AS554 Irreversible Processes in Plasmas
April 13, 2001

Problem Set #6 (due Apr. 16, 2001)

Question 1: Chapman-Enskog-Braginskii procedure

In class I wrote down the first order kinetic correction equation

$$\frac{\partial f_0}{\partial t} + \vec{v} \cdot \frac{\partial f_0}{\partial \vec{x}} + \frac{q}{m} \vec{E}_* \cdot \frac{\partial f_0}{\partial \vec{v}} = -\Omega(\vec{v} - \vec{u}) \times \hat{b} \cdot \frac{\partial f_1}{\partial \vec{v}} + \hat{C} f_1$$

where $\hat{C}$ is the linearized collision operator, Ω is the cyclotron frequency, $\hat{b} = \vec{B}/B$ is the direction of the magnetic field, and $\vec{E}_* = \vec{E} + \vec{u} \times \vec{B}/c$ (the electric field with the dominant $\vec{E} \times \vec{B}$ drift component taken out). f_0 is a Maxwellian specified by the density, flow, and temperature $n(\vec{x}, t)$, $\vec{u}(\vec{x}, t)$, and $T(\vec{x}, t)$. Expand $\partial f_0/\partial t$ and $\partial f_0/\partial \vec{x}$ explicitly in terms of gradients of n, $\vec{u}$, and T. Then use the lowest order fluid equations (i.e., Braginskii's fluid equations but with the closure terms $\vec{q}$ and $\vec{\vec{P}}$ set to zero, since they vanish for a Maxwellian and are thus higher order, and also neglect the collisional friction and heating terms $\vec{R}_i$ and Q_i as they are weak for $i - e$ collisions) to eliminate the time derivatives. Show that the resulting equation can be written as:

$$\begin{aligned} -\Omega \delta\vec{v} \times \hat{b} \cdot \frac{\partial f_1}{\partial \vec{v}} + \hat{C} f_1 = {} & f_0 \left(\frac{m}{2} \frac{|\delta\vec{v}|^2}{T} - \frac{5}{2} \right) \delta\vec{v} \cdot \nabla \ln T \\ & + f_0 \frac{m}{2T} \left(\delta\vec{v}\delta\vec{v} - \frac{|\delta\vec{v}|^2 \vec{\vec{1}}}{3} \right) : \vec{\vec{W}} \end{aligned}$$

where $\vec{\vec{W}} = \nabla\vec{u} + (\nabla\vec{u})^{\mathrm{T}} - (2/3)\vec{\vec{1}}\nabla \cdot \vec{u}$, and $\delta\vec{v} = \vec{v} - \vec{u}(\vec{x}, t)$. [This result is equivalent to Krommes' Ch.31 Eq. 34 generalized to include a magnetic field, Krommes' Ch. 33 Eq. 54, or Braginskii's Eq. 4.15.]

Consider the simple limit of a Krook model collision operator $\hat{C} f_1 = -\nu f_1$, no magnetic field, and no gradients in the flow so $\vec{\vec{W}} = 0$. Calculate the heat flux $\vec{q} = \int d^3v f_1 0.5 m |\delta\vec{v}|^2 \delta\vec{v}$ and find the result in the form $\vec{q} = -n\chi\nabla T$. What value should ν have to match Braginskii's result for the ion heat flux in the $\vec{B} = 0$ limit, on p.38 of the Formulary?

Question 2

Using the Rosenbluth potentials, evaluate the collisional velocity-space flux $\mathbf{J}^{a/b}$, when f_b is a Maxwellian. Express in terms of the relaxation rates $\nu_s^{a/b}$, $\nu_\parallel^{a/b}$, and $\nu_\perp^{a/b}$ (as defined in the NRL Plasma Formulary). Put the answer in a form similar to the one in the Formulary (under Fokker–Planck Equation) (but note that the answer in older editions of the Formulary is wrong!). (I did some of this in class but only for the slowing down $\nu_s^{a/b}$. You should complete the calculation and do it for all terms.)

Useful formulas:

The Rosenbluth potential form derived in class (which is slightly different from the form in the Formulary, where the order of one of the derivatives has been interchanged) can be written as:

$$\mathbf{J}^{a/b} = \nu_0^{a/b} v^3 \left[\frac{m_a}{m_b} (\nabla_{\mathbf{v}} h_b) f_a - \frac{1}{2} (\nabla_{\mathbf{v}} \nabla_{\mathbf{v}} g_b) \cdot \nabla_{\mathbf{v}} f_a \right]$$

where

$$\nu_0^{a/b} = \frac{4\pi e_a^2 e_b^2 \log \Lambda_{ab}\, n_b}{m_a^2\, v^3},$$

(as defined in the NRL Plasma Formulary) and

$$\nabla_{\mathbf{v}}^2 h_b = -4\pi f_b, \qquad \nabla_{\mathbf{v}}^2 g_b = 2h_b, \qquad \int h_b\, d^3\mathbf{v} = 1.$$

Note that in MKS units, $\nu_0^{a/b}$ is:

$$\nu_0^{a/b} = \frac{e_a^2 e_b^2 \log \Lambda_{ab}\, n_b}{4\pi \epsilon_0^2 m_a^2 v^3},$$

If f_b is spherically symmetric, h_b and g_b are found by straightforward integration. For example, if f_b is a Maxwellian with temperature T_b then

$$\frac{dh_b}{dv} = -\frac{\psi(x^{a/b})}{v^2} = -\frac{1}{v^2} \frac{\nu_s^{a/b}}{(1 + m_a/m_b)\nu_0^{a/b}}$$

where $x^{a/b} = v_a^2/(2T_b/m_b)$ and

$$\psi(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, t^{1/2} e^{-t} = \mathrm{erf}(\sqrt{x}) - \sqrt{x} \frac{2}{\sqrt{\pi}} e^{-x}$$

in terms of the error function

$$\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-v^2)\, dv,$$

Question 3

Evaluate this expression for $J^{a/b}$ in a limit appropriate for $i-e$ collisions, $m_a \gg m_b$ (and $x^{a/b} \ll 1$), and ignore collisions with other ions. This is the limit of "Brownian motion", where massive particles (such as dust particles in air or particles in a colloidal suspension) move in response to collisions with much lighter particles. Show that the resulting collision operator can be written in the form

$$C_{ie}(f_i) = \frac{\partial}{\partial \vec{v}} \cdot \left(\nu_{ie} \vec{v} f_i + D \frac{\partial f_i}{\partial \vec{v}} \right)$$

where the velocity space diffusion coefficient satisfies the Einstein relation $D = \nu_{ie} T_e / m_i$. (In the previous homework on the slowing down of beams, the velocity space diffusion due to $i-e$ collisions was neglected since pitch angle scattering by $i-i$ collisions is the dominant process.)

Using this expression for the $i-e$ collision operator in

$$\frac{\partial f_i}{\partial t} = C_{ii}(f_i) + C_{ie}(f_i)$$

calculate the rate of temperature equilibration, given ions and electrons initially at temperatures T_i and T_e. You can assume that $i-i$ collisions (which are $\sqrt{m_i/m_e}$ stronger than $i-e$ collisions) are strong enough to force f_i to always be a Maxwellian. Compare your result with Braginskii's ion heating rate Q_i as given on p.37 of the Formulary. In what limit will the more general expression for thermal equilibration on p. 34 of the Formulary agree with your calculation?