moment model as derived in Ref. 1, i.e., Eqs. (5)–(7) are kept, and the heat flux is chosen to be $q=-n\chi\, \partial T/\partial z$, where $\chi = 2\sqrt{2/\pi}v_t/|k|$. By going to three-moments, we now conserve energy as well, which is needed to study modes such as the ion-temperature-gradient-driven (ITG) instability. As shown in Fig. 3, even better accuracy is achieved by going to the four-moment model given in Ref. 1.

F. Discussion

There are a number of parallels between the phase-mixing paradigm and the usual Landau damping problem. For example, f itself does not decay, while all integrals of f do. More fundamentally, the phase-mixing problem is equivalent to solving for the Green's function, so these fluid models should be applicable to a wide number of problems (as long as the closure coefficients are chosen to conserve energy[1]). A number of results are given in Ref. 1 which illustrate the generality of these equations, at least linearly. An n-moment fluid model is equivalent to an n-pole approximation of the Z function in one dimension, or of the full dispersion relation resulting from the drift-kinetic equation in slab geometry (generalizations to include finite gyroradius and toroidal effects are discussed below), i.e., it reproduces the main features of ITG and drift waves and models inverse-Landau damping as well. The Z function assumes that $f_0(v)$ is Maxwellian. Reference 1 shows that there exists an $f_0(v)$ which is close to Maxwellian and which exactly reproduces the n-pole fluid response function. It is comforting to know that there is a physically meaningful $f_0(v)$ that is equivalent to the assumed fluid closures.

Although we have not proven convergence formally, it does appear from our numerical work (e.g., Figs. 2 and 3) that these fluid models converge to the exact Z function fairly rapidly as the number of moments n increases. Another conjecture is that almost any $f(v)$ could be modeled by keeping enough moments and generalizing somewhat the form of the damping coefficients for the highest moments (so that it is not written in terms of just the Maxwellian v_t). However, the convergence might be slow (or nonexistent) for some types of distribution functions which are not sufficiently smooth.

It should be pointed out that the class of two-pole Z-function approximations of Fried *et al.*[14] reduces to our two-moment result only when their parameter ξ is chosen to be $[2-(\pi/4)]^{1/2}\approx 1.10$, though the most widely cited result from Fried *et al.* is with $\xi=0.60$. In the cold-plasma limit $\zeta = \omega/k\sqrt{2}v_t \gg 1$, the $\xi=0.60$ two-pole approximation is asymptotic to $-1/\zeta-1.8/(2\zeta^3)+\cdots$, while the real Z function is asymptotically $-1/\zeta-1/(2\zeta^3)+\cdots$. One can find the equivalent $f_0(v)$ from the $\xi=0.60$ approximation and show that $\int dv\, f_0(v)$ gives 1.8 times the density of the actual $f_M(v)$. However, by choosing $\xi=1.10$ (or using any of our higher-moment approximations) one finds that the proper coefficient of the $1/\zeta^3$ term is obtained, thus leading to the proper density normalization. In the cold-plasma limit, the $1/\zeta$ term in the Z function just cancels the adiabatic part of the plasma response so that it is actually the $1/\zeta^3$ term that determines the plasma behavior.

We have demonstrated the accuracy of our fluid model only for *linear* Landau damping. This is probably sufficient for the types of problems we are presently interested in, such as drift-wave-type turbulence. Because $k_\parallel/k_\perp \sim \epsilon \ll 1$

in the usual gyrokinetic ordering, drift-wave turbulence is usually assumed to be dominated by the $\mathbf{E}\times B\cdot\nabla f_1$ nonlinearities[12,13,15] and not the $E_\parallel\, \partial f_1/\partial v_\parallel$ nonlinearity (which would be related to nonlinear Landau damping and particle trapping). This should be valid unless for some reason $\partial f_1/\partial v_\parallel$ becomes very large, of order $f_1/(v_t\epsilon)$, which would seem unlikely in a real plasma having waves with a broad spectrum of parallel phase velocities, $\omega/k_\parallel$.

Even if parallel nonlinearities are important, the fluid equations [(5)–(7), (10), and (11)] do contain many nonlinear terms which may capture the relevant physics. Each fluid equation is an exact *nonlinear* conservation law: approximations are introduced only in the highest moment in a way which still preserves the conservative form of the equations. Investigation of the nonlinear limits and capabilities of these equations is still underway. Some types of nonlinearities are already modeled sufficiently accurately with two to four moment equations.[2] It might even be possible to extend these equations to model the local flattening of $f(v)$ caused by particle trapping in nonlinear Landau damping,[16] but that would probably require at least five moment equations. Nevertheless, our fluid model of kinetic phase mixing is only an approximation and so breaks down somewhere. In particular, it would probably be ill-suited for some types of problems which depend sensitively on the velocity space details of a complicated distribution function $f(v)$.

G. Another approach

We have presented a physically motivated rationale for a fluid model of phase mixing with a diffusion coefficient that scales as $D \sim v_t/|k|$, showing that it is equivalent to an n-pole approximation of the Z function. Chang and Callen[17] have presented an alternate approach to this problem in terms of a hybrid fluid-kinetic calculation. In brief, they use the *nonlinear* fluid equations for the lower moments, using the exact *linear* kinetic result to close the higher moments. For example, in the linear definition of $q_1 = m\int dv(f_1 v^3 - 3f_0 u v^2)$, one would use the linear solution of the Vlasov equation (including the driving electric field), $f_1 = (e/m)k\phi\, \partial f_0/\partial v/(\omega - kv)$. This relates q_1 to ϕ. Similarly, other moments (such as T_1) can be expressed in terms of ϕ, allowing one to eliminate ϕ and produce an exact linear expression of the form $q_1 = -n_0\chi(\omega/|k|v_t)ikT_1$, where $\chi(\omega/|k|v_t)$ is a complex heat conductivity function which involves the Z function, and so is linearly exact. [Actually, they employ a Chapman–Enskog-like procedure which eliminates the ϕ dependence from the start.]

In the $\omega \ll |k|v_t$ limit, it turns out that their $\chi(\omega/|k|v_t)$ reduces to our result, $\chi = 2\sqrt{2/\pi}\,(v_t/|k|)$. This is the regime where the phase mixing is important. The opposite limit of $\omega \gg |k|v_t$ is where the fluid equations automatically work already and it does not matter too much what is assumed for the higher moments. Of course there may be some problems for which higher linear accuracy is desired, although the transcendental dependence on ω would have to be approximated in some way for nonlinear initial-value codes. One approach would be to use some kind of instantaneous estimate[13] of ω. Another would be to use an n-pole approximation for $\chi(\omega/|k|v_t)$, which turns out to be equivalent to our approach of just keeping n more fluid moments beyond p. These additional fluid moment equations would also introduce more nonlinearities.

Chang and Callen have also included collisions and fully electromagnetic perturbations (while we have been working primarily in the collisionless electrostatic limit for now). There are some other differences in our work that one should be aware of when making comparisons. From the full Boltzmann equation, they derive three fluid equations for the *particle* density, parallel momentum, and total pressure p. They employ linear closure approximations for the stress tensor Π (which corrects for the differences between $p_\perp$ and $p_\parallel$), and for the parallel flow of total heat. The four fluid equations we derive from the gyrokinetic equation are for the *guiding-center* density, parallel momentum, and *parallel* and *perpendicular* pressures $p_\parallel$ and $p_\perp$, employing linear closures for the parallel flows of both perpendicular and parallel heat. This explains why their equations contain both momentum viscosity and heat conduction terms, while ours only have heat conduction terms. Our approach also has some additional nonlinearities, because we do not assume a linear closure approximation for the stress tensor. Their choice of variables and coordinates was partially motivated by a desire to reduce to the traditional form of the usual Braginskii equations in the collisional limit. However, this has meant that they have focused on longer wavelengths where second-order expansions in $k_\perp\rho$ can be made, while we have found it easier to develop robust finite-gyroradius models by starting with the guiding-center coordinates of the gyrokinetic equation.

IV. EXTENSION TO OTHER PHASE-MIXING PROCESSES

The toroidal electrostatic gyrokinetic equation in conservative form is[18–20]

$$\frac{\partial}{\partial t}(BF) + \nabla\cdot[BF(v_\parallel\hat{b} + \mathbf{v}_d + J_0\mathbf{v}_E)] + \frac{\partial}{\partial v_\parallel}\left[BF\left(-\frac{e\hat{b}\cdot\nabla J_0\phi}{m} + v_\parallel(\hat{b}\cdot\nabla\hat{b})\cdot J_0\mathbf{v}_E - \mu\hat{b}\cdot\nabla B\right)\right], \tag{13}$$

where $F(v_\parallel,\mu,\mathbf{x},t)$ is the total distribution function (containing both equilibrium and perturbed parts) and $\mu = v_\perp^2/(2B)$; $J_0(k_\perp v_\perp/\Omega)$ is an operator that carries out the gyroaveraging of the electric field. The combination BF appears because the magnetic field B is the velocity space Jacobian, i.e., the guiding-center density is

$$n_{\rm gc} \equiv \int d^3v\, F \equiv \int dv_\parallel \int d\mu\, BF. \tag{14}$$

Imposing quasineutrality on the particle density (not the guiding-center density) yields

$$n_e = n_i \equiv \int d^3v\, J_0 F - n_{i0}[1-\Gamma_0(b)]\frac{e\Phi}{T_i}, \tag{15}$$

where the last term is the polarization density due to the variation of the ion density around a gyro-orbit. Note that the particle density n_i is different than the guiding center density n_{gc} because of finite-gyroradius effects.

As we have already found, the $v_\parallel$ free-streaming term in Eq. (13) gives rise to parallel phase mixing (and the parallel Landau resonance) due to the spread in parallel velocities of different particles. Likewise, the v_d and gyroradius-averaged $E\times B$ drifts will also give rise to phase mixing and associated resonances. In the case of the $v_\parallel$ term, the phase mixing rate was $\nu \sim |k_\parallel| v_t$, where v_t is the average spread in parallel velocities. For a simple $1/R$ magnetic field, the ∇B and curvature drift takes the form

$$\mathbf{v}_d = -[(v_\parallel^2 + v_\perp^2/2)/\Omega R]\hat{y}.$$

This term is responsible for toroidal drift resonances. Because there is a range of drift speeds, i.e., high-energy particles drifting faster than lower-energy particles, this term will also produce phase mixing. For example, consider the simple limit $\partial f/\partial t + v_d\, \partial f/\partial y = 0$, and assume an initial density perturbation of the form $n_1(0)\exp(ik_y y)$. The kinetic result is $n_1(t) = n_1(0)/(1 + ik_y\bar{v}_d t/2)\sqrt{1+ik_y\bar{v}_d t}$, where $\bar{v}_d = 2v_t^2/\Omega R$. Proceeding as before and adding a dissipation rate of order $\nu \sim |k_y|\bar{v}_d$ into higher moments of the fluid equations would provide a model of this process, although this fluid model would have an exponential asymptotic dependence instead of the $1/t^{3/2}$ scaling of the kinetic result. This asymptotic difference may not be important for most applications. If it ever is important, it may be possible to improve the situation by allowing the dissipative terms to depend on ω (employing some type of instantaneous approximation) or on time history integrals. A more complete discussion of the fluid models for these toroidal drift resonances can be found in Waltz *et al.*[6]

The gyroradius-averaged $E\times B$ drift will also lead to phase mixing because the gyroaveraging causes the high-energy particles to drift slower than low-energy particles. In the $p_\perp$ fluid equation, this introduces a dissipative term $\nabla\cdot q_\perp \approx n_0|(\Delta J_0 \mathbf{v}_E)\cdot\nabla| T_\perp$ (in analogy to the parallel phase mixing which led to a dissipative term of the form $\nabla\cdot q_\parallel \approx n_0 v_t|\nabla_\parallel| T_\parallel$). Expanding the J_0 for small $k_\perp\rho$, we see that this is a k^4 hyperviscositylike term. This new damping term is intrinsically nonlinear, being most important when $k_\perp\rho \sim 1$ and when the electric field is large enough so that $\mathbf{v}_E\cdot k_\perp$ is comparable to the linear growth rate.

V. THE PROCEDURE FOR DERIVING THE GYROFLUID EQUATIONS

In this section we outline the general procedure for deriving gyrofluid equations by taking moments of the gyrokinetic equation and discuss some of the main issues that arise in the closure approximations and the nonlinear terms. (The complete derivation of the full set of gyrofluid equations will be presented in a later paper.)

There are two gyroaveraging operators J_0 in Eqs. (13)–(15), one is in the gyrokinetic equation itself, and the second is in the quasineutrality equation to handle the transformation from the guiding-center density to the particle density. When these two equations are linearized and solved in a shearless slab, the two J_0's combine to give a dispersion relation with coefficients of the form $\langle J_0^2\rangle_0 \equiv \int d^3v\, f_M(v) J_0^2 = \Gamma_0(b_0) = \exp(-b_0) I_0(b_0)$, where I_0 is the modified Bessel function and $b = \rho_\perp^2 k_\perp^2 = (T_\perp/M\Omega^2)k_\perp^2$. This might suggest that one should operate on the gyrokinetic equation by J_0 before deriving moment equations. However, this leads to complications in the magnetic field shear terms and in the nonlinear terms. As was pointed out by Bellew and Bakshi[21] and by Linsker,[22] $J_0[F\hat{b}(\mathbf{x})] \neq \hat{b}(\mathbf{x}) J_0 F$ in a sheared magnetic field because J_0 and $\hat{b}(\mathbf{x})$ do not commute. The missing terms are of order $k_\perp^2\rho^2$, and are as important as any other second-order gyroradius terms. A proper treatment of this leads to an important gyroaveraging of the effective $k_\parallel(x)$ which can significantly alter the radial eigenmode structure and eigenfrequency. The nonlinear complications arise from the $J_0(FJ_0\mathbf{v}_E)$ term, because the leading J_0 operates on both the F and the v_E, while the second J_0 operates only on v_E. Because of these complications, it is more natural to first take moments of the gyrokinetic equation in guiding-center variables (i.e., do not operate with the leading J_0 yet), resulting in guiding-center fluid equations involving only a single power of $\langle J_0\rangle$. Although the *guiding-center* fluid variables are the ones being advanced in time, we need to know the *particle* density to calculate the fields. One way of doing this is to approximate the integral in Eq. (15) by

$$\int d^3v\, J_0 F(\mathbf{x},\mathbf{v},t) \approx \int d^3v\, F_M(\mathbf{v}) J_0^2[\alpha(\mathbf{x},t) + \beta(\mathbf{x},t)v_\perp^2], \tag{16}$$

where α and β can then be related to the guiding-center density and temperature by the definitions $n_{gc} = \int d^3v\, F$ and $n_{gc}T_{\perp,gc} = \int d^3v\, F m v_\perp^2/2$. This ansatz for the $v_\perp$ dependence of F is suggested by the linear form of F, and is rigorously exact through second order in $k_\perp\rho$ while being well behaved for large $k_\perp\rho$.

Our general goal has been to employ Padé-like approximations which are rigorously correct through second order in $k_\perp\rho$ while still being well behaved for large $k_\perp\rho$. An example of this would be $\langle J_0^2\rangle \approx 1/(1+b) = (1-\rho_\perp^2\nabla_\perp^2)^{-1}$, which leads to Poisson-like equations which can be easily solved numerically [for example, by using fast Fourier transforms (FFT's) in one direction and a tridiagonal solver in the other]. Figure 4 compares various approximations for $\langle J_0^2\rangle$, showing that the usual Taylor-series approximation introduces large errors for $k_\perp\rho > 0.8$, while the others are better behaved. Figure 5 compares the shearless slab $\eta_{crit}(b)$ from fluid equations employing various FLR (finite Larmor radius) approximations, with the exact kinetic result.[23] By using the approximation $\langle J_0\rangle$

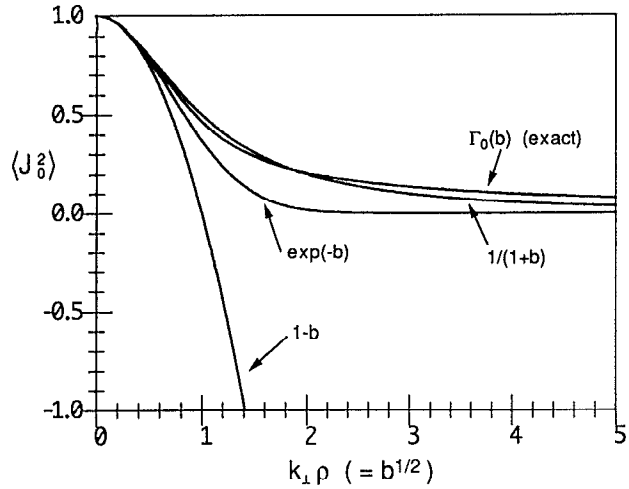

FIG. 4. Comparisons of various types of approximations of the gyroaveraging operator $\langle J_0^2 \rangle$.

$\approx \langle J_0^2 \rangle^{1/2} = \Gamma_0^{1/2}$ (which we have implemented numerically using FFT's), one actually reproduces the exact linear kinetic result. The curve labeled $\langle J_0^2 \rangle = 1 - b$ employs a second-order Taylor series approximation of the usual approach of combining the two J_0 operators, ignoring the Bakshi–Linsker shear effect (which vanishes in this shearless case).

To sketch out the procedure for deriving the gyrofluid equations, consider the slab limit of Eq. (13). Multiplying by $\int d^3v$, we produce a conservation law for the guiding-center density (we drop the gc subscript for now):

$$\frac{\partial n}{\partial t} + \nabla \cdot (n u_\parallel \hat{b}) + \nabla \cdot (n \langle J_0 \rangle \mathbf{v}_E) = 0. \tag{17}$$

Without specifying the exact form of the approximation for $\langle J_0 \rangle$, we will assume it is a is a function only of b

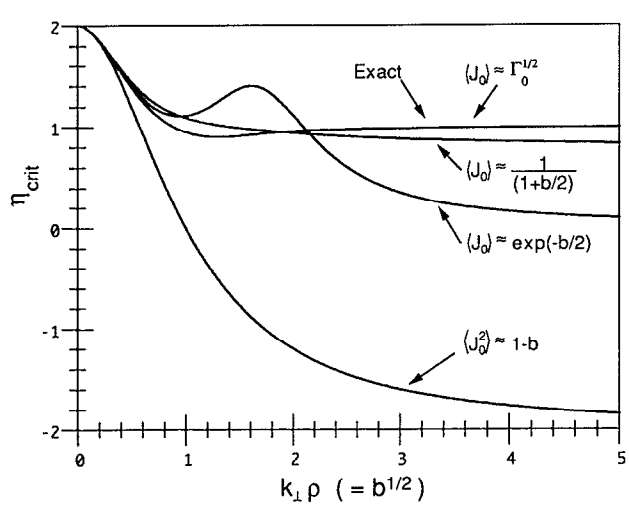

FIG. 5. Comparisons of the slab $\eta_{crit}(k_\perp \rho)$ calculated with different approximations for the gyroaveraging operators $\langle J_0 \rangle$ or $\langle J_0^2 \rangle$. The $\langle J_0 \rangle \approx \Gamma_0^{1/2}$ approximation reproduces the exact kinetic result.

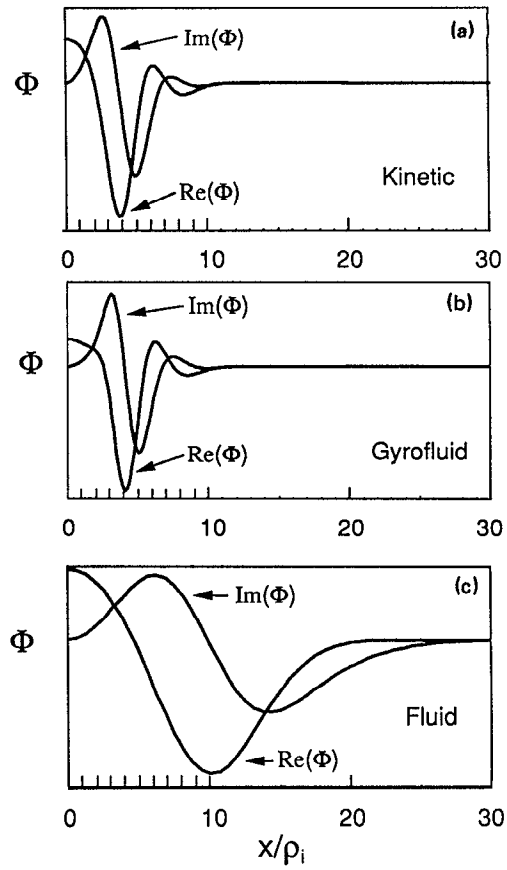

FIG. 6. Radial eigenfunctions for $\eta_i = 2.0$, $k_y \rho_i = 0.67$, $T_i = T_e$, $L_s = 40 L_n$, showing the good agreement between the exact kinetic result (a) and the gyrofluid result (b) but not the previously used fluid equations (c).

$= \rho^2 k_\perp^2$, where the gyroradius $\rho^2 = \rho_0^2 + \rho_1^2 = (T_{\perp,0} + T_{\perp,1})/(m\Omega^2)$ contains equilibrium and perturbed components. Expanding the last term in Eq. (17) in the gyrokinetic ordering leads to

$$\nabla \cdot (n \langle J_0 \rangle \mathbf{v}_E) = \nabla (n_0 + n_1) \cdot \langle J_0 \rangle_0 \mathbf{v}_E - n_0 \nabla (\rho_0^2 + \rho_1^2) \cdot \frac{\partial \langle J_0 \rangle_0}{\partial b_0} \nabla_\perp^2 \mathbf{v}_E.$$

The ∇n_0 term leads to the usual linear ω_* term. The ∇n_1 term gives the nonlinear $E \times B$ convection of the perturbed density, using a gyroaveraged $\mathbf{v}_E$. The $\nabla \rho_0^2$ leads to part of the usual linear $\eta \omega_*$ FLR corrections [the rest of the FLR corrections appear when transforming these guiding-center variables to particle variables, along the lines of Eq. (16), for comparison with the usual fluid equations which are in particle variables]. Finally, the $\nabla \rho_1^2$ term gives rise to nonlinear effects from the perturbations in the average gyroradius. Similar types of terms arise in the derivation of the other gyrofluid equations (for $u_\parallel$, $p_\parallel$, and $p_\perp$).

VI. CONCLUSIONS

Figure 6 compares the radial eigenfunction calculated from the resulting gyrofluid equations with the exact ki-

netic result (calculated with Linsker's integral-eigenvalue code[22]) and with the older fluid equations which had previously been used by a number of researchers. The gyrofluid equations agree very well with the kinetic result, while the old fluid equations predict a growth rate γ which is 3.3 times too fast and a mode width Δ which is two times too large. This produces a mixing length estimate for $\chi \sim \gamma \Delta^2$ that is about 13 times too big. (The eigenfrequencies $\omega L_n/c_s$ calculated by the various equations were gyrokinetic: $-0.086+0.033i$, gyrofluid: $-0.086+0.030i$, and fluid: $-0.275+0.11i$.) The gyrofluid equations agree much better with the kinetic result because of their more careful treatment of three main effects: (1) kinetic phase mixing (Landau damping and its inverse), (2) gyroaveraging of the electrostatic potentials (using Padé-like approximations rather than Taylor-series approximations), and (3) Bakshi and Linsker's gyroaveraging of $k_\parallel(x)$ in a sheared magnetic field.

It is hoped that by providing a more accurate treatment of these important physical effects, the gyrofluid equations will build upon the qualitative understanding of tokamak transport developed using the usual fluid equations, and enable detailed nonlinear numerical simulations to have more quantitative predictive power. In this paper we have only demonstrated the linear accuracy of the gyrofluid equations. Future papers will investigate their nonlinear accuracy, both by numerical simulations and by analytic tests, but there are several reasons to expect that these models will usually continue to work nonlinearly. The gyrofluid equations contain a number of terms resulting from the $E\times B$ nonlinearities, including nonlinear variations in the gyroradius averaging. They are based on moments of the gyrokinetic equation, and hence satisfy important nonlinear conservation laws. Our fluid model of parallel phase mixing is an accurate n-pole approximation of the velocity-integrated Green's function resulting from the linear propagator $(-i\omega + ik_\parallel v_\parallel)^{-1}$, and so should be applicable to a wide variety of problems. For example, a weak-turbulence analysis of the gyrofluid equations in various limits is underway,[24] and has demonstrated that ion Compton scattering (where the ions resonate with a beat wave) can be reproduced.

In this short paper we have given an overview of the issues involved in deriving collisionless gyrofluid equations, placing the main emphasis on our new fluid models of phase mixing. A complete derivation of the electrostatic gyrofluid equations in slab geometry is left to a future paper,[5] as is the extension to toroidal drift resonances.[6] Additional effects which need to be considered for more realistic comparisons with tokamak data include generalization to electromagnetic perturbations (which is considered in a recent paper by Brizard[20] but without our phase-mixing models and with a different choice of FLR models), collisional effects (which have been considered in the recent work of Chang and Callen,[17] see Sec. II F above for a comparison with our work), and trapped-electron effects.

ACKNOWLEDGMENTS

We want to particularly thank R. E. Waltz and R. R. Dominguez for recent collaboration to extend the gyrofluid equations to include toroidal drift resonances, and R. E. Waltz and G. D. Kerbel for helpful discussions regarding the numerical solution of these types of equations. W. W. Lee, J. V. W. Reynders, R. A. Santoro, and M. Beer have helped in comparisons between gyrokinetic and gyrofluid numerical calculations. S. Hamaguchi, W. Horton, and W. Park helped by providing a nonlinear 3-D slab fluid code which we modified to solve the gyrofluid equations. We have had useful discussions with W. M. Tang, T. S. Hahm, Liu Chen, and A. Brizard regarding the physics of the gyrokinetic equation and toroidal effects. We thank S. C. Cowley, Liu Chen, and P. Diamond for intuition regarding slab and toroidal ITG turbulence, and thank H. V. Wong for pointing out the importance of Bakshi and Linsker's gyroaveraging of the shear. We have also had useful discussions about fluid models of Landau damping with H. L. Berk, C. Oberman, Z. Chang, J. D. Callen, A. Dimits, J. Krommes, J. Glanz, and H. Biglari. Many of the calculations for this paper were done with the very useful symbolic math computer programs MATHEMATICA[25] and MAPLE.[26]

This work was supported in part by U.S. Department of Energy Contract No. DE-AC02-76CH03073.

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