Introduction to Plasma Physics:

A 1-hour taste* of key concepts & results for astrophysics graduate students

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AST541: Seminar in Theoretical Astrophysics

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* Can't possibly cover all interesting topics in plasma courses AST55n…, General Plasma Physics I & II, Waves, Irreversible Processes, Turbulence…

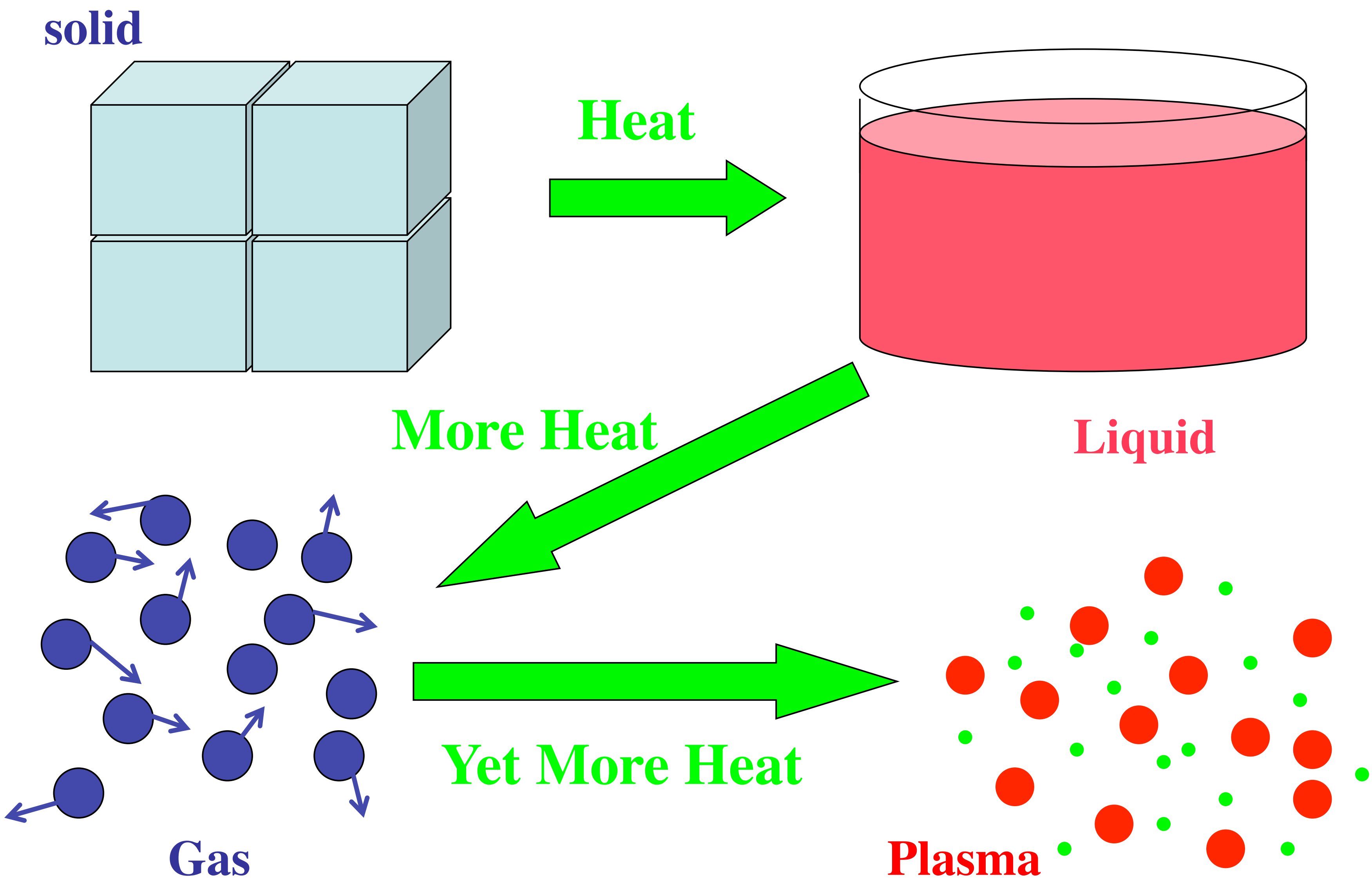
* Can't possible cover all of these slides: some skipped, some briefly skimmed…

acknowledgements: some slides borrowed from Profs. Stone, Fisch, others

Introduction to Plasma Physics

- Fundamentals of plasmas,
  - 4th state of matter
  - weak coupling between pairs of particles, but
  - strong collective interactions: Debye shielding, electron plasma oscillations
- Fundamental Length & Time Scales
  - Debye length, mean free path, plasma frequency, collision frequency
  - hierarchy of length/ & time scales, related to fundamental plasma parameter:
    Λ = # of particles in a Debye sphere
- Single Particle Motion:
  - ExB, grad(B), other drifts, conservation of adiabatic invariant μ, magnetic mirrors
- Kinetic starting point: Vlasov/Boltzmann equation
- MHD Eqs.
  - Braginskii/Chapman-Enskog fluid equations
  - approximations made in getting MHD, properties of MHD
  - Flux-freezing
  - Alfven waves
- Collective kinetic effects: Plasma waves, wave-particle interactions, Landau-damping

Plasma--4th State of Matter



Standard Definition of Plasma

- "Plasma" named by Irving Langmuir in 1920's

- The standard definition of a plasma is as the 4th state of matter (solid, liquid, gas, plasma), where the material has become so hot that (at least some) electrons are no longer bound to individual nuclei. Thus a plasma is electrically conducting, and can exhibit collective dynamics.

- I.e., a plasma is an ionized gas, or a partially-ionized gas.

- Implies that the potential energy of a particle with its nearest neighboring particles is weak compared to their kinetic energy (otherwise electrons would be bound to ions). → Ideal "weakly-coupled plasma" limit. (There are also more-exotic strongly-coupled plasmas, but we won't discuss those.)

- Even though the interaction between any pair of particles is typically weak, the collective interactions between many particles is strong.
  2 examples: Debye Shielding & Plasma Oscillations.

States of Matter

1. Just an approximation, not a material property.

2. Depends on time scales, space scales, and physics of interest (is gravel a solid or a liquid?)

Broader Definition of Plasma

- The electron temperature needs to be above ~0.3-1 eV in order to have most hydrogen ionized in thermal equilibrium. However, at lower temperatures can have weakly ionized plasmas (where plasma effects are still important), single species plasmas (pure electrons or pure ions, so there is no recombination), or non-equilibrium plasmas (at low density it takes a long time to recombine).

- Single-species non-neutral plasmas include intense charged particle beams where the self-interactions of the beam become important relative to external forces.

- A broader definition of a plasma could include matter which is electrically conducting even if the weak-coupling approximation doesn't hold. There are "strongly-coupled plasmas", "plasma crystal" states….

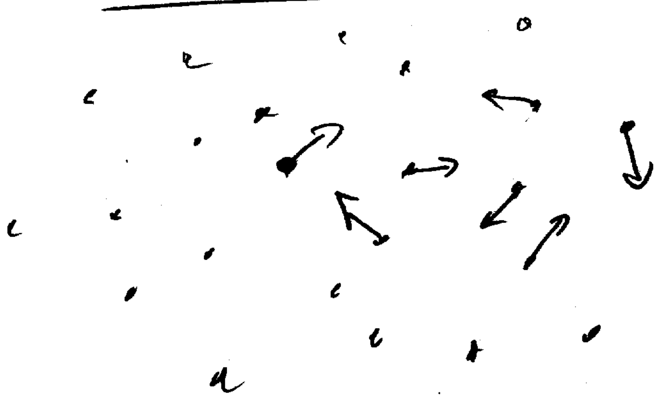
- Unconventional plasma at extreme conditions involving collective effects through the strong nuclear force and not just electric forces: quark-gluon plasma ("Big Bang Goo", NYT headline for article on RHIC, by J. Glanz, plasma physicist turned journalist).

- However, here we will focus on the conventional or ideal limit of "weakly-coupled plasmas"

Interesting Phenomena in Plasmas

- Electrically conducting: try to short out electric fields & give quasineutrality
- Low resistivity → can often approximate plasma as in ideal conducting fluid → magnetic fields are trapped in plasmas.
  Frozen-flux → magnetic fields and plasmas move together.
- Except when they don't: breaking magnetic field lines by reconnection
- Dynamo mechanisms can generate magnetic fields
- Thermal conduction & viscosity very different parallel & perpendicular to magnetic field.
- Collective effects: various types of wave-particle interactions can occur
  - Landau damping: damping of electric fields in a conservative Hamiltonian system: waves lose energy to particles (2010 Fields medal)
  - Particle acceleration mechanisms (cosmic rays)
  - Kinetic instabilities: particles put energy into waves (collisionless shocks)
- All the interesting phenomena of neutral fluid dynamics: instabilities, shocks, turbulence, & chaos, plus electric and magnetic fields (with some surprising differences: e.g., Magneto-Rotational Instability in accretion disks).

(Standard) Plasmas are Weakly Coupled



Electron density $n \approx n_e \approx n_i$

Typical interparticle spacing $L_1 \sim n^{-1/3}$

(i.e., a box of size L_1^3 contains 1 particle on average, $L_1^3 n = 1$).

Interaction between nearest neighbors is typically weak:

$$\frac{\text{Typical Potential Energy of nearest neighbors}}{\text{Typical Kinetic Energy " " "}}$$

$$= \boxed{\frac{\text{P.E.}}{\text{K.E.}} \sim \frac{\frac{e^2}{L_1}}{\frac{1}{2}mv^2} \approx \frac{e^2 n^{1/3}}{T} \ll 1}$$

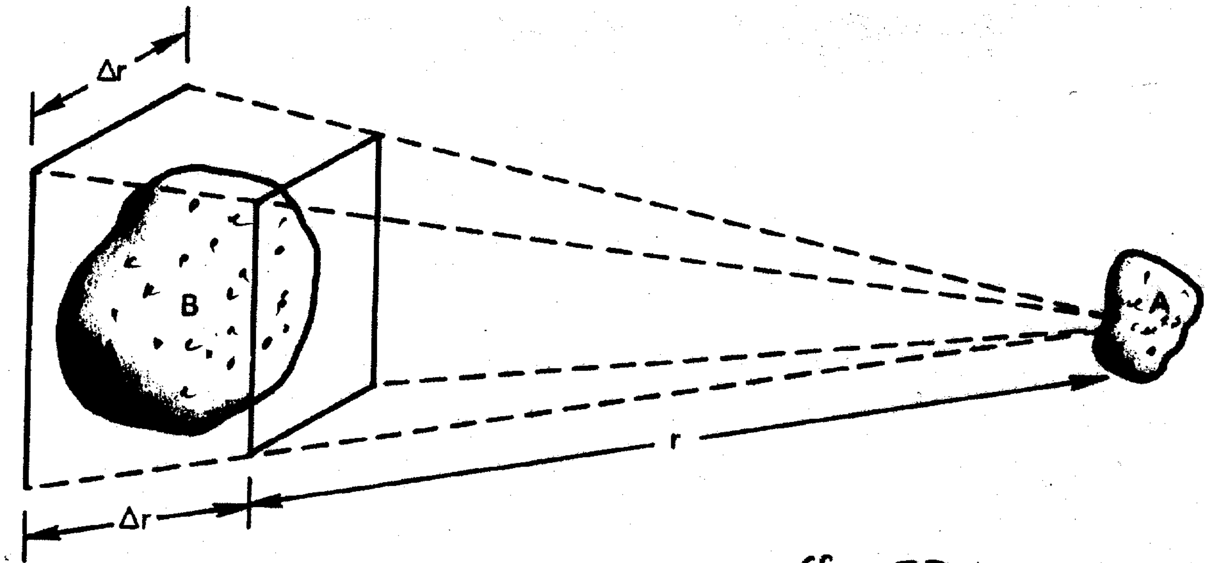
for a 'standard' ("weakly-coupled") plasma.

(Plasma literature measures T in energy units, so Boltzmann's constant $k_B = 1$.)

To be a plasma, must be sufficiently hot and/or sufficiently rarified.

	MFE	SW Solar Wind @ 1 AU (satellite measurements)	GC Galactic Center @ Bondi Accretion Radius. (measured by Chandra X-ray telescope.)
n_e (cm^{-3})	10^{14}	10	10^{2}
T_e	10 keV	10 eV	2 keV
$\frac{P.E.}{K.E} \sim \frac{e^2 n^{1/3}}{T}$	10^{-6}	10^{-8}	10^{-10}

Collective Long Range Forces Dominate



(from F.F. Chen, Intro to Plasma Physics)

FIGURE 1-1 Illustrating the long range of electrostatic forces in a plasma.

$$\text{Force} \propto \underset{\sim}{E} \propto \frac{1}{r^2} * (\#\text{ of particles})$$

$$\propto \frac{1}{r^2} * (\Delta r)^3$$

For fixed solid angle $\dfrac{\Delta r}{r} \approx \text{const.}$

$$\propto r$$

Many particles far away dominate over few particles near by.
(Mixture of electrons + ions, w/ Debye shielding, will reduce this.)

Don't need to know exactly where the distant particles are with great precision. Sufficient to treat them as an averaged, smooth charge density & current density (a kind of smeared-out cloud), instead of discrete particles.

$$n_e(\underset{\sim}{x}, t) = \int d^3v \, \underbrace{f_e(\underset{\sim}{x}, \underset{\sim}{v}, t)}$$

electron density

("fluid theory")

electron distribution function

("kinetic theory")

$f_e(\underset{\sim}{x}, \underset{\sim}{v}, t) \, (\Delta x)^3 (\Delta v)^3$

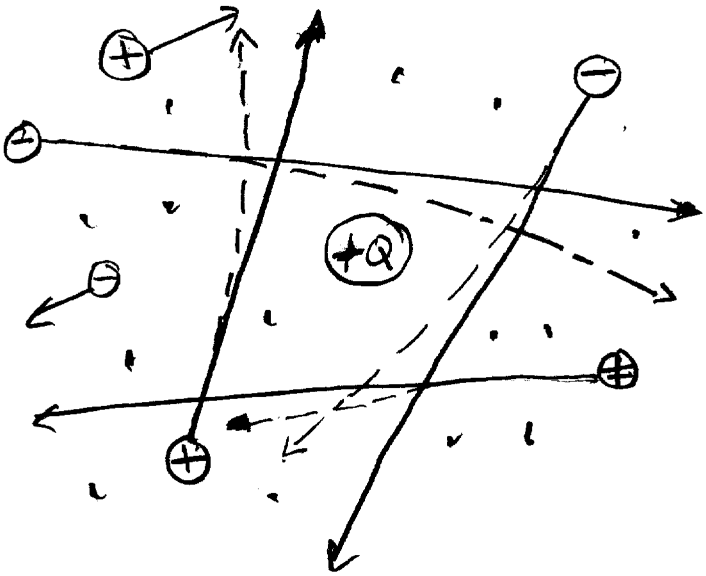
= # of electrons

within Δx of $\underset{\sim}{x}$ &

within Δv of $\underset{\sim}{v}$

at time t.

Debye Shielding



Add a charge $+Q$. Will attract (slightly) electrons + repel (slightly) ions producing a negative cloud that shields the charge $+Q$.

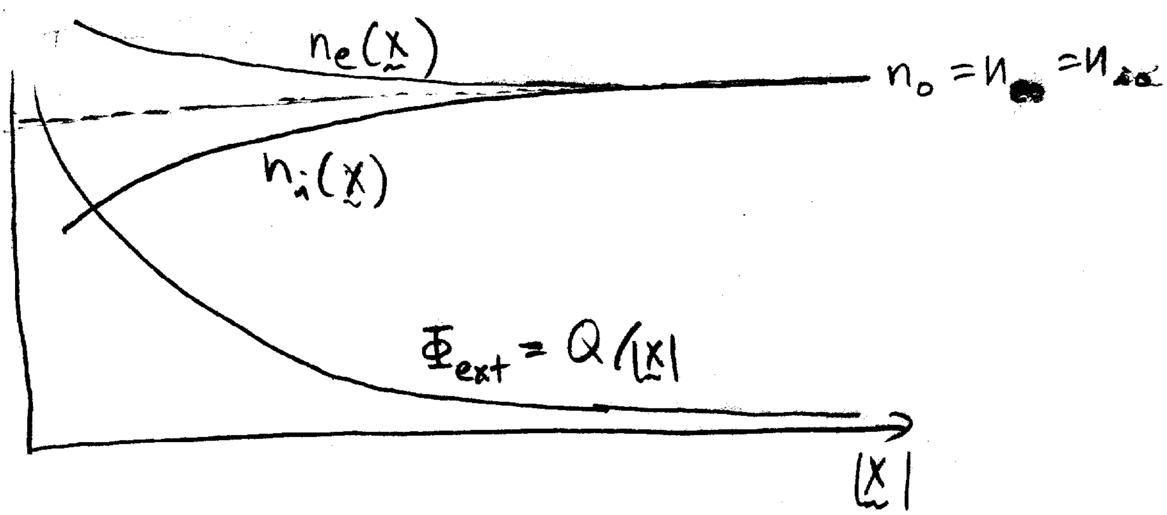
Electrons + ions in Boltzmann/Gibbs thermal equilibrium:

$$f_s(\underset{\sim}{x},\underset{\sim}{v}) \propto e^{-H/T} = C\, e^{-\left(\frac{1}{2}m_s v^2 + q_s \Phi(\underset{\sim}{x})\right)/T}$$

$$n_s(\underset{\sim}{x}) \propto e^{-q_s \Phi(\underset{\sim}{x})/T}$$

$$\left(\frac{q\Phi}{T} \ll 1\right)$$

$$n_s(\underset{\sim}{x}) \approx n_{s0}\left(1 - \frac{q_s \Phi}{T}\right)$$



$$\nabla \cdot \underset{\sim}{E} = 4\pi \sum_s q_s n_s + 4\pi Q\, \delta(\underset{\sim}{x})$$

$$-\nabla^2 \Phi = -\underbrace{\sum_s \frac{4\pi n_{s0} q_s^2}{T}}_{\frac{1}{\lambda_D^2}} \Phi + 4\pi Q\, \delta(\underset{\sim}{x})$$

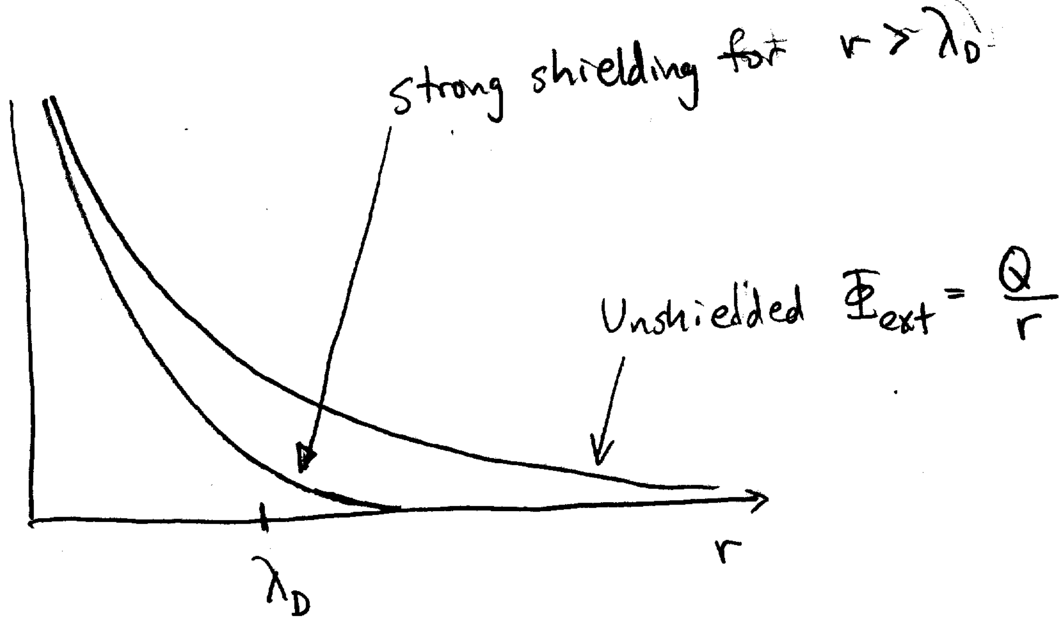
$\Leftarrow$ often just the electron contribution is used, λ_{De}

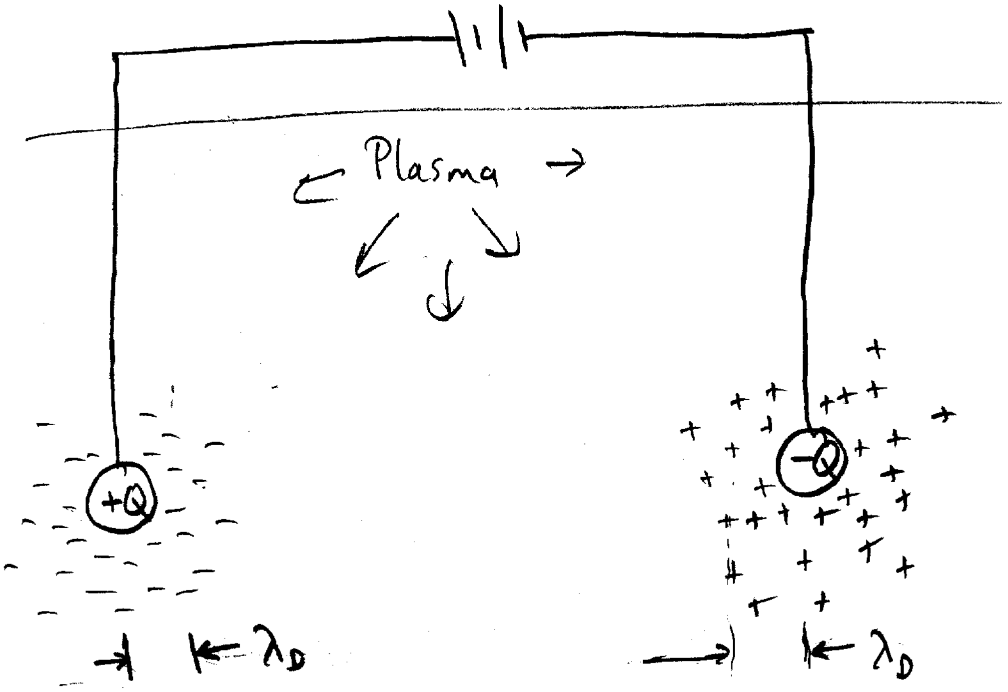
Fourier transform:

$$\left(k^2 + \frac{1}{\lambda_D^2}\right) \Phi_{\underline{k}} = 4\pi Q$$

Inverse transform (or verify by direct substn):

$$\Phi(\underset{\sim}{r}) = \frac{Q}{r}\, e^{-r/\lambda_D}$$





Debye shielding very effective,

Debye shielding radius λ_D usually very tiny:

NRL Formulary p. 28:	Magnetic Fusion	Solar Wind (SW)	Galactic Center (GC)
λ_D (cm)	$7*10^{-3}$	$7*10^{2}$	$3*10^{3}$
ω_{pe} (s^{-1})	$6*10^{11}$	$2*10^{5}$	$6*10^{5}$

Associated frequency scale $\omega_{pe} = \dfrac{v_{te}}{\lambda_{De}} = \sqrt{\dfrac{4\pi n_e e^2}{m_e}} \propto \sqrt{n_e}$

= inverse of time scale for electrons to travel λ_{De} & set up Debye shielding.

ω_{pe} = freq. of plasma oscillations.

ω_{pe} or cyclotron frequency Ω_{ce} are usually the highest frequency for collective dynamics or microinstabilities.

Aside: Every particle in a plasma is simultaneously being shielded by other particles, and is (weakly) participating in the shielding of other particles.

(BBGKY hierarchy ⇒ Liouville Eq. or Klimontovich Eq. expand to give Vlasov Eq. + correction term (Collision operator) for weak correlations between particles.)

The Plasma Parameter

In order for smooth density $n(\vec{x})$ approximation to hold (or equivalently, for the weak-coupling assumption to hold), need many particles within a Debye sphere:

$$\Lambda = n_e \frac{4\pi}{3}\lambda_{De}^3 = 1.7\times 10^9 T_{eV}^{3/2} n_e^{-1/2}$$

n_e in cm^{-3} (handy formulas: NRL p. 28…)

$=$ "The number of particles in a Debye Sphere" a.k.a. "The Plasma Parameter"

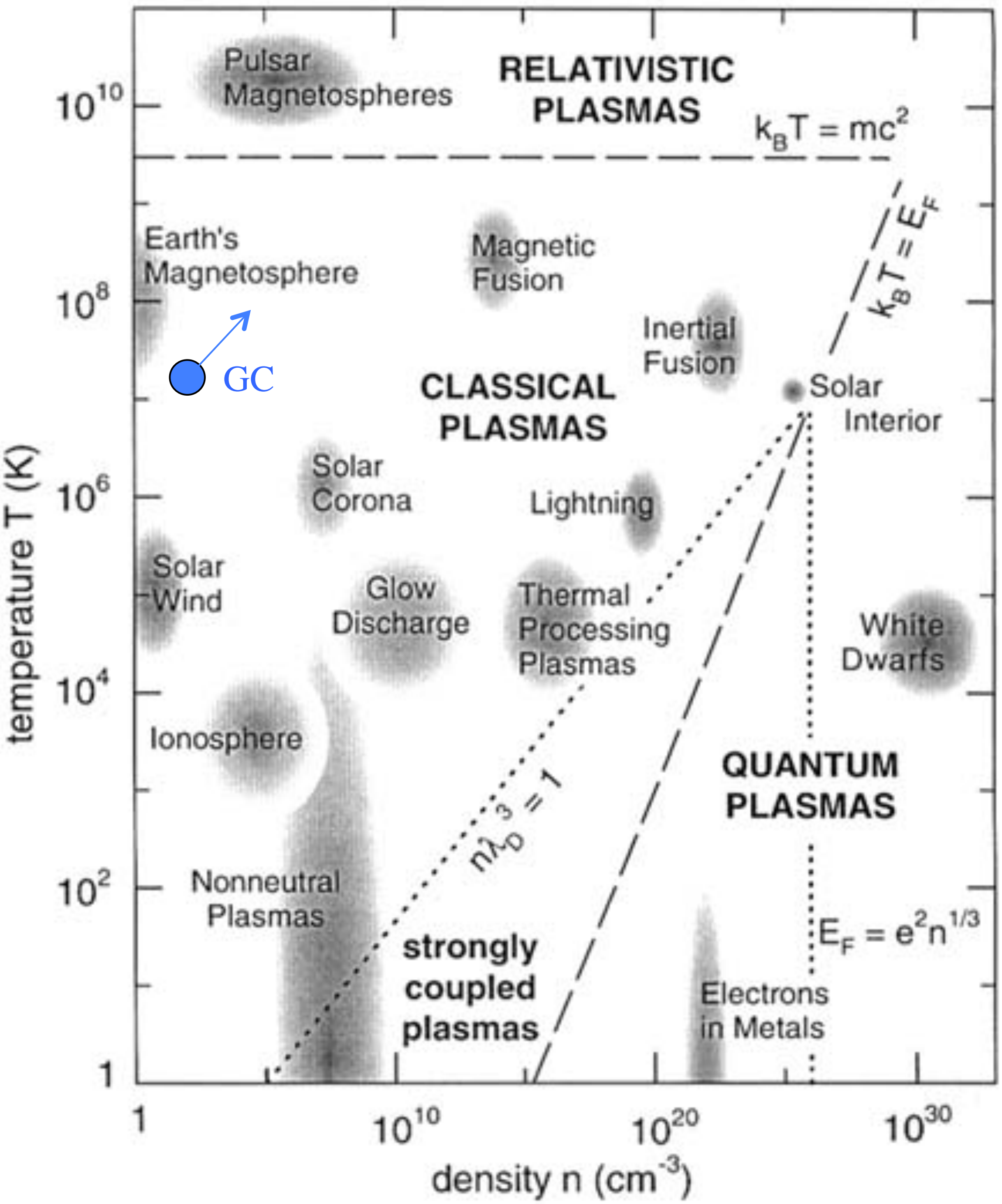
	Magnetic Fusion (MFE)	Solar Wind (SW)	Galactic Center (GC)
$\Lambda =$	$\sim 10^8$	$\sim 10^{10}$	$\sim 10^{13}$

It turns out that many key parameters can be expressed in terms of the number of particles in a Debye sphere. For example, the ratio of the potential energy between typical nearest neighbor particles to their typical kinetic energy:

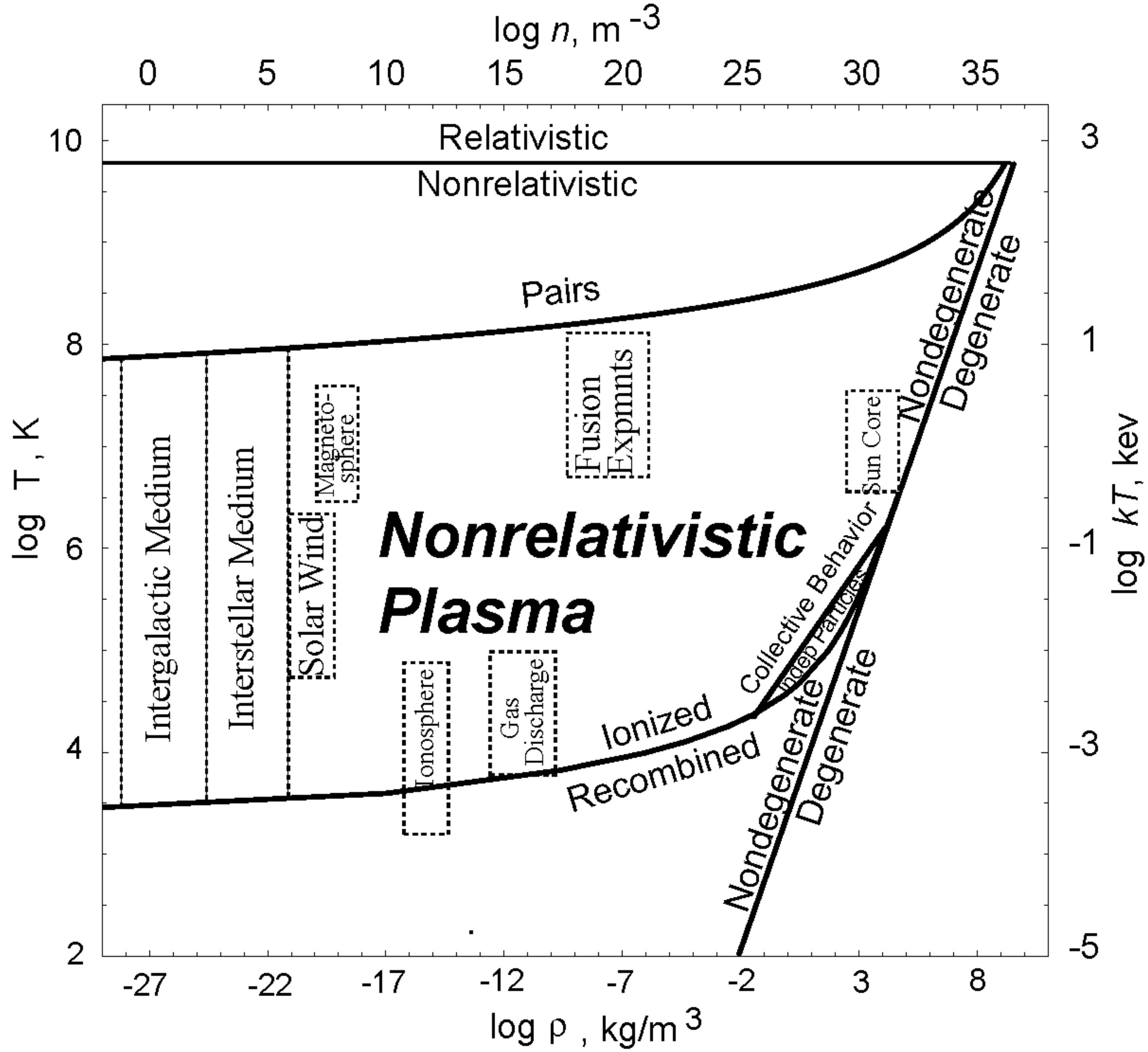
$$\frac{\text{Potential Energy of nearest neighbors}}{\text{Kinetic Energy}} \approx \frac{e^2/n^{-1/3}}{T} = \frac{1}{(36\pi)^{1/3}\Lambda^{2/3}}$$

We will find that $\Lambda >> 1$ also implies that the mean free path between collisions is long compared to the Debye length.

Plasma Zoology: (n,T) Plot

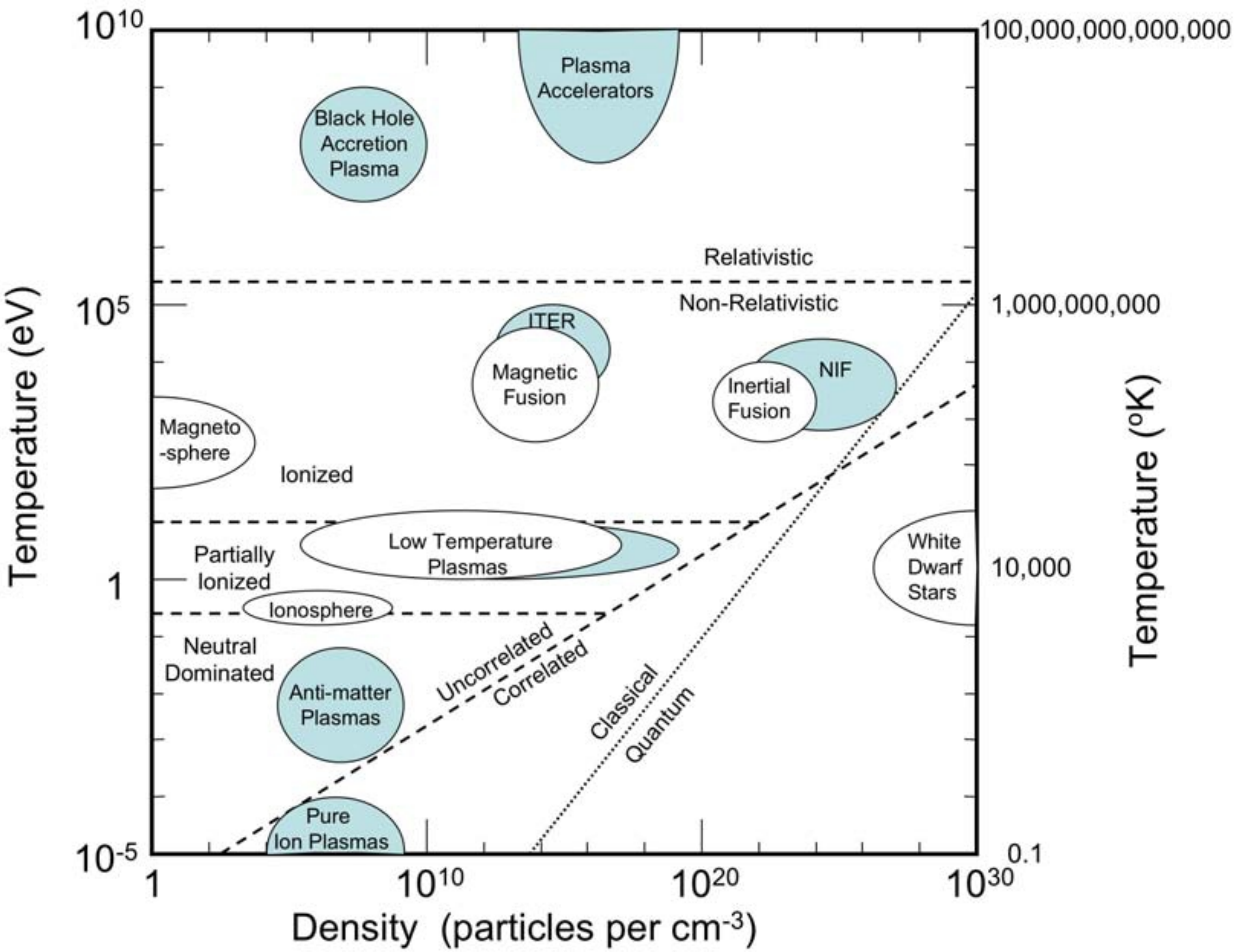


"Plasmas that occur naturally or can be created in the laboratory are shown as a function of density (in particles per cubic centimeter) and temperature. The boundaries are approximate and indicate typical ranges of plasma parameters. Distinct plasma regimes are indicated. For thermal energies greater than that of the rest mass of the electron ($T > m_ec^2$), relativistic effects are important. At high densities, where the Fermi energy is greater than the thermal energy ($E_F > k_BT$), quantum effects are dominant [i.e., electron degeneracy pressure exceeds thermal pressure]. In strongly coupled plasmas (i.e., $n\lambda_D^3 < 1$, where λ_D is the Debye screening length), the effects of the Coulomb interaction dominate thermal effects; and when $E_F > e^2 n^{1/3}$, quantum effects dominate those due to the Coulomb interaction (i.e., the Fermi energy exceeds the potential energy of typical nearest-neighbor particles], resulting in nearly ideal quantum plasmas. At temperatures less than about 10^5K, recombination of electrons and ions can be significant, and the plasmas are often only partially ionized." [From National Research Council Decadal Review, Plasma Science: From Fundamental Research to Technological Applications (1995) [explanations added] http://www.nap.edu/catalog.php?record_id=4936 .]



Blandford and Thorne, Applications of Classical Physics, http://www.pma.caltech.edu/Courses/ph136/yr2006/text.html

Wide range of possible plasma parameters.
Plasmas above the line marked "Uncorrelated-Correlated" correspond to $\Lambda >> 1$



Plasma Science: Advancing Knowledge in the National Interest (2007), National Research Council,
http://books.nap.edu/openbook.php?record_id=11960&page=9

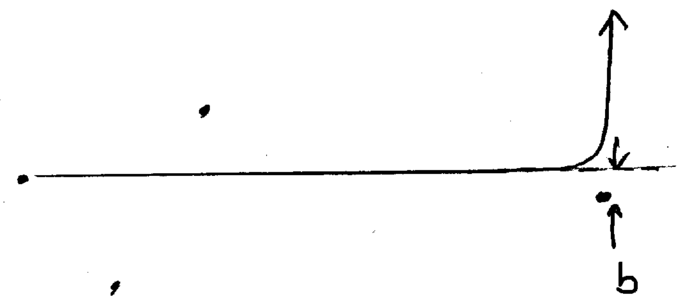
See also NRL Formulary version of this plot.

APPROXIMATE MAGNITUDES IN SOME TYPICAL PLASMAS

Plasma Type	n cm^{-3}	T eV	ω_{pe} sec^{-1}	λ_D cm	$n\lambda_D{}^3$	ν_{ei} sec^{-1}
Interstellar gas	1	1	6×10^4	7×10^2	4×10^8	7×10^{-5}
Gaseous nebula	10^3	1	2×10^6	20	8×10^6	6×10^{-2}
Solar Corona	10^9	10^2	2×10^9	2×10^{-1}	8×10^6	60
Diffuse hot plasma	10^{12}	10^2	6×10^{10}	7×10^{-3}	4×10^5	40
Solar atmosphere, gas discharge	10^{14}	1	6×10^{11}	7×10^{-5}	40	2×10^9
Warm plasma	10^{14}	10	6×10^{11}	2×10^{-4}	8×10^2	10^7
Hot plasma	10^{14}	10^2	6×10^{11}	7×10^{-4}	4×10^4	4×10^6
Thermonuclear plasma	10^{15}	10^4	2×10^{12}	2×10^{-3}	8×10^6	5×10^4
Theta pinch	10^{16}	10^2	6×10^{12}	7×10^{-5}	4×10^3	3×10^8
Dense hot plasma	10^{18}	10^2	6×10^{13}	7×10^{-6}	4×10^2	2×10^{10}
Laser Plasma	10^{20}	10^2	6×10^{14}	7×10^{-7}	40	2×10^{12}

From NRL Plasma Formulary (very useful)

2 Key Length Scales for Collisions



Occasionally 2 particles will get close enough to cause a 90° scatter.

b = "distance of closest approach" $\quad \frac{e^2}{b} \sim T$

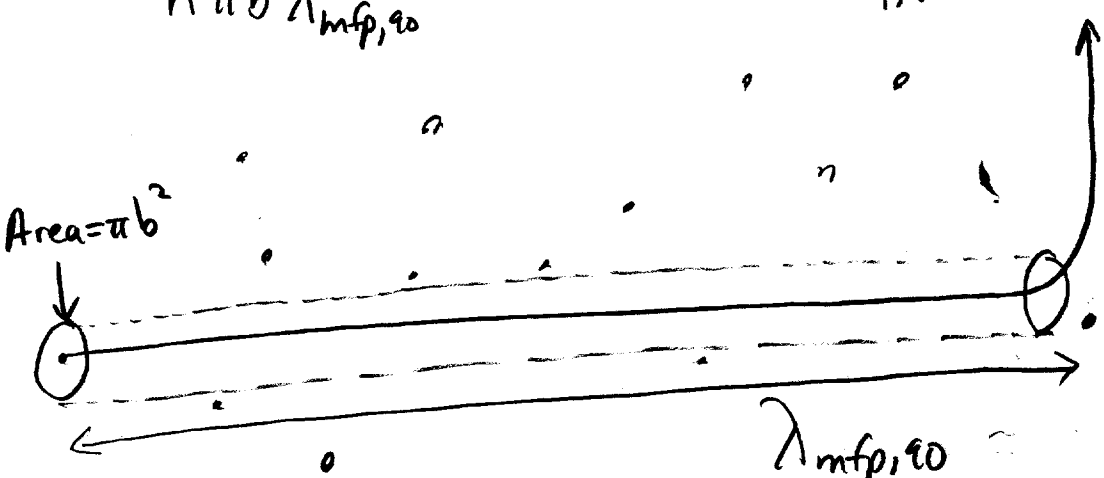
b is small compared to typical interparticle spacing

$$\frac{b}{n^{-1/3}} \sim \frac{1}{\Lambda^{2/3}} \sim 10^{-4} \ll 1$$

Cross-section for 90° collision $= \pi b^2$

Mean-free Path for 90° collision $= \lambda_{mfp,90}$

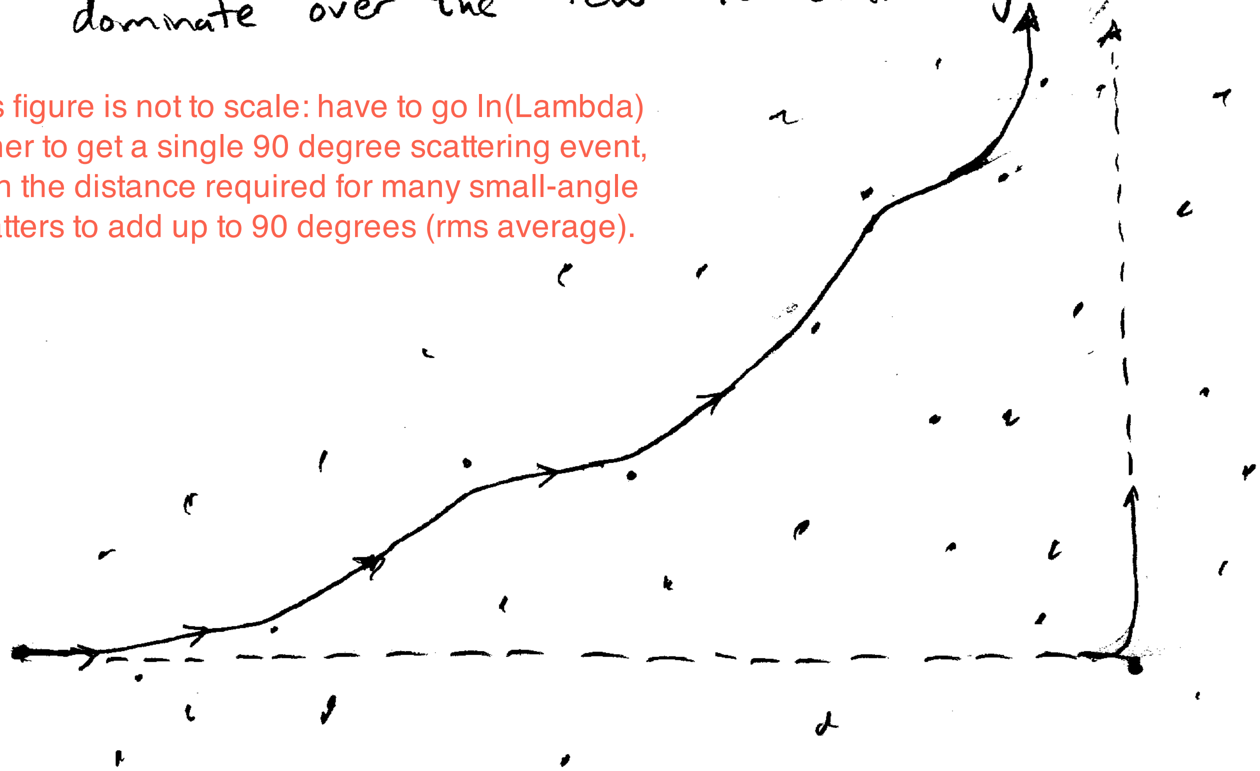
$$n \pi b^2 \lambda_{mfp,90} = 1 \quad \Rightarrow \quad \lambda_{mfp,90} = \lambda_{Debye} \Lambda$$



Small-angle Collisions Dominate

However, cumulative effect of many small-angle scatters dominate over the few 90° scattering events

This figure is not to scale: have to go ln(Lambda) further to get a single 90 degree scattering event, than the distance required for many small-angle scatters to add up to 90 degrees (rms average).



$$\text{Effective } \lambda_{mfp} \cong \frac{\lambda_{mfp,90}}{\ln \Lambda} \simeq \lambda_D \frac{\Lambda}{\ln \Lambda} \sim \lambda_D \frac{10^8}{10}$$

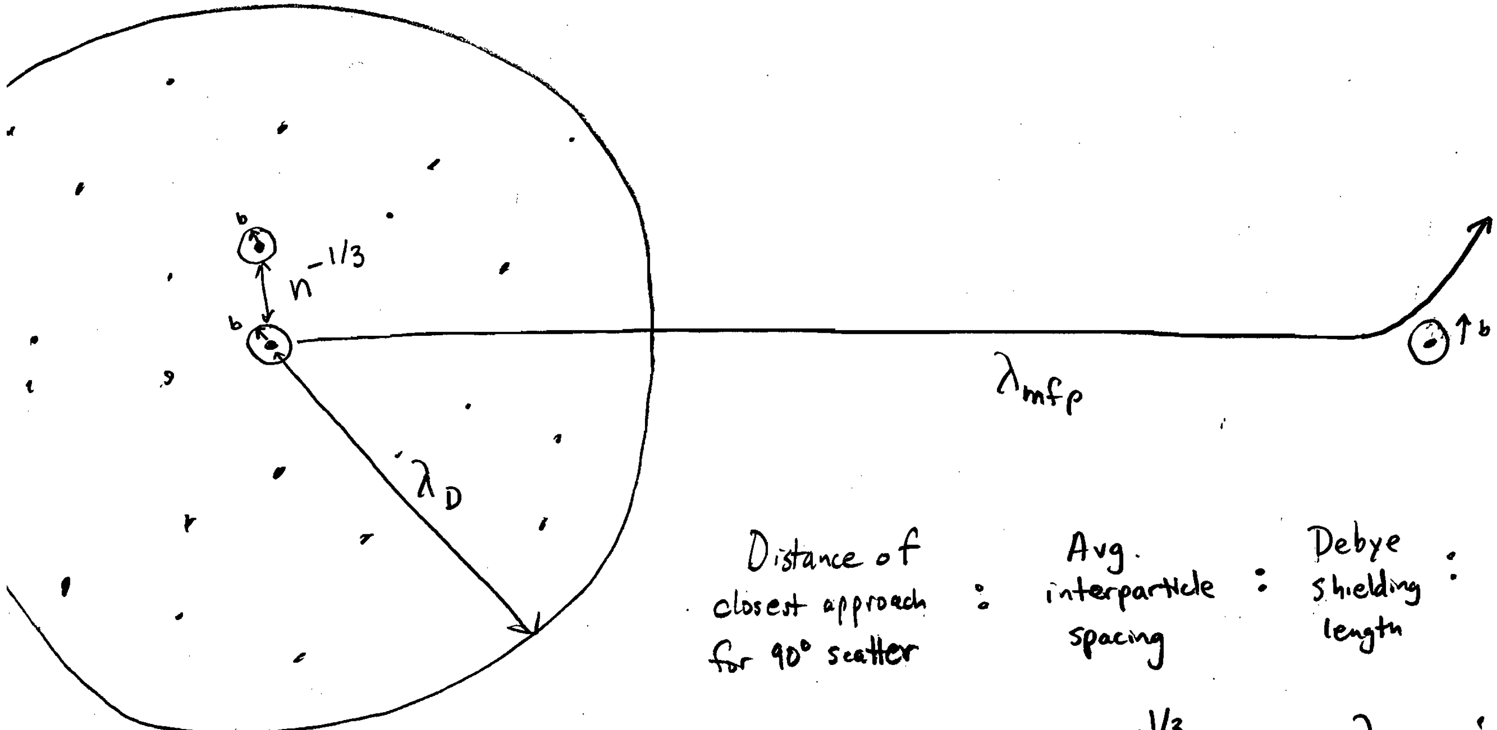
electron Collision frequency: $\nu_e = \dfrac{v_{te}}{\lambda_{mfp}} = 2.91 * 10^{-6} \dfrac{n_e}{T_e^{3/2}} \ln \Lambda$

(collision time $\tau_e \sim 1/\nu_e$)

Ion colliz. freq. $\nu_{ii} \sim \nu_i \sim \sqrt{\dfrac{m_e}{m_i}}\, \nu_i$ $\qquad \nu \downarrow$ as $T_e \uparrow$

$$\boxed{\frac{\nu_e}{\omega_{pe}}} \approx \boxed{\frac{v_{te}}{\lambda_{mfp}} \frac{\lambda_D}{v_{te}}} \approx \boxed{\frac{\ln \Lambda}{\Lambda} \approx 10^{-7}}$$

Review Fundamental Length Scales of a Plasma



Distance of closest approach for 90° scatter	:	Avg. interparticle spacing	:	Debye shielding length	:	$\equiv$ Average Mean ~~free~~ path between collisions.
b	:	$n^{-1/3}$	:	λ_D	:	λ_{mfp}
Λ^{-1}	:	$\Lambda^{-1/3}$	:	1	:	$\frac{\Lambda}{\ln \Lambda}$
10^{-6}	:	10^{-2}	:	1	:	$\frac{10^6}{15} \sim 10^5$

$\Lambda \sim n \lambda_D^3$

$= \#$ of particles in Debye Sphere

$$\left.\begin{array}{l} \lambda_D \sim 10^{-2} \text{ cm} \\ \lambda_{mfp} \sim 100 \text{ m} \end{array}\right\} \text{TFTR}$$

Typical Plasma Parameters: Mean-Free Path Often Large

	Magnetic Fusion	Solar Wind @ 1 AU (L=1 earth radius) 1 AU = 1.5×10^{13}	Galactic Center @ Bondi accretion radius
T_e (eV):	1.00E+04	10	2.00E+03
n_e (/cm^3):	1.00E+14	10	1.00E+02
L macro scale (cm)	1.00E+02	6.40E+05	2.20E+17
beta	2.87E-02	10	10
A_amu	2.5	1	1
(potential energy)/(kinetic energy)	4.5E-07	2.1E-08	2.2E-10
B (g)	5.3E+04	2.8E-05	1.3E-03
L_Debye (cm)	7.4E-03	7.4E+02	3.3E+03
L_Debye/L	7.4E-05	1.2E-03	1.5E-14
# of particles in Debye Sphere	1.7E+08	1.7E+10	1.5E+13
log(Lambda)	1.9E+01	2.4E+01	3.0E+01
log(Lambda) for collisions	2.0E+01	2.5E+01	3.2E+01
Plasma frequency (rad/s)	5.6E+11	1.8E+05	5.6E+05
ion collision frequency (/s)	6.2E+01	3.8E-07	1.7E-09
lambda_mfp (cm)	1.0E+06	8.2E+12	2.6E+16
ion Cyclotron Frequency (rad/s)	2.0E+08	2.7E-01	1.2E+01
rho_i ion gyroradius (cm)	3.0E-01	1.1E+07	3.6E+06
rho_i/L	3.0E-03	1.8E+01	1.6E-11
lambda_mfp/rho_i	3.3E+06	7.2E+05	7.1E+09
lambda_mfp/L	1.0E+04	1.3E+07	1.2E-01

Particle Motion in Uniform $\underset{\sim}{B}$ Field

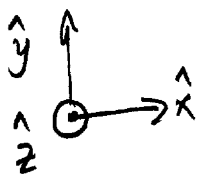
Lorentz Force Law:

$$m \frac{d\underset{\sim}{v}}{dt} = q\left(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\right)$$

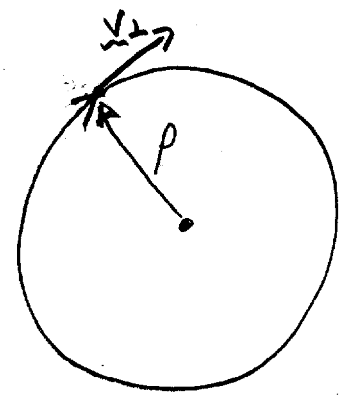
$$\underset{\sim}{v}\cdot \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\,\underset{\sim}{v}\cdot\underset{\sim}{E}$$

only $\underset{\sim}{E}$ can change particle's energy,

$\underset{\sim}{B}$ can only change particle's direction.



$$\underset{\sim}{B} = B\hat{z}$$

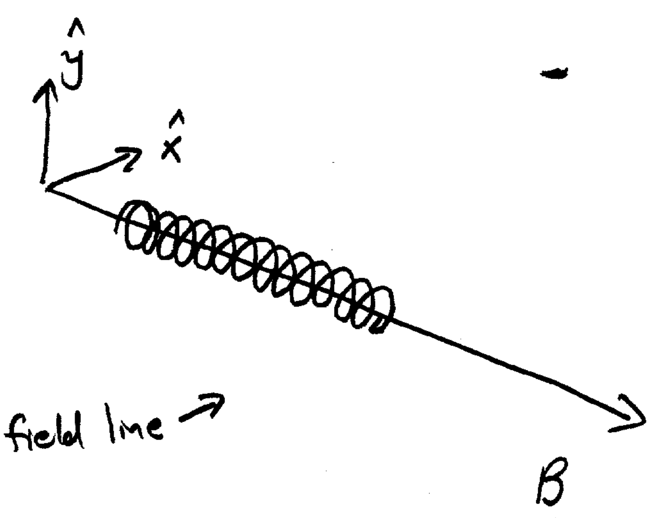


$$\frac{d\underset{\sim}{v}}{dt} = \Omega_c\,\underset{\sim}{v} \times \hat{z}$$

$$\Omega_c = \frac{qB}{mc} = \text{cyclotron freq. or gyro freq.}$$

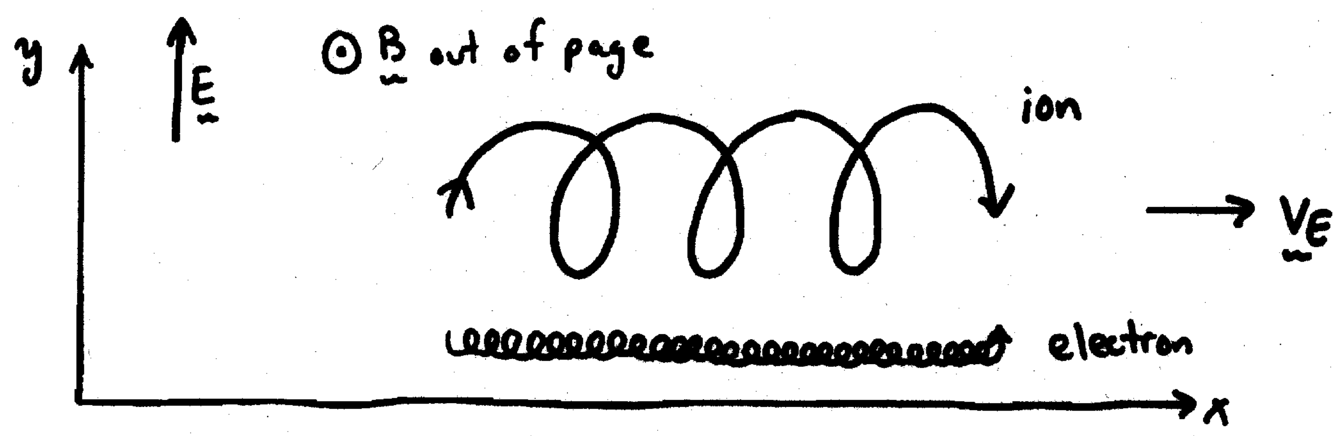
Gyroradius ρ:

$$\rho = \frac{V_\perp}{\Omega_c}$$



Free to move along magnetic field line

Physical Picture of $\underset{\sim}{E} \times \underset{\sim}{B}$ drift:



$\underset{\sim}{E}$ acceleration causes $v_\perp$ to be bigger on top half of orbit for ion (or for electrons on bottom half of orbit. But electrons gyrate around $\underset{\sim}{B}$ in the opposite direction, $\Omega_{ce} < 0$, $\Omega_{ci} > 0$, since $q_e = -q_i$, so net $E \times B$ drift is in same direction for electrons & ions)

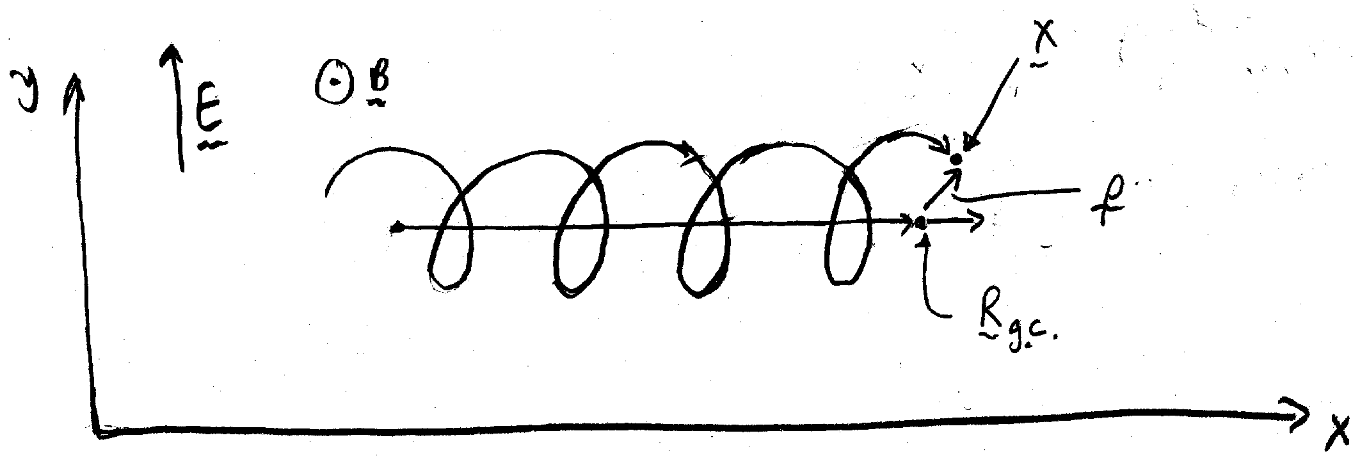
Gyroradius $\rho = \frac{v_\perp}{\Omega}$ is $\therefore$ bigger on top half of orbit (for ions)

$\Rightarrow$ Net drift to right

$$V_{\underset{\sim}{E} \times \underset{\sim}{B}} = \frac{c}{B^2} \underset{\sim}{E} \times \underset{\sim}{B}$$ is the same for electrons & all types of ions (V_E independent of q & m of particle species)

Surprise! Net particle motion is not in the direction of $\underset{\sim}{E}$! (for constant $\underset{\sim}{E}$, $\underset{\sim}{E} \perp \underset{\sim}{B}$)

Particles gyrate rapidly around a slowly drifting "guiding center"



$$\underset{\sim}{X}(t) = \underset{\sim}{R}_{g.c.}(t) + \underset{\sim}{\rho}(t)$$

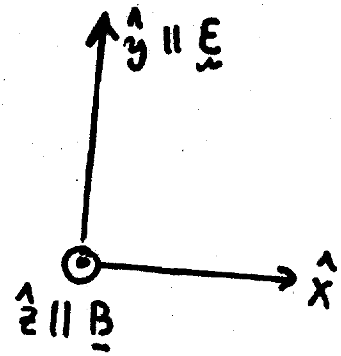
particle position = $\underbrace{\text{guiding center position}}_{\text{slowly drifts.}}$ + rapidly gyrating gyrovector

$$\underset{\sim}{\rho} = \frac{\hat{b} \times \underset{\sim}{v}(t)}{\Omega_c}$$

(actually, $\frac{c}{B^2} \underset{\sim}{E} \times \underset{\sim}{B}$ can be fast, but ∇B & other drifts are slow...)

$\mathbf{E}\times\mathbf{B}$ Drift

$\mathbf{E} = \text{const} = E_0\,\hat{y}$ $\qquad$ $\mathbf{B} = \text{const} = B_0\,\hat{z}$



$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right]$$

Substitute:

$$\mathbf{v} = \underbrace{\frac{c}{B^2}\,\mathbf{E}\times\mathbf{B}} + \mathbf{v}'$$

$$\frac{\times\mathbf{B}}{c} = \frac{1}{B^2}\left[\left(\mathbf{E}\times\mathbf{B}\right)\times\mathbf{B}\right] = -\,\mathbf{E}_\perp$$

(Component of $\mathbf{E} \perp \mathbf{B}$)

Leaves

$$\frac{d\mathbf{v}'}{dt} = \frac{qB}{mc}\,\mathbf{v}'\times\hat{z}$$

same oscillatory motion as before with $\mathbf{E}=0$.

Equivalent to a relativistic transformation to a frame of reference where $\mathbf{E}$ vanishes.

Fast gyration: $\qquad$ E×B drift:

$$V_x = V_\perp \sin(\Omega t + \alpha) + \frac{c}{B}E_y$$

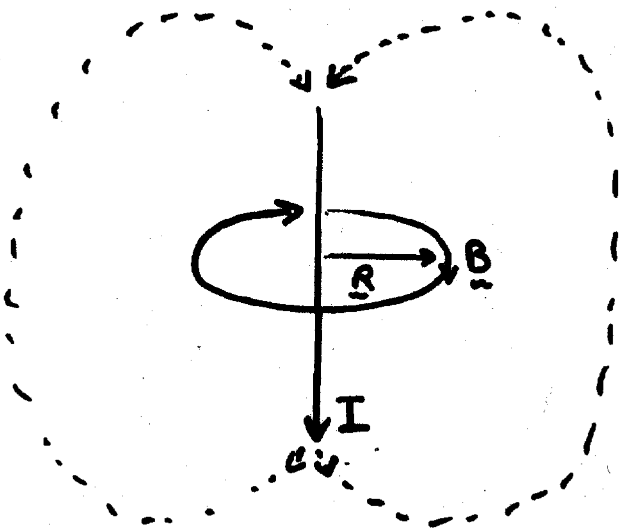
$$V_y = V_\perp \cos(\Omega t + \alpha)$$

$$V_z = V_\parallel .$$

More generally, guiding-center moves at the velocity:

$$\frac{d\mathbf{R}_{g.c.}}{dt} = V_\parallel \frac{\mathbf{B}}{B} + \underbrace{\frac{c}{B^2}\,\mathbf{E}\times\mathbf{B}}_{\mathbf{V}_{E\times B}} + \text{higher-order drifts}$$

Particle Drift Due to ∇B

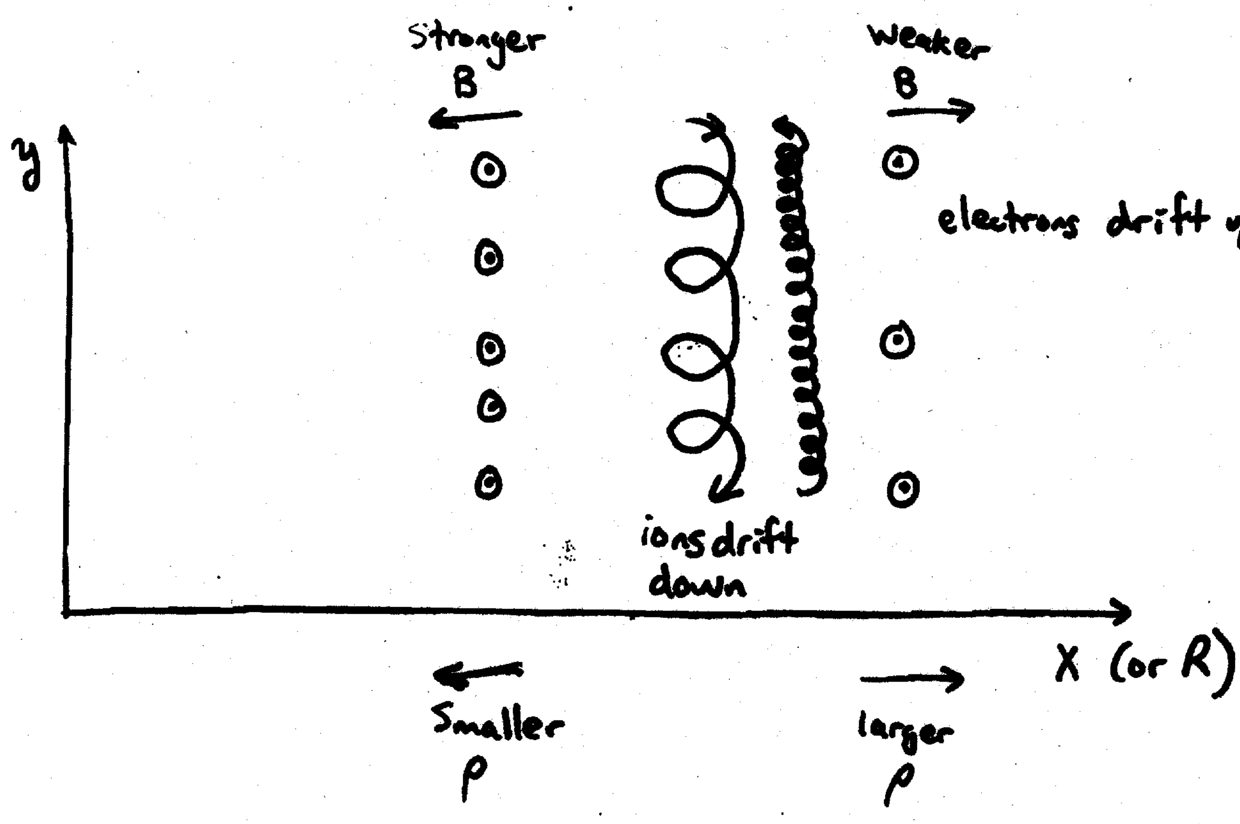


$$\nabla \times \underset{\sim}{B} = \frac{4\pi}{c} \underset{\sim}{j}$$

$$\oint d\underset{\sim}{\ell} \cdot \underset{\sim}{B} = \frac{4\pi}{c} I$$

$$2\pi R B = \frac{4\pi}{c} I$$

$$B \propto \frac{1}{R} \propto \frac{1}{X}$$



$$\rho = \frac{V_\perp}{\Omega} \propto \frac{1}{B} \propto R \propto X$$

$\boxed{\text{"}\nabla B \text{ drift"}}$

$$\underset{\sim}{V}_{\nabla B} = \frac{1}{2} \frac{V_\perp^2}{\Omega_c} \frac{\hat{b} \times \nabla B}{B} \sim V_\perp \underbrace{\frac{\rho}{R}}_{\ll 1}$$

Adiabatic Invariance of μ

$\underset{\sim}{B} = B_0(t)\,\hat{z}$

$\underset{\sim}{B}$ is uniform, but time-varying

(From here on, heavily borrowed from Charles Karney's lecture notes...)

Assume variation of $\underset{\sim}{B}$ is slow:

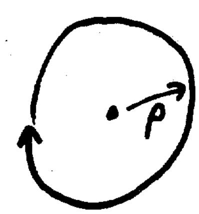
$$\frac{1}{B}\frac{\partial B}{\partial t} << \Omega$$

Faraday's law: $\nabla \times \underset{\sim}{E} = -\frac{1}{c}\frac{\partial \underset{\sim}{B}}{\partial t}$

$\Rightarrow \underset{\sim}{E} \Rightarrow$ energy of particle is not conserved.

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = m\underset{\sim}{v_\perp}\cdot\frac{d\underset{\sim}{v_\perp}}{dt} = \underset{\sim}{v_\perp}\cdot q\underset{\sim}{E}$$

Integrate over 1 cyclotron period $\frac{2\pi}{\Omega}$



$$\Delta W_\perp = \Delta \frac{1}{2}mv_\perp^2 = \oint \underset{\sim}{v_\perp}\cdot q\underset{\sim}{E}\,dt$$

$$= q\oint \underset{\sim}{E}\cdot d\underset{\sim}{s}$$

$$= q\int_A \nabla\times\underset{\sim}{E}\cdot d\underset{\sim}{A} \qquad \text{Stokes theorem}$$

$$= -\frac{q}{c}\int_A \frac{\partial\underset{\sim}{B}}{\partial t}\cdot d\underset{\sim}{A} \qquad \text{Faraday}$$

$$\frac{\Delta B}{\Delta \tau} \quad = \Delta B\,\frac{\Omega}{2\pi}$$

$$A = -\pi\rho^2 = -\pi\left(\frac{v_\perp}{\Omega}\right)^2$$

Sense of area is negative for positive ions.

$$\Delta W_\perp = + \frac{q}{c} \frac{\Delta B}{m} \frac{\cancel{\Omega}}{2\cancel{\pi}} \cancel{\pi} \frac{m v_\perp^2}{\Omega^{\cancel{2}}}$$

$$\Rightarrow \quad \Delta W_\perp = \Delta B \, \frac{W_\perp}{B}$$

$$\frac{\Delta W_\perp}{W_\perp} = \frac{\Delta B}{B}$$

Or $\quad \Delta\left(\frac{W_\perp}{B}\right) = 0$

$$\Delta(\ln W_\perp) = \Delta \ln B$$

$$\Delta(\ln W_\perp - \ln B) = 0$$

$\frac{W_\perp}{B} = \mu$ - adiabatic invariant. $\Delta \ln\left(\frac{W_\perp}{B}\right) = 0$

$\mu \equiv$ magnetic moment

Note: $\mu \propto \pi \rho^2 B$ = flux through gyro-orbit.

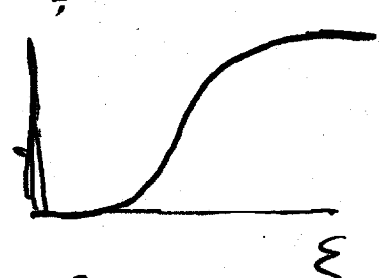
Gyrating particle ring acts like a perfectly conducting ring, conserving flux through its orbit very accurately, if the rate of change of B is slow compared to Ω.

Requires $\omega \gtrsim \Omega_c$ resonant perturbations to efficiently change μ. If $\frac{\omega}{\Omega_c} \sim \varepsilon \ll 1$, then μ is well conserved:

Kruskal: $\quad \frac{\Delta \mu}{\mu} \sim e^{-\frac{1}{\varepsilon}}$ $\varepsilon \sim \frac{\omega}{\Omega} \sim 10^{-3}$

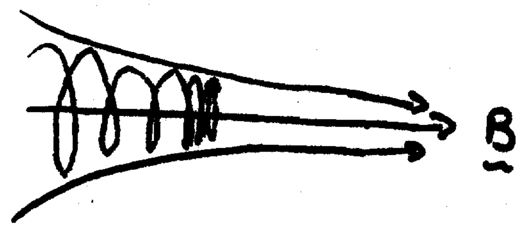
$$e^{-(10^3)}$$

$$e^{-\frac{1}{\varepsilon}}$$



$$f(\varepsilon) = f(0) + \varepsilon \frac{\partial f}{\partial \varepsilon} + \frac{1}{2}\varepsilon^2 \frac{\partial^2 f}{\partial \varepsilon^2} + \cdots$$

Mirror Effect



$\frac{\partial B}{\partial z} > 0$

$\underset{\sim}{E} = 0$

$\Rightarrow$ Kinetic Energy = W = constant

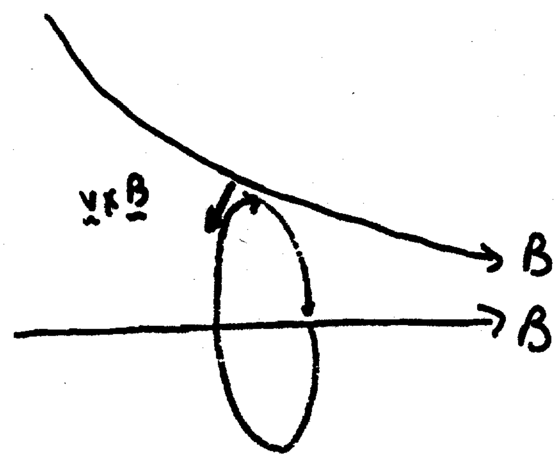
But μ conservation $\Rightarrow$ $W_\perp \propto B$

$W =$ constant

$= \frac{W_\perp}{B} B + W_\parallel$

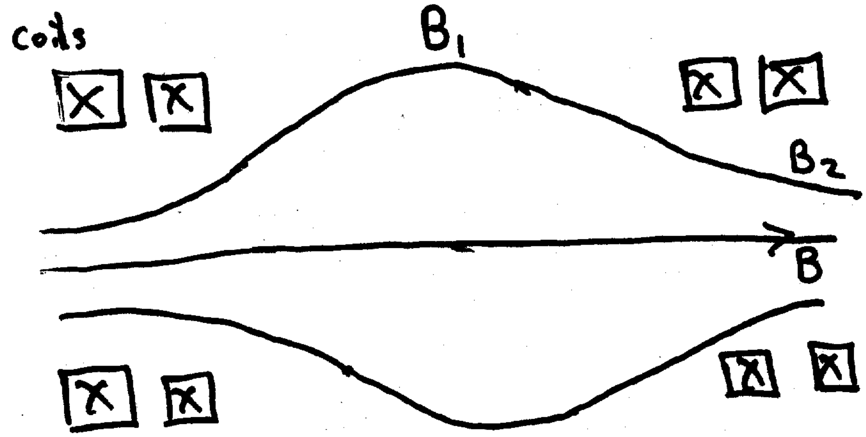
$W = \mu B + \frac{1}{2} m v_\parallel^2$

$\Rightarrow$ $W_\parallel$ becomes 0 at some point
- particle is reflected



z component of force comes from radial component of $\underset{\sim}{B}$

Mirror Machine



Mirror force reflects particles near mirror throat.

Conservation of μ & W:

Particle which reflects at B_2 $(v_{\parallel} = 0)$ has energy $W = W_{\perp 2} = \mu B_2$

At B_1 it has the same energy $W = W_{\perp 1} + W_{\parallel 1}$

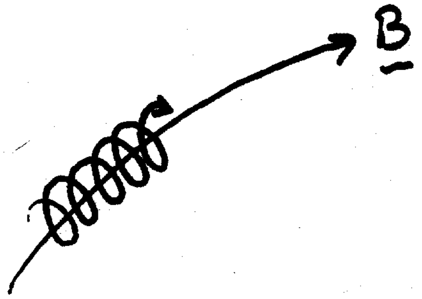
$$= \mu B_1 + \frac{1}{2} m v_{\parallel 1}^2$$

Can show that all particles with

$$\left.\frac{v_\perp}{v}\right|_{\text{at } B_1} > \sqrt{\frac{B_1}{B_2}}$$

are reflected.

Guiding Center Motion Summary



$$\Omega = \frac{qB}{mc} \quad \text{gyrofrequency}$$

$$\rho = \frac{v_\perp}{\Omega} \quad \text{gyroradius}$$

$$\underset{\sim}{r} = \underset{\sim}{R}_{gc} + \underset{\sim}{\rho}$$

Particle Position — Guiding-center position — Rapidly gyrating gyro-radius vector

For static fields (& $\underset{\sim}{E}$ not too strong...), the dominant guiding center drifts are:

$$\frac{d\underset{\sim}{R}_{gc}}{dt} = v_\parallel \hat{b} + \underset{\sim}{v}_d$$

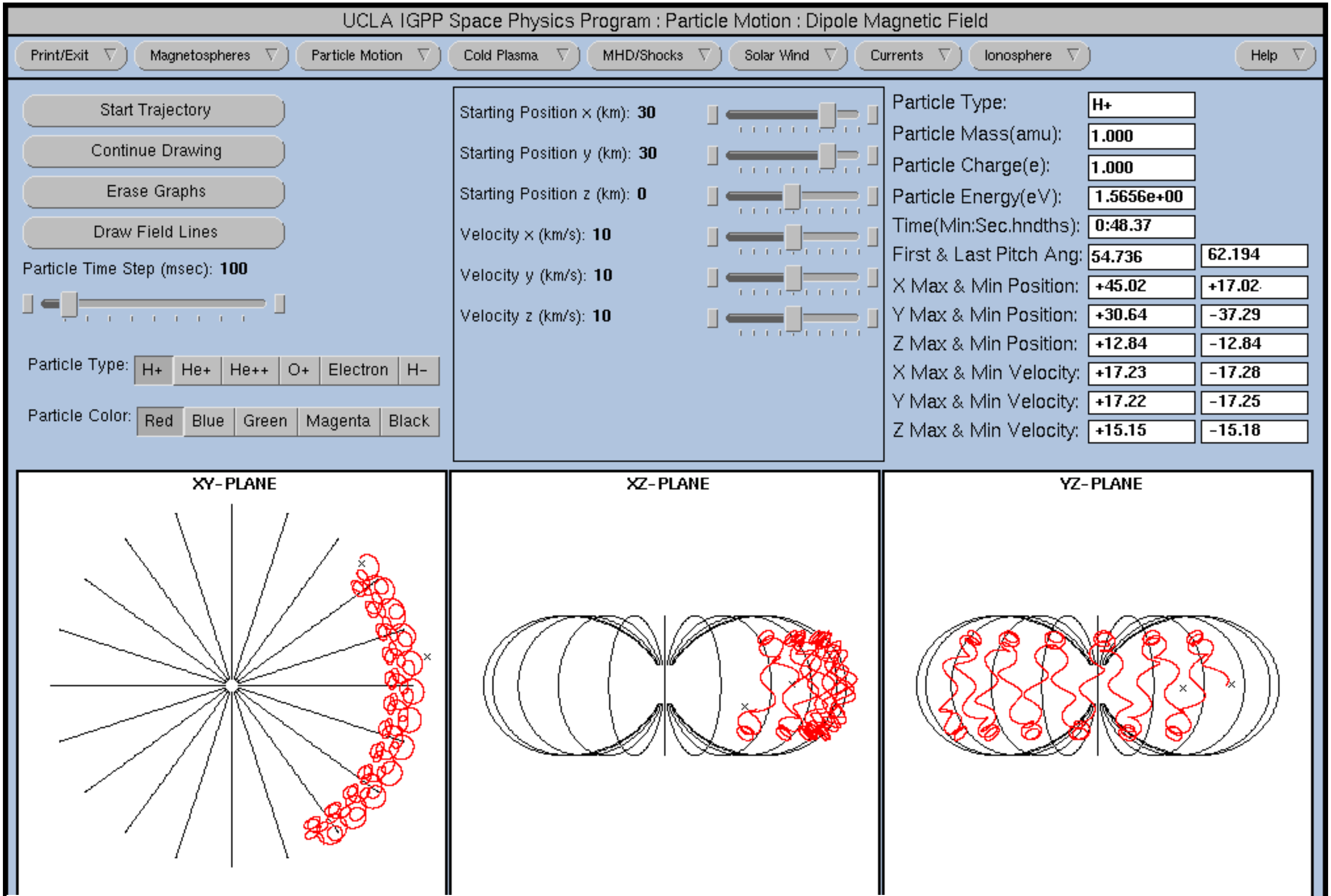
$$\hat{b} = \frac{\underset{\sim}{B}}{B} \qquad \mu = \frac{W_\perp}{B} = \text{constant}$$

$$W = \mu B + q\Phi + \tfrac{1}{2} m v_\parallel^2 = \text{constant}$$

$$\underset{\sim}{v}_d = \frac{c}{B^2}\underset{\sim}{E} \times \underset{\sim}{B} + \frac{\frac{1}{2}v_\perp^2}{\Omega B}\hat{b} \times \nabla B + \frac{v_\parallel^2}{\Omega}\hat{b} \times (\hat{b} \cdot \nabla \hat{b})$$

"E x B" — "∇B" — "Curvature" drift (see textbooks...)

polarization drift $\propto \frac{d\underset{\sim}{E}}{dt}$ important for understanding instabilities & waves from particle viewpoint.



Particle motion in dipole magnetic field, showing bounce motion along magnetic field between magnetic mirrors, and slower grad(B) drift across magnetic field.

http://www-ssc.igpp.ucla.edu/ssc/spgroup_edu.html

This method can be extended to

Systematic Derivation of Complete Guiding Center Drifts

For details, see: Banos, J. Plasma Physics 1 (1965), 305.

George Schmidt, Physics of High Temperature Plasmas (1979)
Kenro Miyamoto, Plasma Physics for Nuclear Fusion (1976)

(Most textbooks derive particle drifts piecemeal, not systematically...)

$\boxed{\text{Modern approach: Hazeltine \& Waelbroeck, The Framework of Plasma Physics (1998)}}$

Also: Cary & Brizard, RMP 2009

Basic idea:

$$\underset{\sim}{r} = \underset{\sim}{R}_{gc} + \underset{\sim}{\rho}$$

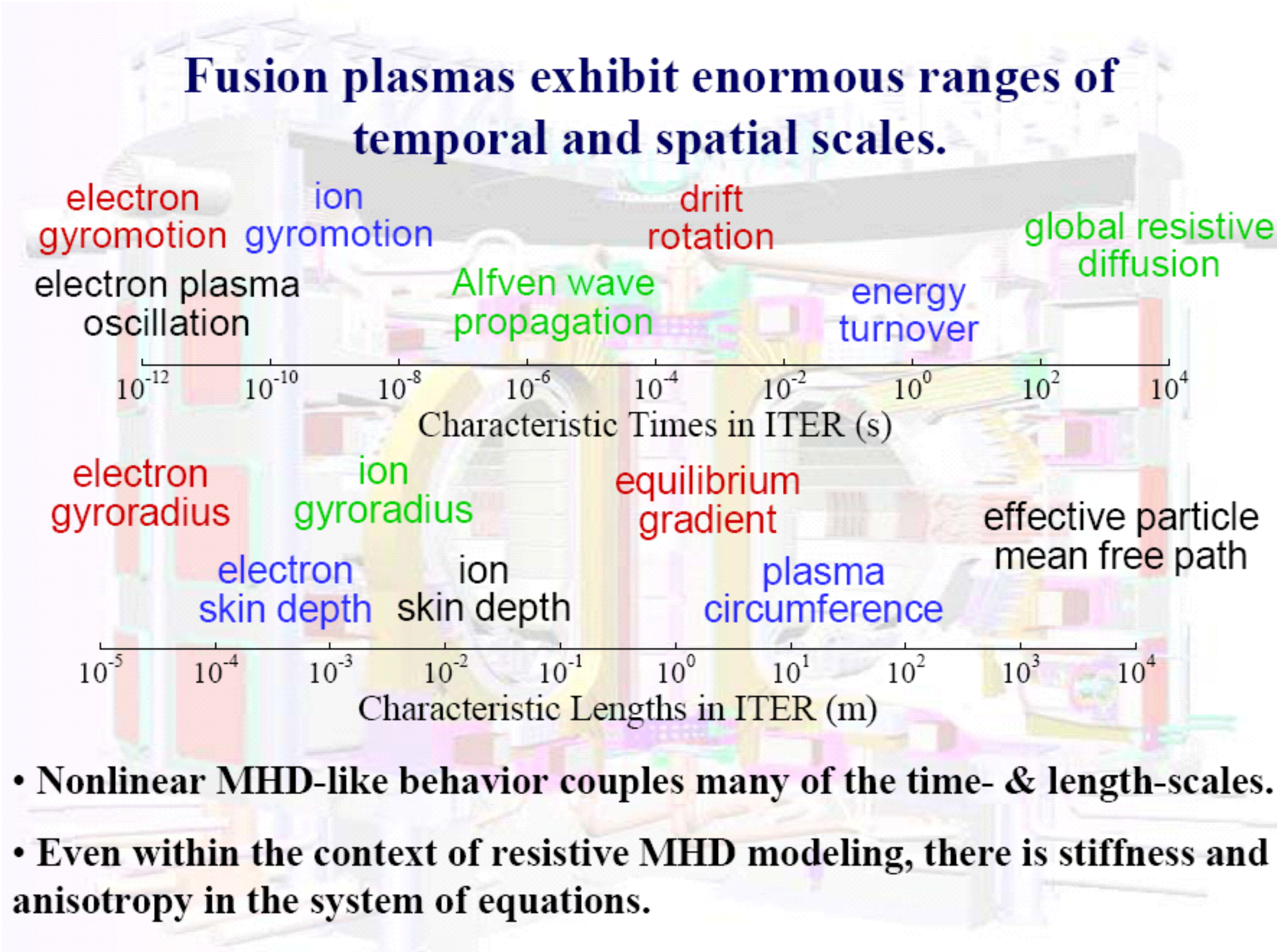
Particle Position — Slowly moving guiding center — fast gyro-motion

Assume gyroradius ρ is small & expand: $\underset{\sim}{B} \approx \underset{\sim}{B}(\underset{\sim}{R}_{gc}, t) + \underset{\sim}{\rho} \cdot \nabla \underset{\sim}{B}$ etc.

Assume Ω is very large & average equations of motion over fast gyro-motion to obtain an equation for the slower moving guiding-center...

Subtleties: various ways to expand $\underset{\sim}{E}$ → MHD ordering / macro-instabilities
→ gyrokinetic ordering / micro-instab.

Alternative: Modern Hamiltonian methods, non-canonical Lie perturbation theory, useful for going to higher order, investigate accuracy of adiabatic invariants, etc...



Even with the most powerful computers expected in the next 20 years, there are many problems with such an extreme range of scales that they can't be directly solved…

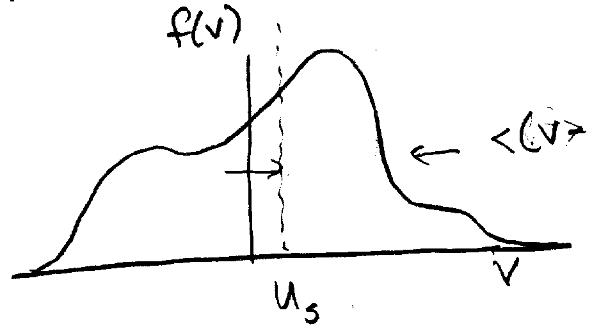
Carl Sovinec, http://www.csm.ornl.gov/workshops/SciDAC2005/SovinecTalk/scidac05_talk.pdf

Vlasov-Boltzmann Kinetic Eq. for Self-Consistent Collective Dynamics

$f_s(\underset{\sim}{x}, \underset{\sim}{v}, t)$ = particle distribution function in phase space, position $\underset{\sim}{x}$, velocity $\underset{\sim}{v}$ at time t for species s.

$$n_s(\underset{\sim}{x}, t) = \int d^3v\, f_s$$

$$n_s u_s = \int d^3v\, f_s\, \underset{\sim}{v}$$



$\leftarrow \langle (v-u^2) \rangle \equiv v_{ts}^2 \equiv \frac{T_s}{m_s}$

defines effective temperature.

$$\frac{\partial f_s}{\partial t} + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \Big[\underbrace{\underset{\sim}{v}}_{\dot{\underset{\sim}{x}}}\; f_s \Big] + \frac{\partial}{\partial \underset{\sim}{v}} \cdot \Big[\underbrace{\frac{q_s}{m_s}\Big(\underset{\sim}{E} + \frac{\underset{\sim}{v} \times \underset{\sim}{B}}{c}\Big)}_{\dot{\underset{\sim}{v}}} f \Big] = C[f]$$

Conservation law in phase-space.

$C[f_s]$ = Landau collision operator
= velocity diffusion & drag Fokker-Planck Eq.
(small-angle collisions dominate)
Causes f to relax towards Maxwellian.
(Similar operator for collisions of stars in galaxies
gravity & electrostatics both long-range $\frac{1}{r^2}$ forces)

+ Maxwell's Eqs, need

charge density $\sigma = \sum_s q_s n_s$

current density $\underset{\sim}{j} = \sum_s q_s n_s \underset{\sim}{u}_s$

Vlasov+Maxwell ⇒ starting point for most kinetic theory

Conservation laws follow from $\underset{\sim}{v}$-integration!

$\int d^3v *$: $$\frac{\partial n_s}{\partial t} + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left(n_s \underset{\sim}{u}_s \right) = 0$$ particle conservation

$\int d^3v\, m_s \underset{\sim}{v} \times$ ⇒ momentum conservation

$$\frac{\partial}{\partial t}\left(m_s n_s \underset{\sim}{u}_s\right) + \frac{\partial}{\partial \underset{\sim}{x}} \cdot \left[m_s n_s \underset{\sim}{u}_s \underset{\sim}{u}_s \right] = - \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\sim}{\mathbb{P}}_s + q_s n_s \left(\underset{\sim}{E} + \frac{\underset{\sim}{u}_s \times \underset{\sim}{B}}{c} \right)$$

Sum over species & use quasineutrality ($\sigma \underset{\sim}{E} \to 0$) to get standard MHD. (Need $\sigma \underset{\sim}{E}$ for relativistic MHD.)

$$\mathbb{P}_{s,ij} = m_s \int d^3v \, f_s \left(v_i - u_{s,i}\right)\left(v_j - u_{s,j}\right)$$

If collisions are sufficiently rapid,

$f_s \approx$ Maxwellian + small corrections

($\Rightarrow$ Braginskii thermal conduction & viscosity...)

$$\mathbb{P}_{s,ij} = p_s \delta_{ij} \qquad \frac{\partial}{\partial \underset{\sim}{x}} \cdot \underset{\approx}{\mathbb{P}}_s = \frac{\partial p_s}{\partial \underset{\sim}{x}} = \nabla p_s$$

$\int d^3v \, \frac{1}{2} m \underset{\sim}{v} \underset{\sim}{v} \, *$

$\vdots$ $\leftarrow$ energy conservation

& infinite chain of moments hierarchy

closed if collisions are rapid.

Otherwise need kinetic theory

or model thereof

Generalized Ohm's Law

(not completely general - dropped $\frac{m_e}{m_i}$ terms & electron inertia).

$$\boxed{\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c} = \frac{\mathbf{j}\times\mathbf{B}}{n_e e c} - \frac{\nabla\cdot\mathbf{P}_e}{e n_e} + \eta\,\mathbf{j}}$$

"Hall term"

"FLR terms"

Finite Larmor Radius (Gyro)

In "MHD", this is usually simplified further to:

$$\boxed{\underbrace{\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}}_{=\mathbf{E}' \text{ in frame moving with fluid}} = \eta\,\mathbf{j}}$$

"Resistive MHD"

or

$$\boxed{\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c} = 0}$$

"Ideal MHD"
perfectly conducting fluid.

Fundamental Ordering Assumptions of MHD

Magneto - Hydro - Dynamics

Look for phenomena which are :

slow compared to cyclotron frequencies Ω_{ce}, Ω_{ci}

large scale compared to gyroradii ρ_i, ρ_e

$$\frac{\partial}{\partial t} \sim \omega \qquad \nabla \sim \frac{1}{L}$$

$$\boxed{\frac{\omega}{\Omega_{ci}} \sim \frac{\rho_i}{L} \sim \varepsilon \ll 1}$$

Allow flow velocity of plasma $u \sim \frac{cE}{B} \sim V_{ti}$

+ allow $\beta \sim \frac{p}{B^2/8\pi}$ $O(1)$

(subsidiary orderings, $\beta \ll 1$, allowed later)

Later do self-consistency checks to see if these are satisfied.

$\frac{\partial \underline{B}}{\partial t}$ in the MHD Approximation

$$\frac{\partial \underline{B}}{\partial t} = -c\,\nabla \times \underline{E} = \nabla \times (\underline{u} \times \underline{B}) - c\,\nabla \times (\eta\, \underline{j})$$

using Ohm's Law:

$$\underline{E} + \frac{\underline{u} \times \underline{B}}{c} = \eta\, j$$

$$\nabla \times \underline{B} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\,\underline{j}$$

Displacement current is ignored in the MHD approximation.

$$\frac{\frac{1}{c}\frac{\partial E}{\partial t}}{\nabla \times \underline{B}} \sim \frac{\frac{1}{c}\omega E}{\frac{B}{L}} \sim \frac{\omega L}{c^2}\,\frac{cE}{B} \sim \overbrace{\frac{u^2}{c^2} \ll 1}^{\text{assumption}}$$

So in MHD, will make the "Magnetostatic" approximation

$$\nabla \times \underline{B} = \frac{4\pi}{c}\,\underline{j}$$

← order Light waves out of our Eqs.

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{u} \times \underline{B}) - \frac{c^2}{4\pi}\nabla \times (\eta\, \nabla \times \underline{B})$$

If $\eta \approx$ const, use NRL Vector ID # 14:

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B}) + \underbrace{\frac{c^2 \eta}{4\pi}}_{\text{Magnetic Diffusion coefficient}} \left[\nabla^2 \underset{\sim}{B} + \nabla \underbrace{(\nabla \cdot \underset{\sim}{B})}_{0} \right]$$

If $\underset{\sim}{u} = 0$, $\underset{\sim}{B}$ decays on a time scale

$$\frac{1}{\tau_{skin}} \sim \frac{c^2 \eta}{4\pi L^2} \quad \Rightarrow \quad \tau_{skin} \sim 8 \times 10^{-8} L_{cm}^2 T_{eV}^{3/2} \text{ sec}$$

Where L is a "typical" length scale for $\underset{\sim}{B}$ gradients.

$\tau_{skin} \sim$ 1 sec for a copper sphere of 1 cm radius

$\sim 10^4$ years for the molten core of the earth

$\sim 10^{10}$ years for a typical magnetic field in the Sun
(J. D. Jackson, 2nd Edition, p. 473)

~ 25 sec for a tokamak with $L \sim 100$ cm + $T \sim 1$ keV

$\sim 10^{27}$ years for a galaxy with $L \sim 10^{21}$ cm + $T \sim 1$ eV
"Important problems in Astrophysics"
R. Kulsrud, Phys. Plasmas 2, p. 1735 (1995).

However, regions of sharp gradients (such as shocks, or magnetic tearing layers) can have $L \rightarrow 0$ + undergo magnetic diffusion more quickly.

Math Precursor

$$\Phi_B(t) = \int_{x_0(t)}^{x_1(t)} dx \quad B(x,t)$$

$$\frac{d\Phi_B}{dt} = \int_{x_0(t)}^{x_1(t)} dx \ \frac{\partial B}{\partial t} + \frac{dx_1}{dt} B(x_1,t) - \frac{dx_0}{dt} B(x_0,t)$$

Alt. proof:

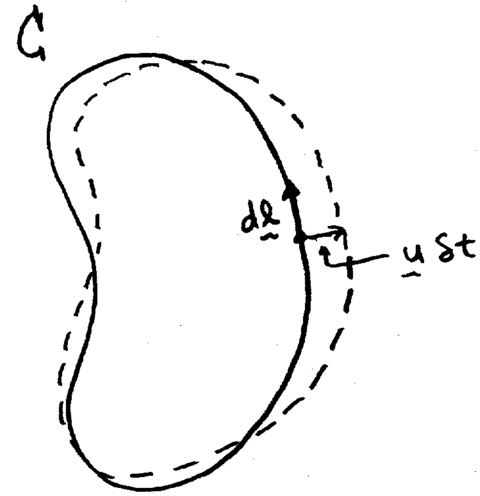
$$\Phi_B(t) = \int_{-\infty}^{\infty} dx \quad H(x_1(t)-x)\ H(x-x_0(t)) \quad B(x,t)$$

$$\frac{d\Phi_B}{dt} = \int_{x_0}^{x_1} dx \ \frac{\partial B}{\partial t} + \int_{-\infty}^{\infty} dx \left\{ \underbrace{H'(x_1-x)}_{\delta(x_1(t)-x)} \frac{\partial x_1}{\partial t} - \underbrace{H'(x-x_0)}_{\delta(x-x_0(t))} \frac{\partial x_0}{\partial t} \right\} B(x,t)$$

Q.E.D.

Flux-conservation

The magnetic flux through any closed contour C that moves with the fluid remains unchanged:



$$\Phi_B = \int d\mathbf{S} \cdot \mathbf{B}$$

$$\frac{\delta\, d\mathbf{S}}{\delta t} = d\mathbf{l} \times \mathbf{u}$$

$$\frac{d\Phi_B}{dt} = \int d\mathbf{S} \cdot \frac{\partial \mathbf{B}}{\partial t} + \oint_C \frac{\partial\, d\mathbf{S}}{\partial t} \cdot \mathbf{B}$$

$$= \underbrace{\int d\mathbf{S} \cdot \nabla \times (\mathbf{u} \times \mathbf{B})}_{= \oint_C d\mathbf{l} \cdot \mathbf{u} \times \mathbf{B}} - \oint_C d\mathbf{l} \times \mathbf{u} \cdot \mathbf{B}$$

cancel

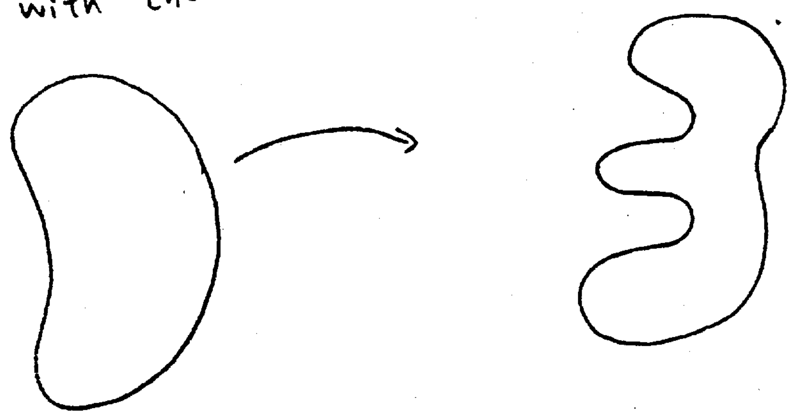
$$\boxed{\frac{d\Phi_B}{dt} = 0}$$

Frozen-in Field Lines (for ideal MHD ~~with proof~~)

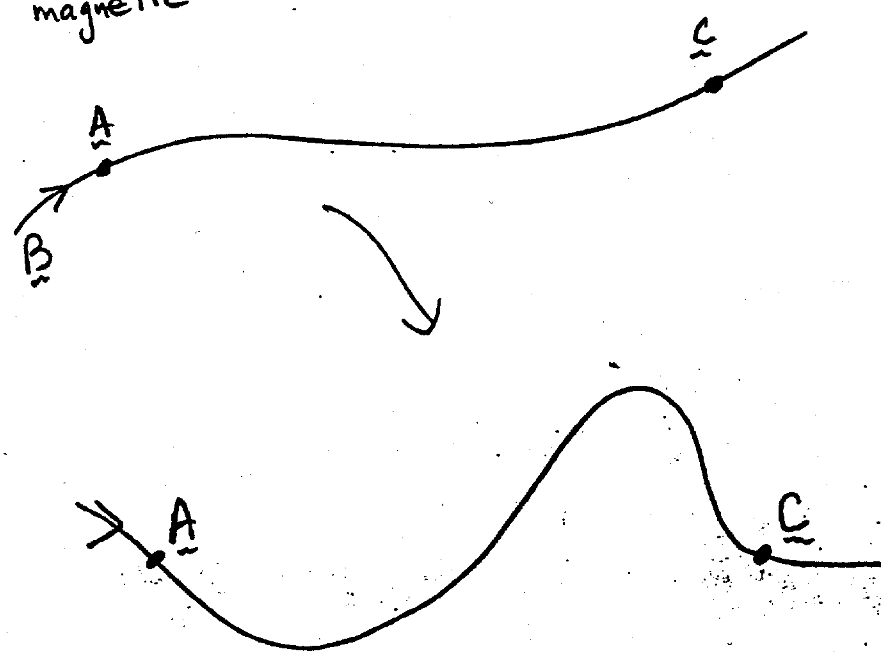
(See S. von Goeler, Lecture XIV for proof) ☆★
or Goldston & Rutherford, Sec. 8.5 first + then Sec. 8.4.

2 parts:

(a) The magnetic flux through a closed contour C that moves with the fluid remains unchanged.

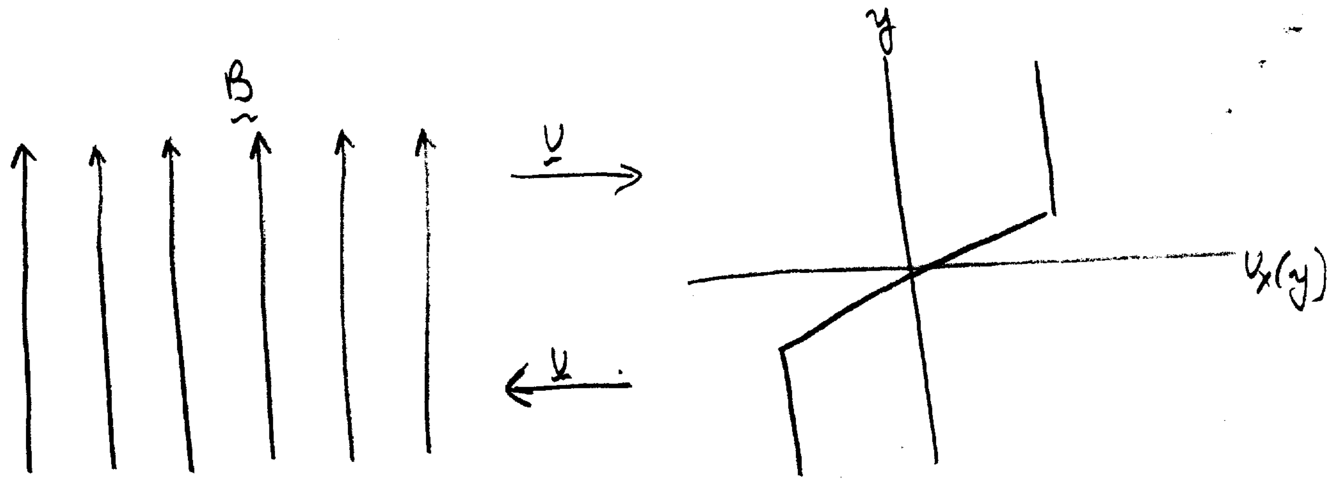


(b) two fluid elements that initially lie on a magnetic field line continue to lie on a field line.

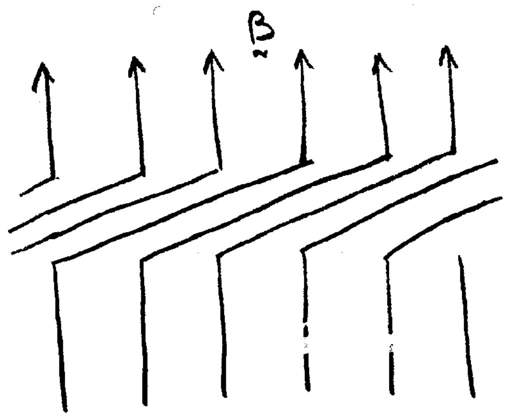


Theorem that Field Lines are frozen in to a Perfectly Conducting Fluid proposed by Alfvén.

Even without plasma compression, can get magnitude of $\underset{\sim}{B}$ to be enhanced :



Later



$\leftarrow |\underset{\sim}{B}|$ is larger here!

"Dynamo" generation of B : earth? Sun? galactic ~~magnet~~ & intergalactic magnetic fields?

The Fluid Universe

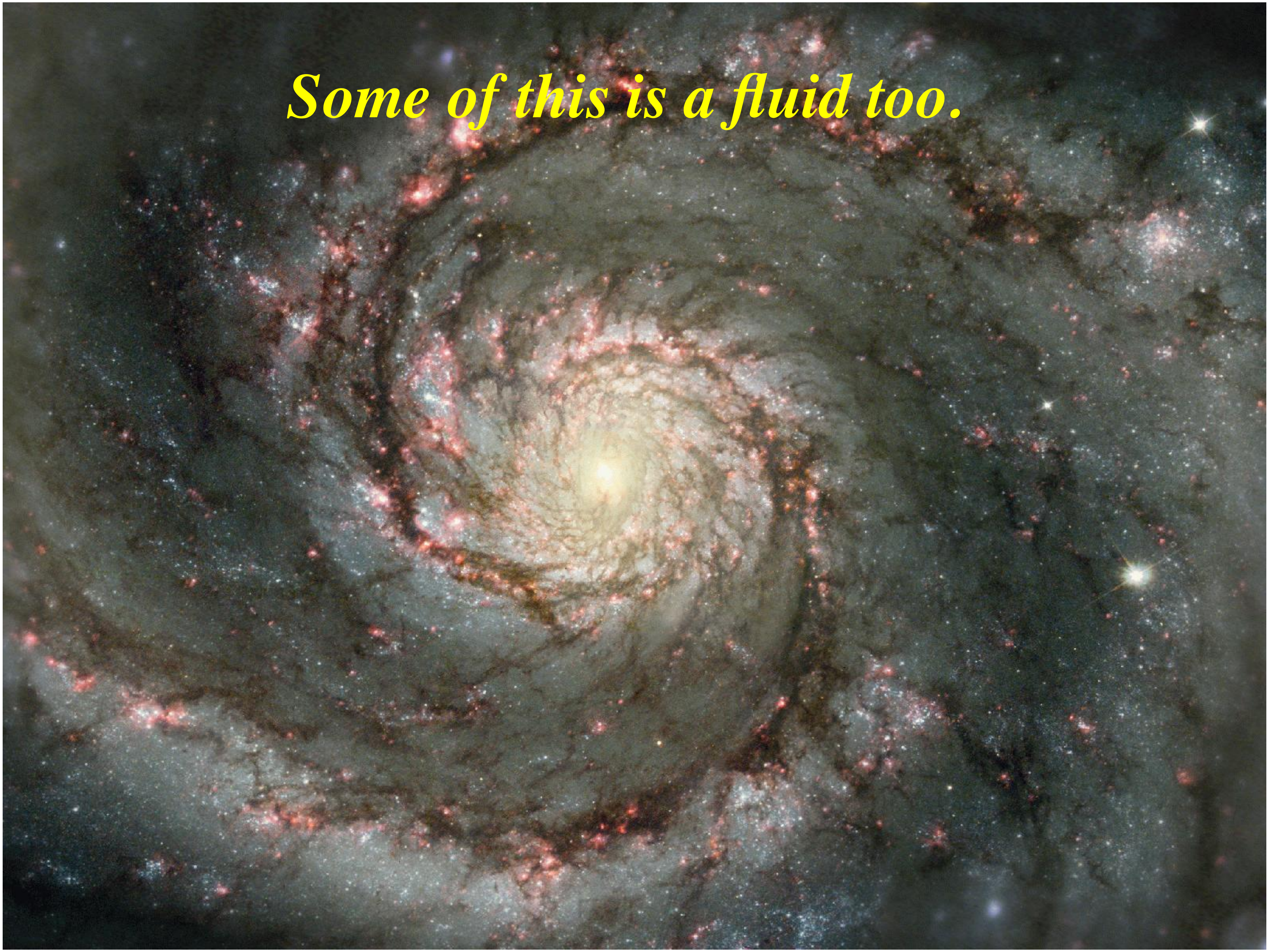
Most of the "big" questions in astrophysics require studying the fluid dynamics of the visible matter.

- How do galaxies form?
- How do stars form?
- How do planets form?

This requires solving the equations of radiation magneto-hydrodynamics (MHD).

Slides in this section from J. Stone


It's a fluid!


Some of this is a fluid too.

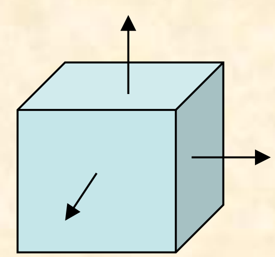
MHD equations: conservation laws

…for mass, momentum, energy, and magnetic flux.

Mass conservation:

Rate of change of mass in a volume is divergence of fluxes through surface

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$



ρ = mass density

$\mathbf{v}$ = velocity

$\frac{\partial}{\partial t}$ = Eulerian derivative (at a fixed point in space)

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$ = Lagrangian derivative (moving with flow)

Momentum conservation:

Rate of change of momentum within a volume is divergence of stress on surface of volume (no viscous stress)

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B} + P^*] = 0$$

Energy conservation:

Rate of change of total energy density E is equal to the divergence of energy flux through the surface

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*)\mathbf{v} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = 0$$

$E = \rho\ v^2/2 + e + B^2/2$ is total energy

$P^* = P + B^2/2$ is total pressure (gas + magnetic)

<u>Flux conservation:</u>

Given by Maxwell's equations:

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E}$$

$$\nabla \cdot \mathbf{B} = 0$$ (constraint rather than evolutionary equation)

From Ohm's Law, the current and electric field are related by

$$\mathbf{J} = \sigma \left(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right)$$

For a fully conducting plasma, $\sigma \longrightarrow \infty$

So c**E** = -(**v** x **B**).

$$\boxed{\frac{\partial B}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0}$$

The results are the equations of compressible inviscid ideal MHD:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B} + P^*] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*)\mathbf{v} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = 0$$

$$\frac{\partial B}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

Where $E = \rho\, v^2/2 + e + B^2/2$ is total energy

$P^* = P + B^2/2$ is total pressure (gas + magnetic)

Plus an equation of state $P = P(\rho, T)$

Warning: used units so that $\mu=1$

Can also be written in non-conservative form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot \rho \mathbf{v}\mathbf{v} = -\nabla p + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$\frac{\partial e}{\partial t} + \nabla \cdot e\mathbf{v} = -\frac{p}{\rho}\nabla \cdot \mathbf{v}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Plus an equation of state $P = P(\rho, T)$

Useful form for numerical methods based on operator splitting (lecture 2)

Equation of state

Usually adopt the ideal gas law $\boxed{P=nkT}$

In thermal equilibrium, each internal degree of freedom has energy (kT/2). Thus, internal energy density for an ideal gas with m internal degrees of freedom

$$e = nm(kT/2).$$

Combining, $\boxed{P = (\gamma\text{-}1)e}$ where $\gamma = (m+2)/m$

For monoatomic gas (H), $\gamma=5/3$ (m=3)
diatomic gas (H_2), $\gamma=7/5$ (m=5)

Also common to use isothermal EOS $\boxed{P = C^2\rho}$ where C=isothermal sound speed when (radiative cooling time) << (dynamical time)

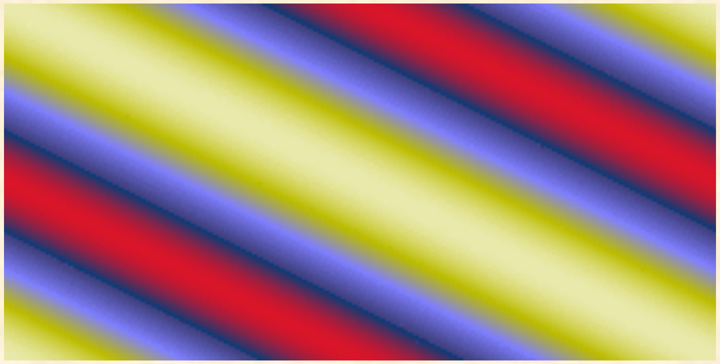
In some circumstances, an ideal gas law is not appropriate, and must use more complex (or tabular) EOS (e.g. for degenerate matter)

Sound waves

Another important characteristic of hyperbolic PDEs is they admit solutions of the form:

$$a = a_0 + a_1 \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t) \quad \text{(WAVES)}$$

When $a_1/a_0 << 1$; waves are small amplitude; linear
When $a_1/a_0 > 1$, waves are large amplitude, nonlinear (in this case, plane wave solution does not persist, for example nonlinear terms cause steepening)



Movie of density in linear sound wave

Linear waves are produced by small amplitude disturbances, with v < C (sound waves)

Dispersion relation for *hydrodynamic* waves.

Substitute solution for plane waves into hydrodynamic equations. Assume a uniform homogeneous background medium, so a_0=constant, and v_0=0. Keep only linear terms. Fluid equations become:

$$-i\omega\rho_1 = -i\rho_0 \mathbf{k}\cdot\mathbf{v_1}$$

$$-i\omega\mathbf{v_1} = -i\frac{1}{\rho_0}\mathbf{k}P_1$$

$$-i\omega P_1 = -i\gamma P_0 \mathbf{k}\cdot\mathbf{v_1}$$

Linear system with constant coefficients! Solutions require det(A) =0, which requires

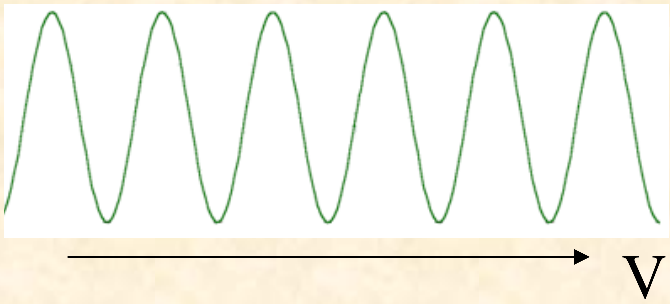
$\boxed{\omega^3(\omega^2 - C^2k^2) = 0}$ where $C^2 = \gamma P_0/\rho_0$ is the adiabatic sound speed

Apparently 5 modes; 3 advection modes and 2 sound waves with

$$\omega/k = \pm C$$

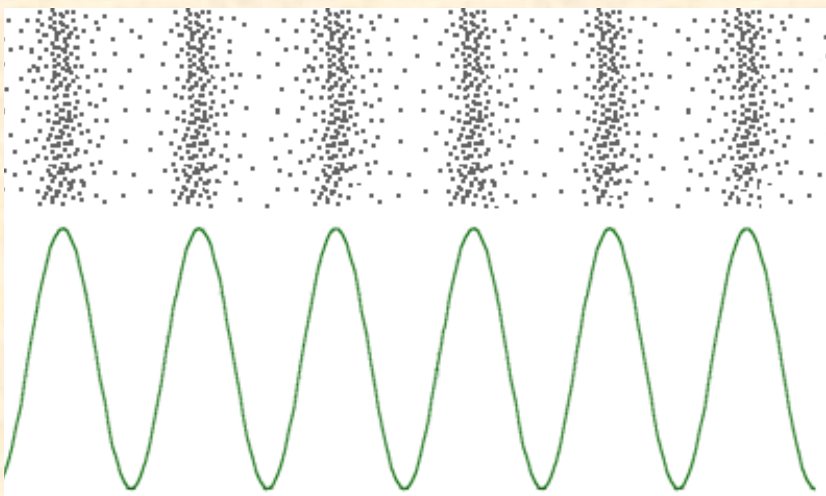
Summary of wave modes in *hydrodynamics*:

1. Entropy waves. Advect constant density field at V.



e.g. advection of sinusoidal density profile

2. Sound waves. Density, velocity, and pressure fluctuations that propagate at V+C and V-C.



Dispersion relation for MHD waves.

Substitute solution for plane waves into MHD equations. Assume a uniform homogeneous background medium, so a_0=constant, and v_0=0. Get a much more complicated dispersion relation (derivation is non-trivial! see Jackson):

$$[\omega^2 - (\mathbf{k}\cdot\mathbf{v}_A)^2][\omega^4 - \omega^2 k^2 (v_A^2 + C^2) + k^2 C^2 (\mathbf{k}\cdot\mathbf{v}_A)^2] = 0$$

Where $\mathbf{v}_A = \frac{\mathbf{B}}{\sqrt{4\pi\rho}}$ is the Alfven speed

$C^2 = \gamma P_0/\rho_0$ is the sound speed

There are three modes (only one in hydrodynamics!):

Alfven wave propagates at V_A

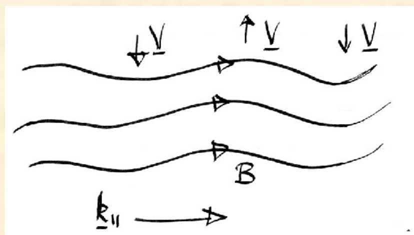
Slow and fast magnetosonic waves propagating at C_s and C_f

(Of course, the entropy mode is also present in both cases)

MHD Wave Modes.

1. Alfven Waves

Zero-frequency when k perpendicular to B (propagate along B), incompressible. Represent propagating transverse perturbations of field.



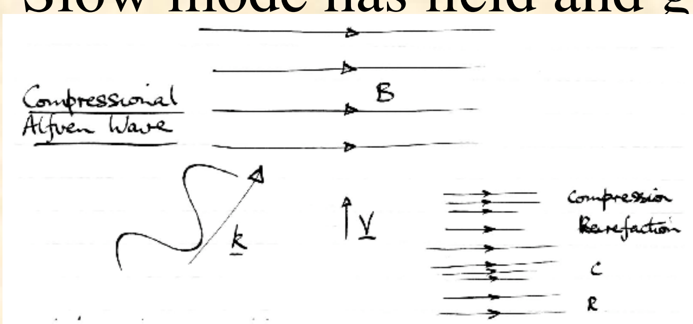
$$\mathbf{V}_A = \frac{\mathbf{B}}{\sqrt{4\pi\rho}}$$

2. Fast and Slow Magnetosonic Waves

Compressible perturbations of both field and gas.

Fast mode has field and gas compression in phase

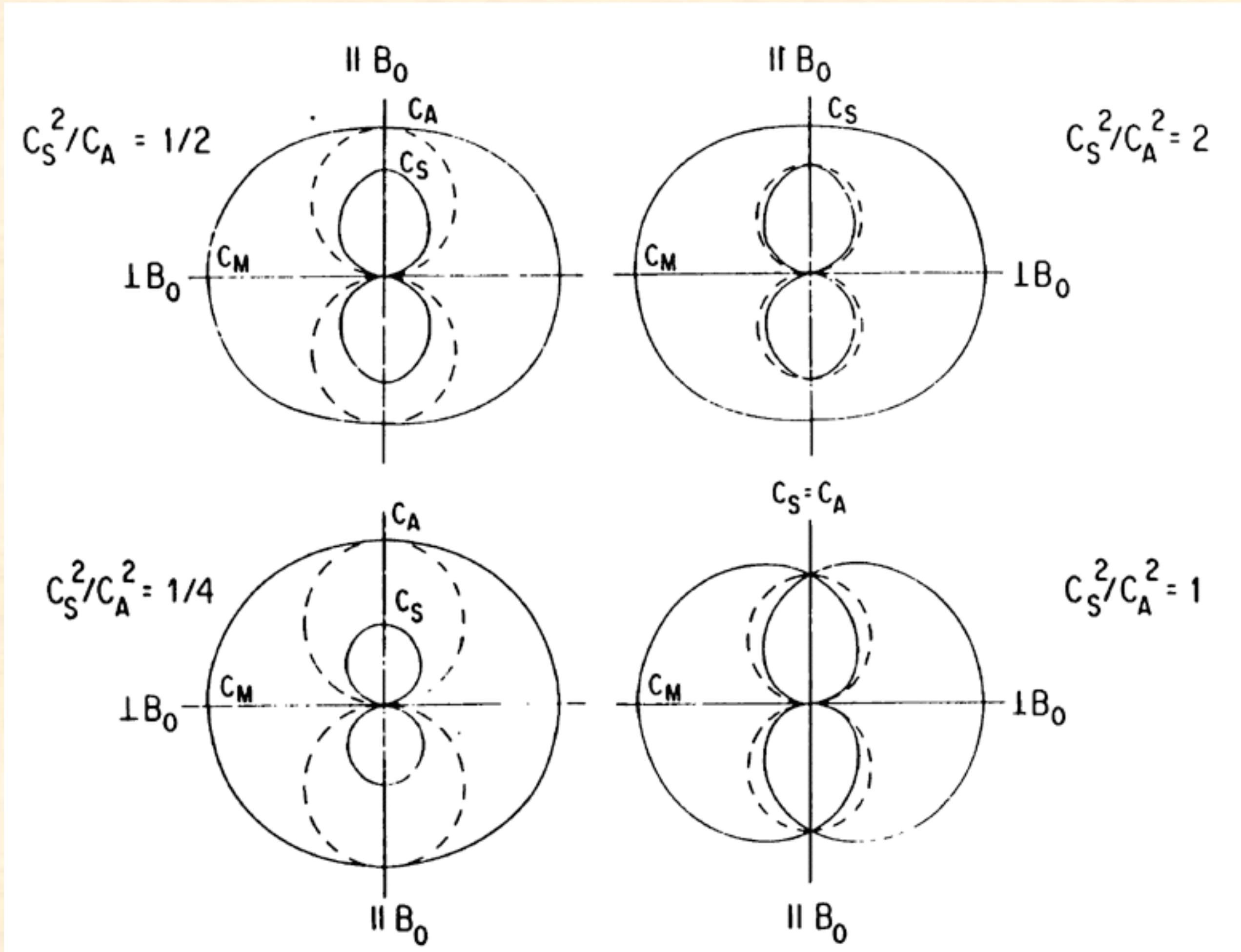
Slow mode has field and gas compression out of phase.



$$C_{f,s}^2 = \frac{1}{2}\left([C^2 + C_A^2] \pm \sqrt{[C^2 + C_A^2]^2 - 4C^2 C_{Ax}^2} \right)$$

$$C_A^2 = (B_x^2 + B_y^2 + B_z^2)/i(4\pi\rho), \qquad C_{Ax}^2 = B_x^2/(4\pi\rho)$$

Phase velocities of MHD waves: Friedrichs diagrams.



Note for in some cases, modes are degenerate. Eigenvalues of linearized MHD equations are not always linearly independent. MHD equations are not *strictly hyperbolic*.

Linear Instabilities

Going beyond the study of waves and shocks in fluids requires learning about the zoo of MHD instabilities in fluids.

See monographs by Chandrasekhar 1965
Drazin & Reid 1981

Probably the most important are:

1. Gravitational instability.
2. Thermal instability.
3. Rayleigh-Taylor (RT) instability.
4. Richtmyer-Meshkov (RM) instability.
5. Kelvin-Helmholtz (KH) instability.
6. Magneto-rotational instability (MRI)
7. Kink instability (current driven)
8. Sausage/Ballooning instability (pressure-gradient driven)

Further Plasma References

- NRL Plasma Formulary: http://wwwppd.nrl.navy.mil/nrlformulary/
- Plasma Science: Advancing Knowledge in the National Interest (2007), National Research Council, http://books.nap.edu/openbook.php?record_id=11960&page=9
- www.pppl.gov
- Workshop on Opportunities in Plasma Astrophysics, Jan. 18-21, 2010 at PPPL: http://www.pppl.gov/conferences/2010/WOPA/
- many more…

- Textbooks:
- F. F. Chen simplest introduction with many physical insights
- Goldston & Rutherford, somewhat more advanced, but still for beginning graduate student or upper level undergraduate
- Kulsrud, Plasma Physics for Astrophysics
- Many others, some much more mathematical or advanced:
  Hazeltine & Waelbroeck, Friedberg, Boyd & Sanderson, Dendy, Bittencourt, Wesson, Krall & Trivelpiece, Miyamoto, Ichimaru, Spitzer (elegantly brief), Stix, others
- Blandford & Thorne's draft book has chapters on plasma physics:
  http://www.pma.caltech.edu/Courses/ph136/yr2006/text.html