Notes for Astrophysical Sciences 554: Irreversible Processes Plasmas

G. W. Hammett, et.al.
Princeton University Plasma Physics Laboratory
P.O. Box 451, Princeton, NJ 08543 USA

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Abstract

These are some notes to supplement my lectures for the second half of this course. Another major resource (from which I am heavily borrowing) is Prof. Krommes' extensive notes at
ftp://ftp.pppl.gov/pub/krommes/AST-554/notes.dvi.

1 Simple 1-D Krook Illustration of the Chapman-Enskog Method for Deriving Transport Coefficients

(I actually did the next section on Rosenbluth potentials first, but in these notes I'll start with this simpler topic first.)

1.1 Almost-trivial case: Krook model conserving only particles

We will eventually do a more complete treatment of the Braginskii-Chapman-Enskog equations for a plasma with magnetic fields, but we start with this very simple (or even trivial) limit to illustrate the essential ideas with a minimum of algebra.

Start with a simple 1-D kinetic equation for a gas (no magnetic or electric field) with a simple number-conserving Krook operator (also known as a BGK model or Bhatnagar-Gross-Krook model):

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\nu(f - f_M \int dv f)$$

[In my senior year in college in 1980, I took an applied math course from Prof. Krook, who was at that time a friendly elderly man. Which is not to say he wasn't friendly as a young man also.] Here $f_M = \exp(-v^2/(2v_t^2))/\sqrt{2\pi v_t^2}$ is a stationary Maxwellian with unit density. This Krook model collision operator causes f to relax to a stationary Maxwellian of a fixed temperature, i.e., it doesn't conserve momentum

or energy. This is actually an appropriate model for a case where f is for some trace species which is colliding with some background thermal bath of particles. The momentum lost by these trace particles goes into the much larger bath of background particles, with which the trace particles eventually come into thermal equilibrium. We consider this as a "trace species" so that the momentum or energy lost from the trace species (a.k.a. 'test-particles") causes a negligible change in the momentum and energy of the background bath species.

(?? A somewhat more standard notation would be to absorb the density into the definition of $f_M = n \exp(-v^2/(2v_t^2))/\sqrt{2\pi v_t^2}$, with $n = \int dv f$. I could rewrite all the notes to use this form.)

Integrating the kinetic equation over all velocity gives:

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0$$

where $n = \int dv f$, and $nu = \int dv f v$. To calculate how the density n evolves in time, we need to know the flow u. We could operate on the kinetic equation with $\int dv v$ and get an equation for $\partial(nu)/\partial t$, but that would require knowing the $\int dv f v^2$ moment. This gives rise to an infinite chain of moment equations, i.e. to a "closure problem" which occurs in many contexts.

In the high collisionality limit, the Chapman-Enskog procedure provides a rigorous asymptotic method for truncating the chain of equations and deriving a closure for a higher moment in terms of lower moments. To make the ordering expansion we will be doing a little clearer, introduce the expansion parameter ϵ into the kinetic equation as

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = -\frac{\nu}{\epsilon}(f - f_M \int dv f) \tag{1}$$

and expand $f = f_0 + \epsilon f_1 + \ldots$. [The parameter ϵ is just used to help keep track of the order of various terms, and in the end we can set ϵ to 1.] To lowest order as $\epsilon \to 0$, we have:

$$0 = -\frac{\nu}{\epsilon}(f_0 - f_M \int dv f)$$

so $f_0 = f_M n = f_M(v) n(x,t)$. Note that the velocity dependence of f_0 is determined, but the density is an arbitrary function of time and space at this order, and will not be determined until higher order equations. (Also note that I've kept the full f (instead of expanding it) in the velocity integral in the collision operator, so n contains the full density. If I had just used f_0, then the density would still be arbitrary, but higher order components might contain density also?? Using the full f as I've done here is a better analog to the particle conserving properties of the full collision operators involving conservative derivatives. Perhaps I am belaboring this point.)

The next order equation is

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial x} = -\nu f_1$$

substituting $f_0 = F_M(v)n$ and using the fluid equation $\partial n/\partial t = -\partial(nu)/\partial x$, we get

$$-f_M \frac{\partial(nu)}{\partial x} + f_M v \frac{\partial n}{\partial x} = -\nu f_1 \tag{2}$$

However,

$$\begin{aligned} nu &= \int dv v f \\ &= \int dv v (f_0 + \epsilon f_1 + \ldots) \\ &= 0 + \epsilon \int dv v f_1 + \ldots \end{aligned}$$

and so the nu term in Eq. (2) is higher order (i.e., the lowest order equation for the density is $\partial n/\partial t = 0 + \mathcal{O}(\epsilon)$. Then Eq. (2) simplifies to lowest order to give

$$-\nu f_1 = f_M v \frac{\partial n}{\partial x}$$

This is sometimes called the correction equation. With a more rigorous integro-differential collision operator, it is more complex to invert the collision operator to solve for f_1, though with the Krook model used here the inversion is trivial. We can now solve to find the particle flux

$$\begin{aligned} nu &= \int dv v f_1 \\ &= -\frac{1}{\nu} \int dv f_M v^2 \frac{\partial n}{\partial x} \\ &= -D \frac{\partial n}{\partial x} \end{aligned}$$

where the diffusion coefficient

$$D = \frac{v_t^2}{\nu} = \nu \lambda_{\text{mfp}}^2$$

has the proper dimensions of a random-walk diffusion coefficient, and can be clearly understood from random-walk arguments where the step size is λ_{mfp} and the step time is $1/\nu$. [The mean free path is defined by $\lambda_{\text{mfp}} = v_t/\nu$.]

Now that all of the higher moments (in this case just nu) have been completely specified in terms of lower moments ($nu = -D\partial n/\partial x$), we have a closed set of equations (just a single equation actually) that can be solved to find the evolution of density over time:

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2}$$

It is important to remember the fundamental ordering assumptions being used here: time scales slow compared to a collision frequency $(\partial/\partial t)/C \ll 1$ and mean

free paths short compared to gradient scale lengths $(v_t\partial/\partial x)/\nu \sim \lambda_{\rm mfp}/L \ll 1$. In a strong magnetic field, the restrictions on the gradients perpendicular to the field can be relaxed somewhat, and only have to long compared to the gyroradius, not the mean free path. You should read the section on p. 38-39 of the NRL, which is a nice summary of the assumptions in Braginskii.

1.2 Extension to a momentum-conserving Krook operator

Now extend the procedure of the previous section to include a particle and momentum conserving Krook operator:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = C(f) = -\frac{\nu}{\epsilon}(f - nf_M) \tag{3}$$

where $n(x,t) = \int dv v f$ and f_M is now a shifted Maxwellian

$$f_M = \frac{1}{\sqrt{2\pi v_t^2}} e^{-\frac{(v-u(x,t))^2}{2v_t^2}}$$

and $nu = \int dv f v$ so that nf_M contains the same amount of momentum and particles as f. (?? Could add a sketch of $f(v)$ and $f_0(v)$, showing that they contain the same density and average momentum, and that $f_1(v) = f(v) - f_0(v)$ has no net density or momentum.) The first two fluid moment equations of Eq. (3) are

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0$$

$$mn\frac{du}{dt} = mn\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x}$$

we keep 2 moment equations (for particle and momentum density) in this section, while in the previous section we only kept 1 equation because in this section our collision operator has two conserved quantities while in the previous section it had only 1. It is not until the pressure moment that this collision operator can start to have any effect, which is calculated by the "correction equation" for f_1 below and used to provide a closure for p. (This is related to the projection operation that Krommes talks about: evaluating the collision operator $g = Cf$ leads to a g which has no density or momentum, i.e., it is a projection onto a a subspace of all possible $g(v)$. In other problems with similarities to this, some people refer to annihilation operators.)

As before, expanding $f = f_0 + \epsilon f_1$, to lowest order we have $f = f_0 = nf_M$. To next order we have

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial x} = -\nu f_1$$

Use

$$\frac{\partial f_0}{\partial t} = \frac{1}{n}\frac{\partial n}{\partial t} f_0 + \frac{(v-u)}{v_t^2}\frac{\partial u}{\partial t} f_0$$

Substituting the fluid equations for $\partial n/\partial t$ and $\partial u/\partial u$, and using the lowest order expression for the pressure $p = \int dv f m(v-u)^2 = \int dv f_0 m(v-u)^2 = nT_0$ (where $T_0 = mv_t^2$), after a little bit of algebra one can get a closed "correction equation" which defines f_1 in terms of gradients of n and u. Plugging this expression for f_1 in to find the next order corrections to the pressure, and carrying out a bit of algebra (I'm getting tired of typesetting the algebra), in particular using the useful i.d. in 1 dimension

$$\langle v^{2n} \rangle \equiv \frac{1}{\sqrt{2\pi v_t^2}} \int dv e^{-\frac{v^2}{2v_t^2}} v^{2n} = v_t^{2n}(2n-1)!!$$

(where the double factorial $n!! = n(n-2)(n-4)\cdots(3)(1)$ for odd n), eventually leads to the result:

$$p = \int dv m(v-u)^2(f_0 + \epsilon f_1) = nT_0 - \epsilon nm \frac{2v_t^2}{\nu} \frac{\partial u}{\partial x}$$

and the final momentum equation is

$$mn\frac{du}{dt} = -T_0 \frac{\partial n}{\partial x} + \frac{\partial}{\partial x}\left(nm\eta_0 \frac{\partial u}{\partial x} \right)$$

where the viscosity $\eta_0 = 2v_t^2/\nu$ again makes senses as a random walk diffusion coefficient. But note that the density equation is still

$$\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} = 0$$

i.e., there is no diffusion directly in this equation. There is particle transport, but it is hidden self-consistently in the higher moment equations.

To study the magnitude of the transport associated with this viscosity, one can linearize the equations for small amplitude perturbations of the form $n = n_0 + n_1(t)\cos kx$, and find that the mode frequency is

$$\omega = \frac{-i\eta_0 k^2 \pm \sqrt{-\eta k^2 + 4v_t^2 k^2}}{2}$$

Thus the density gradients decay away in time, as they would for diffusion, but the damping is combined with sound wave phenomena.

1.3 Higher-order corrections, "Burnett equations"

This section is advanced or speculative material that can/should be skipped.

Here we try to carry out this procedure to second order in ϵ. We run into troubles, which might be related to difficulties in the higher order "Burnett equations" that are sometimes discussed in the literature. The steps I take here I'm guessing are conceptually similar to the derivation of the Burnett equations, though here we use a simple Krook collision operator that only conserves particles.

We go back to Eq. (1), with the simple Krook model that conserves only particles, and make the substitution $f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \ldots$ (except in the last $\int dv f \equiv n$ term of the Krook collision operator, which we keep to all orders to simplify some matters). Use the lowest order result that $f_0 = f_M(v)n(x,t)$, and we are left with

$$\frac{\partial(f_0 + \epsilon f_1 + \ldots)}{\partial t} + v\frac{\partial(f_0 + \epsilon f_1 + \ldots)}{\partial x} = -\nu(f_1 + \epsilon f_2 + \ldots) \tag{4}$$

Substituting $f_0 = nf_M$ and using $\partial n/\partial t = -\partial(nu)/\partial x = -\partial(\int dv f v)/\partial x$, gives through first order in ϵ:

$$-f_M\frac{\partial}{\partial x}\int dv v\epsilon f_1 + \epsilon\frac{\partial f_1}{\partial t} + v\frac{\partial n}{\partial x}f_M + \epsilon v\frac{\partial f_1}{\partial x} = -\nu(f_1 + \epsilon f_2 + \ldots) \tag{5}$$

The terms independent of ϵ give the results $f_1 = -(v/\nu)F_M\partial n/\partial x$ and the first order flux $(nu)_1 = \int dv v f_1 = -D\partial n/\partial x$, as we got in the previous section. Substituting this expression for f_1 into the terms in Eq. (5) proportional to ϵ^1 give the result

$$\nu^2 f_2 = v f_M\frac{\partial^2 n}{\partial t\partial x} + (v^2 - v_t^2)f_M\frac{\partial^2 n}{\partial x^2} \tag{6}$$

Integrating to find the second order flux gives $(nu)_2 = \int dv v f_2 = (v_t/\nu)^2\partial^2 n/\partial t\partial x$, and using this in the closure for the density equation gives:

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} - \frac{v_t^2}{\nu^2}\frac{\partial}{\partial t}\frac{\partial^2 n}{\partial x^2}. \tag{7}$$

With a little rearranging to

$$\left(1 + \frac{v_t^2}{\nu^2}\frac{\partial^2}{\partial x^2}\right)\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} \tag{8}$$

And we see that this just doesn't look well behaved. For example, Fourier-transform in x:

$$\frac{\partial n_k}{\partial t} = -\frac{Dk^2}{1 - \lambda_{\rm mfp}^2 k^2}n_k \tag{9}$$

There is a singularity at $k\lambda_{\rm mfp} = 1$, and modes with $k\lambda_{\rm mfp} > 1$ are actually unstable (with the growth rate ν at large k). One might try to ad-hoc patch up these equations by noting that, to the extent that $k^2\lambda_{\rm mfp}^2 \ll 1$ is the fundamental Chapman-Enskog ordering assumption, then one can approximate the denominator in this expression by

$$\frac{\partial n_k}{\partial t} = -Dk^2(1 + \lambda_{\rm mfp}^2 k^2)n_k \tag{10}$$

which after Fourier transforming back leads to a better behaved hyperdiffusion term:

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} - D\lambda_{\rm mfp}^2\frac{\partial^4 n}{\partial x^4} \tag{11}$$

I suppose another way of trying to rationalize this is to say it is equivalent to just substituting the lowest order result $\partial n/\partial t = D\partial^2 n/\partial x^2$ into the last term of Eq. (7). Still, the fact we had to make such modifications seems a bit strange. The papers mentioned in the following discussion might help clarify the situation.

To summarize and comment on the higher order corrections:

At zeroth order (ignoring the transport coefficients), the fluid equations are the dissipationless "Euler equations". The Chapman-Enskog procedure is usually carried out to first order in ν/ω or $\lambda_{\rm mfp}/L$, yielding the standard viscosities etc. that are in the Navier-Stokes fluid equations (or in Braginskii's fluid equations for plasmas). Attempts to work to second order in Kn= $\lambda_{\rm mfp}/L$ (the "Knudson number"), lead to what are usually called the Burnett equations or variations thereof. Some papers have reported fundamental difficulties with the Burnett equations (such as violation of the second-law of thermodynamics due to negative dissipation or a heat flux in an isothermal gas), or problems with the higher-order boundary conditions required, while other papers have reported improved results with the Burnett equations in some regimes or ways to fix the Burnett equations (see for example D.W. Mackowski, et.al., Phys. Fluids 11, 2108 (1999), "Comparison of Burnett and DSMC predictions of pressure distributions and normal stress in one-dimensional, strongly nonisothermal gases", and references therein, or "Numerical Simulation of the flow around a flying vehicle with high speed at high altitude", K.L. Guo and G.S. Liaw, (http://library.redstone.army.mil/hsvsim/papers/hsc013.pdf) and the papers by Balakrishnan and others they cite, and other recent papers).

Recent papers have been written on these topics, motivated in part by the growing importance of longer mean-free-path effects in various applications (such as micromachinery, space shuttle reentry, thermal transport in the vapors used for crystal growth, etc.).

The Chapman-Enskog procedure is an asymptotic expansion, and, as is well known in the theory of asymptotics, asymptotic series are not necessarily guaranteed to converge as more terms are added at a fixed value of the expansion parameter (it is only guaranteed that a finite number of terms in the expansion will converge to the right answer in the limit as the expansion parameter is taken to its limit). In fact, sometimes keeping more higher-order terms causes the approximation to get worse if the expansion parameter isn't small enough. (Give a simple example??:)

Instead, it seems to me that an expansion procedure is needed that can robustly interpolate between the short and long mean-free-path limits. The Landau-fluid closure models that I and others have worked on are designed to handle the long mean-free-path limit, and extensions to include collisions can then interpolate smoothly between the two regimes with a robust Padé-type of approximation (see for example Eq. 51 in "Landau fluid models of collisionless magnetohydrodynamics," P.B. Snyder, G.W. Hammett, and W. Dorland, Phys. Plasmas 4, 3974 (1997) or sections in Stephen Smith's thesis or papers, or "Transport theory in the collisionless limit", R.D. Hazeltine, Phys. Plasmas 5, 3282 (1998)). In k space, the crudest 1-moment Landau-fluid

model is of the form

$$\frac{\partial n_k}{\partial t} = -\frac{v_t^2 k^2}{\nu + v_t|k|} n_k$$

Comparing with Eq. (10), we see that this is better behaved, and provides a smooth transition between the collision dominated result at small k, and damping at a phase-mixing rate $\sim v_t|k]$ at high k.

Of course my work has focused on plasmas in the strong magnetic field limit with a 4 or 6-moment method, while in a neutral gas the formulation would be closer to Grad's 13 moment approach (modified to include phase-mixing/Landau-damping types of terms). (Others who have worked on such Landau fluid closures include Callen and Chang, Dorland, Beer, Waltz, Smith, Snyder, Mattor and Parker, etc. and recent papers by Sugama et.al.).

These Landau-fluid approximations work well and are useful for some problems, but may have inaccuracies for certain types of problems. To be really accurate in a longish mean-free-path regime, one should either use fully kinetic treatments, such as particle-in-cell methods or phase-space continuum or "Vlasov" methods.

2 From the Landau Collision Operator to the Rosenbluth Potential Form

This mostly followed Krommes' Chapter 14 Sec. VI.

The Rosenbluth Potential form of the Collision operator has the advantage that one can use the results of potential theory (from gravitation and electrostatics) to greatly simplify the integrals if the species being collided with is spherically symmetric in velocity space. In particular it makes it easy to show that if one is colliding with a Maxwellian species, the collision operator reduces to the form on p. 36 of the NRL Plasma Formulary (the section on "Fokker-Planck Equation"). This Maxwellian limit makes clear the physics of a (directed velocity) slowing down term, a pitch-angle scattering term, and an energy diffusion term, and is an easier way to prove the formulas in the NRL for these various rates.

To summarize the resulting formulas:

The Rosenbluth potential form derived in class (which is slightly different from the form in the Formulary, where the order of one of the derivatives has been interchanged) can be written as:

$$\left(\frac{\partial f_a}{\partial t}\right)_{\text{coll}} = \sum_b C(f_a, f_b) = -\sum_b \frac{\partial}{\partial \vec{v}} \cdot \mathbf{J}^{a/b}$$

(note that the NRL convention used here for the sign of C is the reverse of Krommes', and note that where the NRL uses $\alpha\backslash\beta$ subscripts, I use a/b subscripts, which is faster to type), where

$$\mathbf{J}^{a/b} = \nu_0^{a/b} v^3 \left[\frac{m_a}{m_b}(\nabla_{\mathbf{v}} h_b) f_a - \frac{1}{2}(\nabla_{\mathbf{v}}\nabla_{\mathbf{v}} g_b) \cdot \nabla_{\mathbf{v}} f_a\right]$$

where

$$\nu_0^{a/b} = \frac{4\pi e_a^2 e_b^2 \log\Lambda_{ab}\, n_b}{m_a^2\, v^3},$$

(as defined in the NRL Plasma Formulary) and

$$\nabla_{\mathbf{v}}^2 h_b = -4\pi f_b, \qquad \nabla_{\mathbf{v}}^2 g_b = 2h_b, \qquad \int f_b\, d^3\mathbf{v} = 1.$$

Note that in MKS units, $\nu_0^{a/b}$ is:

$$\nu_0^{a/b} = \frac{e_a^2 e_b^2 \log\Lambda_{ab}\, n_b}{4\pi\epsilon_0^2 m_a^2 v^3},$$

If f_b is spherically symmetric, h_b and g_b are found by straightforward integration. For example, if f_b is a Maxwellian with temperature T_b then

$$\frac{dh_b}{dv} = -\frac{\psi(x^{a/b})}{v^2} = -\frac{1}{v^2}\frac{\nu_s^{a/b}}{(1+m_a/m_b)\nu_0^{a/b}}$$

where $x^{a/b} = v_a^2/(2T_b/m_b)$. (A minor point: This is the definition of $x^{a/b}$ as given on p. 31 of the NRL formulary, and the a subscript reminds us which velocity to use. In our context, or when used in the operator on NRL p. 34 for collisions with Maxwellian species, $v_a = v$, where $\vec{v}$ is the argument of f_a.) We can express $\psi(x)$ in the NRL form or in a form using the standard error function

$$\psi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, t^{1/2} e^{-t} = \mathrm{erf}(\sqrt{x}) - \sqrt{x} \frac{2}{\sqrt{\pi}} e^{-x}$$

where

$$\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-v^2)\, dv,$$

As a useful reference (from notes by Karney), we collect here some of the equations in terms of the error function (instead of the $\psi(x)$ function the NRL uses):

$$\begin{aligned}
\frac{dh_b}{dv} &= -\frac{1}{v^2}\Big(\mathrm{erf}(u) - u\,\mathrm{erf}'(u)\Big) = -\frac{1}{v^2} \frac{\nu_s^{a/b}}{(1 + m_a/m_b)\nu_0^{a/b}} \\
\frac{dg_b}{dv} &= \frac{1}{2}\left(\left(2 - \frac{1}{u^2}\right)\mathrm{erf}(u) + \frac{\mathrm{erf}'(u)}{u}\right) = \frac{1}{2}\frac{\nu_\perp^{a/b}}{\nu_0^{a/b}} \\
\frac{d^2 g_b}{dv^2} &= \frac{1}{v}\left(\frac{\mathrm{erf}(u)}{u^2} - \frac{\mathrm{erf}'(u)}{u}\right) = \frac{1}{v}\frac{\nu_\parallel^{a/b}}{\nu_0^{a/b}} \\
\mathrm{erf}(u) &= \frac{2}{\sqrt{\pi}} \int_0^u \exp(-x^2)\, dx, \qquad \mathrm{erf}'(u) = \frac{2}{\sqrt{\pi}} \exp(-u^2) \\
u &= v/\sqrt{2T_b/m_b}
\end{aligned}$$