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A Landau fluid model for electromagnetic plasma microturbulence

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A fluid model is developed for the description of microturbulence and transport in magnetized, long mean-free-path plasmas. The model incorporates both electrostatic and magnetic fluctuations, as well as finite Larmor radius and kinetic effects. Multispecies Landau fluid equations are derived from moments of the electromagnetic gyrokinetic equation, using fluid closures which model kinetic effects. A reduced description of electron dynamics, appropriate for the study of microturbulence on characteristic ion drift and Alfvén scales, is derived via an expansion in the electron to ion mass ratio. The reduced electron equations incorporate curvature, ∇B, and linear and nonlinear $E\times B$ drift effects, needed to model the electron contribution to the drive and damping of ion gyroradius scale instabilities in tokamaks. The Landau fluid model is linearly benchmarked against gyrokinetic codes, and found to reproduce the toroidal finite beta ion temperature gradient and kinetic ballooning instabilities. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1374238]

I. INTRODUCTION

The development of an accurate and numerically efficient model of plasma microturbulence and transport in the kinetic, long mean-free-path regime characteristic of the core of magnetic fusion devices is a long standing challenge. Progress has been made via the development of the nonlinear gyrokinetic equation,[1–4] and its numerical solution using direct,[5,6] particle in cell,[2,7–9] and ''gyrofluid''[10–14] methods. Gyrofluid models take velocity space moments of the five-dimensional gyrokinetic equation to produce a reduced three-dimensional fluid description. Kinetic effects are modeled via appropriately chosen fluid closures. Here we use the term ''Landau fluid,'' which emphasizes the use of fluid closures which model Landau damping, interchangeably with ''gyrofluid,'' which emphasizes that the fluid equations are moments of the gyrokinetic equation in gyrocenter space.

The importance of incorporating magnetic fluctuations (also called finite β effects, where β is the ratio of plasma pressure to magnetic pressure) in descriptions of ion gyroradius scale dynamics, has been identified by numerous authors. Magnetic fluctuations impact the growth rates of predominantly electrostatic linear instabilities, for example the finite β stabilization of the collisionless toroidal ion temperature gradient (ITG) mode,[15,16] and introduce electromagnetic instabilities, such as the kinetic ballooning mode (KBM).[17–22] In addition, magnetic fluctuations are expected to significantly impact nonlinear dynamics and zonal flow generation.[23,24] Linear and nonlinear electromagnetic effects are well documented in the extensive literature on collisional plasmas. The strong impact of magnetic fluctuations in collisional Braginskii[25] simulations (see Refs. 23, 26–29 and references therein for details on recent work in electromagnetic edge turbulence) motivates the development of models for the dynamics of kinetic, long mean-free-path plasmas which include magnetic fluctuations and nonadiabatic electrons. It should be noted that the collisionless ITG mode and the ion drift resonance driven KBM instability, considered in Sec. VI, both require kinetic effects not present in the standard Braginskii model for an accurate description.

Here we develop an extension of earlier electrostatic gyrofluid models[30,31] to incorporate magnetic fluctuations and nonadiabatic passing electron dynamics. (This model can alternately be viewed as an extension of Waltz *et al.*[14] to include more ion moments, the mirror force, different models of toroidal kinetic effects, and a numerically efficient reduced electron model.) A set of general multispecies electromagnetic gyrofluid equations are derived from velocity space moments of the nonlinear gyrokinetic equation in Sec. III. The moment hierarchy is truncated using a set of closures derived to model kinetic effects, including collisionless phase mixing due to parallel streaming and toroidal drifts, as well as linear and nonlinear finite-Larmor-radius (FLR) effects. The general set of gyrofluid equations can be used to describe electron as well as ion dynamics. However, for many problems a more numerically efficient reduced model is appropriate for the electrons. The derivation of a reduced electron model which can be implemented in practical numerical simulations of electromagnetic ion drift and Alfvén scale turbulence is a key result of this paper. Section IV describes the physical motivation and mathematical derivation of the reduced electron equations. These reduced electron equations include the effects of electron temperature and density gradients, electron $\mathbf{E}\times\mathbf{B}$ motion, Landau damping, electron–ion collisions and the parallel electron currents which, along with parallel ion currents, give rise to the parallel magnetic potential. The system of equations is completed with the gyrokinetic Poisson equation and parallel Ampere's Law in Sec. V. The Landau fluid system of equations is then benchmarked with linear gyrokinetic theory in Sec. VI. Linear growth rates and frequencies are compared

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for both the toroidal finite β collisionless ITG mode and the kinetic ballooning mode. Additional benchmarks and results from nonlinear toroidal turbulence simulations using the electromagnetic Landau fluid model developed here are presented in Ref. 32.

II. THE GYROKINETIC EQUATION

The starting point for the derivation of the fluid equations is the nonlinear electromagnetic gyrokinetic equation of Brizard,[33] based upon earlier gyrokinetic work by many authors.[1–4,34,35]

The standard gyrokinetic ordering is invoked as follows:

$$\frac{\omega}{\Omega_i}\sim\frac{k_\parallel v_{ti}}{\Omega_i}\sim\frac{e\phi}{T}\sim\frac{\delta B}{B}\sim\frac{F_1}{F_0}\sim\frac{\rho_i}{L}\sim\varepsilon\ll 1,\quad k_\perp\rho_i\sim 1, \tag{1}$$

where ω is a characteristic frequency of the fluctuations, and $k_\parallel$ and $k_\perp$ are typical fluctuation wave numbers parallel and perpendicular to the equilibrium magnetic field. Ω_i is the ion cyclotron frequency, $v_{ti}=\sqrt{T_i/m_i}$ is the ion thermal speed, and $\rho_i=v_{ti}/\Omega_i$ is the thermal ion gyroradius. L is a typical equilibrium scale length, such as the density scale length $L_n=-\nabla(\ln n_0)^{-1}$, the temperature scale length $L_T=-\nabla(\ln T_0)^{-1}$, or the plasma minor radius (a) or major radius (R). T and B are typical equilibrium temperatures and magnetic fields, and F_0 is the equilibrium distribution. F_1 is the fluctuating distribution function, ϕ is the electrostatic potential (which is assumed to have no equilibrium component), and δB is the fluctuating component of the magnetic field.

Gyrokinetics averages over the fast gyromotion of the particles around a strong magnetic field, reducing the kinetic equation from three to two velocity space dimensions, and leaving the magnetic moment μ as a rigorously conserved quantity. The gyrokinetic ordering takes advantage of the spatial anisotropy created by the strong magnetic field. Parallel to the field, particles can stream freely, and fluctuating wavelengths are long, $k_\parallel L\sim 1$. Perpendicular to the field, particle motion is strongly restricted, and wavelengths scale with the gyroradius $k_\perp\rho_i\sim 1$.

The fluctuating distribution function is ordered small compared to the equilibrium distribution, which here is taken as a Maxwellian. Nonetheless, perpendicular gradients of fluctuating quantities are the same order as perpendicular gradients of the equilibrium $(k_\perp F_1\sim F_0/L)$, and hence the perpendicular nonlinearities due to the $\mathbf{E}\times\mathbf{B}$ drift and field line bending are kept, while parallel nonlinearities are small, and are ordered out here.

Brizard's electromagnetic gyrokinetic equation can be written in the form

$$\frac{\partial F}{\partial t}+\dot{\mathbf{X}}\cdot\nabla F+\dot{v}_\parallel\frac{\partial F}{\partial v_\parallel}=C(F), \tag{2}$$

where F is the gyrocenter distribution function in the gyrocenter phase space coordinates $(\mathbf{X},v_\parallel,\mu,\zeta)$. Within the gyrokinetic ordering $(\omega\ll\Omega_i)$, the gyrophase angle ζ is effectively averaged over, and does not appear explicitly $(\partial F/\partial\zeta=0)$. The gyrocenter magnetic moment $\mu=v_\perp^2/2B+\mathcal{O}(\varepsilon)$ is exactly conserved and enters the equations only as a parameter. An as yet undefined collision operator $C(F)$ has been added to the right-hand side.

Equation (2) is solved through $\mathcal{O}(\varepsilon^2)$ in the gyrokinetic ordering defined above. When ordering terms in the gyrokinetic equation, all frequencies are compared to Ω_i, and all lengths to ρ_i. Hence, for example, $\partial F/\partial t\sim\omega F_1$ is $\mathcal{O}(\varepsilon^2)$, because $\partial F_0/\partial t=0$, $F_1/F_0\sim\varepsilon$, and $\omega/\Omega_i\sim\varepsilon$. Any gradient operator acting on F_0 or B is $\mathcal{O}(\varepsilon)$ because $\rho_i/L\sim\varepsilon$. A parallel gradient on F_1 is $\mathcal{O}(\varepsilon^2)$ because $k_\parallel\rho_i\sim\varepsilon$. However, a perpendicular gradient acting on F_1 is $\mathcal{O}(\varepsilon)$ because $k_\perp\rho_i\sim 1$. Because ∇F is $\mathcal{O}(\varepsilon)$, $\dot{\mathbf{X}}$ is needed only to $\mathcal{O}(\varepsilon)$, while $\dot{v}_\parallel$ must include terms through $\mathcal{O}(\varepsilon^2)$.

The fluctuating magnetic field $\delta\mathbf{B}$ is described to lowest order in terms of a magnetic potential along the equilibrium field, $\delta\mathbf{B}=\nabla\times A_\parallel\hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is a unit vector along the equilibrium field. Note that $A_\parallel$ and $A_\perp$ are fluctuating quantities. The equilibrium magnetic field is denoted by $\mathbf{B}$ or $B\hat{\mathbf{b}}$, never as a magnetic potential. The perturbation along the equilibrium field $(\delta B_\parallel)$ is small for $\beta\ll 1$, as can be seen from perpendicular force balance, and $\delta B_\parallel$ is neglected here.

The gyrocenter velocity is then given by

$$\dot{\mathbf{X}}=v_\parallel\left[\hat{\mathbf{b}}+\frac{\langle\delta\mathbf{B}_\perp\rangle}{B}\right]+\mathbf{v}_E+\mathbf{v}_d, \tag{3}$$

where the angular brackets denote gyroangle averages. The first term on the right represents free streaming along the total magnetic field. The second term is the gyroaveraged $\mathbf{E}\times\mathbf{B}$ drift velocity, $\mathbf{v}_E=(c/B)\hat{\mathbf{b}}\times\nabla\langle\phi\rangle$. $\mathbf{v}_d$ is the combined curvature and ∇B drift velocity. In general, $\mathbf{v}_d$ can be written

$$\begin{aligned}\mathbf{v}_d&=\frac{v_\parallel^2}{\Omega}\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})+\frac{\mu}{\Omega}\hat{\mathbf{b}}\times\nabla B\\&=\frac{v_\parallel^2+\mu B}{\Omega B^2}\mathbf{B}\times\nabla B+\frac{v_\parallel^2}{\Omega B^2}\hat{\mathbf{b}}\times(\nabla\times\mathbf{B}\times\mathbf{B}).\end{aligned} \tag{4}$$

Using the equilibrium relations $\nabla p=(1/c)\mathbf{J}\times\mathbf{B}$ and $\nabla\times\mathbf{B}=(4\pi/c)\mathbf{J}$, this can be written

$$\mathbf{v}_d=\frac{v_\parallel^2+\mu B}{\Omega B^2}\mathbf{B}\times\nabla B+\frac{v_\parallel^2}{\Omega B^2}\hat{\mathbf{b}}\times\nabla p. \tag{5}$$

The second term on the right is small for $\beta\ll 1$,[36] and is neglected here for simplicity and to maintain consistency with neglecting $\delta B_\parallel$. A cancellation occurs between the ∇p term in $\mathbf{v}_d$ and a finite $\delta B_\parallel$ term.[17,37] Hence it does not improve accuracy to keep the ∇p term until $\delta B_\parallel$ has been fully included. The definition

$$\mathbf{v}_d\doteq\frac{v_\parallel^2+\mu B}{\Omega B^2}\mathbf{B}\times\nabla B, \tag{6}$$

is used henceforth.

The gyrocenter parallel acceleration can be written

$$\begin{aligned}\dot{v}_\parallel=&-\frac{e}{mc}\frac{\partial\langle A_\parallel\rangle}{\partial t}-\frac{e}{m}\left(\hat{\mathbf{b}}+\frac{\langle\delta\mathbf{B}_\perp\rangle}{B}\right)\cdot\nabla\langle\phi\rangle\\&-\mu\left(\hat{\mathbf{b}}+\frac{\langle\delta\mathbf{B}_\perp\rangle}{B}\right)\cdot\nabla B+v_\parallel(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E.\end{aligned} \tag{7}$$

The first two terms on the right-hand side represent the total parallel electric field, which includes both a magnetic induction term $-1/c(\partial\langle A_\parallel\rangle/\partial t)$, and an electrostatic term evaluated along the total magnetic field. The next term is the total mirror force, and the final term is important for phase space conservation.[30,38]

Using the definition $\delta\mathbf{B}=\nabla\times A_\parallel\hat{\mathbf{b}}$, the term $\delta\mathbf{B}_\perp$ can be written as follows:

$$\delta\mathbf{B}_\perp=\hat{\mathbf{b}}\times(\delta\mathbf{B}\times\hat{\mathbf{b}})=-\hat{\mathbf{b}}\times\nabla A_\parallel+\hat{\mathbf{b}}\times\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\,A_\parallel, \tag{8}$$

or upon gyroaveraging

$$\langle\delta\mathbf{B}_\perp\rangle=-\hat{\mathbf{b}}\times\nabla\langle A_\parallel\rangle+\hat{\mathbf{b}}\times\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\,\langle A_\parallel\rangle. \tag{9}$$

The second term on the right hand side is $\mathcal{O}(\varepsilon^2)$ and does not enter Eq. (2) to the required order.

The gyroangle averages are expressed in terms of a gyroaveraging operator J_0 as follows:

$$\langle\phi\rangle=J_0(\alpha)\phi,\quad \langle A_\parallel\rangle=J_0(\alpha)A_\parallel,$$

where α is the operator defined by

$$\alpha\doteq-i\frac{\sqrt{2\mu B}}{\Omega_i}\nabla_\perp,$$

or, in Fourier space

$$\alpha=\frac{\sqrt{2\mu B}}{v_{ti}}k_\perp\rho_i.$$

The operator J_0,

$$\begin{aligned}J_0(\alpha)&=\frac{1}{2\pi}\int_0^{2\pi}d\zeta\exp(i\alpha\cos\zeta)\\&=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{i\alpha}{2}\right)^{2n}\\&=\sum_{n=0}^{\infty}\frac{1}{(n!)^2}\left(\frac{\sqrt{2\mu B}}{2\Omega}\right)^{2n}\nabla_\perp^{2n},\end{aligned} \tag{10}$$

is a simple Bessel function in Fourier space. In real space, J_0 does not in general commute with other operators, and must be manipulated with care. J_0 operates only on the electrostatic potential ϕ and the parallel magnetic potential $A_\parallel$.

Defining the unit vector along the total magnetic field $\tilde{\mathbf{b}}=\hat{\mathbf{b}}+(\langle\delta\mathbf{B}_\perp\rangle/B)$ and the total parallel electric field $\tilde{E}_\parallel=-(1/c)(\partial/\partial t)J_0A_\parallel-\tilde{\mathbf{b}}\cdot\nabla J_0\phi$, the gyrokinetic equation can be written

$$\begin{aligned}&\frac{\partial F}{\partial t}+(v_\parallel\tilde{\mathbf{b}}+\mathbf{v}_E+\mathbf{v}_d)\cdot\nabla F\\&\quad+\left[\frac{e}{m}\tilde{E}_\parallel-\mu\tilde{\mathbf{b}}\cdot\nabla B+v_\parallel(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right]\frac{\partial F}{\partial v_\parallel}=C(F).\end{aligned} \tag{11}$$

III. ELECTROMAGNETIC GYROFLUID EQUATIONS

Gyrofluid equations are derived by taking velocity space moments of Eq. (11), and implementing closures to model kinetic effects. For simplicity of notation, the derivation for a single ion species is presented here. The subscript i is omitted in this section, and all quantities (v_t,Ω,T, etc.) are taken to refer to the single ion species unless otherwise noted. The full, normalized equations for the multispecies case are given in Sec. III E.

Because $k_\perp\rho_i\sim1$, finite-Larmor-radius (FLR) effects must be accounted for, both in the moment equations and in the closures.

In order to simplify the process of taking velocity space moments, all functions of velocity space ($F,J_0,\mu,v_\parallel$, etc.) are moved to the same side of the spatial and temporal operators.

The first two terms in Eq. (11), $B(\partial F/\partial t)=(\partial/\partial t)FB$ and $Bv_\parallel\hat{\mathbf{b}}\cdot\nabla F=\mathbf{B}\cdot\nabla(FBv_\parallel/B)$ are easily put in a form suitable for taking moments. The next three terms require modification.

Noting that spatial derivatives are taken with μ and $v_\parallel$ fixed, we can write for any field A

$$\nabla J_0A=J_0\nabla A+A\nabla J_0, \tag{12}$$

where

$$\nabla J_0(\alpha)=\frac{\partial J_0}{\partial\alpha}\nabla\alpha=J_1(\alpha)\frac{\alpha}{2B}\nabla B. \tag{13}$$

The term representing free streaming along the fluctuating magnetic field, $-v_\parallel/B(\hat{\mathbf{b}}\times\nabla J_0A_\parallel)$, can be combined with the $\mathbf{E}\times\mathbf{B}$ drift by introducing the operator notation:

$$\phi'=\phi-\frac{v_\parallel}{c}A_\parallel,\quad \mathbf{v}_E'=\frac{c}{B}\hat{\mathbf{b}}\times\nabla J_0\phi',\quad \mathbf{v}_\phi'=\frac{c}{B}\hat{\mathbf{b}}\times\nabla\phi'. \tag{14}$$

Using Eqs. (12) and (13)

$$B\mathbf{v}_E'\cdot\nabla F=B\frac{c}{B}\hat{\mathbf{b}}\times\left(J_0\nabla\phi'+J_1\frac{\alpha}{2B}\phi'\nabla B\right)\cdot\nabla F. \tag{15}$$

The J_1 term above can be neglected as it is $\mathcal{O}(\varepsilon^3)$ due to the presence of ϕ', ∇B, and ∇F, each of which are $\mathcal{O}(\varepsilon)$. Noting that $J_0\nabla F=\nabla J_0F-(\alpha/2B)FJ_1\nabla B$, and introducing the operator notation

$$i\omega_d\doteq\frac{v_t^2}{\Omega B^2}\mathbf{B}\times\nabla B\cdot\nabla, \tag{16}$$

allows us to write

$$\begin{aligned}&B\frac{c}{B}\hat{\mathbf{b}}\times J_0\nabla\phi'\cdot\nabla F\\&\quad=B\frac{c}{B}\hat{\mathbf{b}}\times\nabla\phi'\cdot\nabla(J_0F)-F\frac{c\alpha}{2B}J_1\hat{\mathbf{b}}\times\nabla\phi'\cdot\nabla B\\&\quad=\mathbf{v}_\phi'\cdot\nabla(J_0FB)+\frac{e}{T}FB\left(J_0+J_1\frac{\alpha}{2}\right)i\omega_d\phi'.\end{aligned} \tag{17}$$

Invoking the approximation outlined in Eqs. (4)–(6), and noting that $i\omega_dB=0$, the ∇B and curvature drift term can be written

$$B\mathbf{v}_d\cdot\nabla F=i\omega_d[FB(v_\parallel^2+\mu B)]. \tag{18}$$

Turning now to the $\dot{v}_\parallel(\partial F/\partial v_\parallel)$ terms, we note first that all components of $\dot{v}_\parallel$ except the lowest order mirror force $-\mu\hat{\mathbf{b}}\cdot\nabla B$ are $\mathcal{O}(\varepsilon^2)$ and therefore involve only the equilibrium distribution, which is taken to be

$$F_0=\frac{n_0}{(2\pi v_t^2)^{3/2}}e^{-v_\parallel^2/2v_t^2-\mu B/v_t^2}. \tag{19}$$

The electric field terms can be written as follows to $\mathcal{O}(\varepsilon^2)$

$$\begin{aligned}-B\frac{e}{m}\hat{\mathbf{b}}\cdot\nabla(J_0\phi)\frac{\partial F}{\partial v_\parallel}&=-\frac{e}{m}\frac{\partial F_0}{\partial v_\parallel}B\hat{\mathbf{b}}\cdot\nabla(J_0\phi)\\&=-\frac{e}{m}\hat{\mathbf{b}}\cdot\nabla\left(\frac{\partial F_0}{\partial v_\parallel}BJ_0\phi\right)+\frac{e}{m}J_0\phi\frac{\partial F_0}{\partial v_\parallel}\\&\quad\times B\left(1-\frac{\mu B}{v_t^2}\right)\hat{\mathbf{b}}\cdot\nabla\ln B,\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{\partial F}{\partial v_\parallel}\frac{e}{m}(\hat{\mathbf{b}}\times\nabla J_0A_\parallel)\cdot\nabla J_0\phi&=\frac{e}{m}\frac{\partial F_0}{\partial v_\parallel}(\hat{\mathbf{b}}\times J_0\nabla A_\parallel)\cdot J_0\nabla\phi\\&=\frac{e}{m}\frac{\partial F_0}{\partial v_\parallel}J_{0A_\parallel}J_{0\phi}(\hat{\mathbf{b}}\times\nabla A_\parallel)\cdot\nabla\phi,\end{aligned} \tag{21}$$

where the notation $J_{0A_\parallel}$ and $J_{0\phi}$ is used to indicate the field on which the Bessel function operator acts. All J_1 terms above have been dropped as they are $\mathcal{O}(\varepsilon^3)$.

The mirror force terms can be written

$$\frac{\partial F}{\partial v_\parallel}B(-\mu\hat{\mathbf{b}}\cdot\nabla B)=-\mu B^2\frac{\partial F}{\partial v_\parallel}\hat{\mathbf{b}}\cdot\nabla\ln B, \tag{22}$$

$$\begin{aligned}\frac{\partial F}{\partial v_\parallel}\mu(\hat{\mathbf{b}}\times\nabla J_0A_\parallel)\cdot\nabla B&=\frac{\partial F_0}{\partial v_\parallel}\frac{\mu}{B}J_0(\nabla B\times\mathbf{B})\cdot\nabla A_\parallel\\&=-\frac{\partial F_0}{\partial v_\parallel}BJ_0\frac{e\mu B}{cT}i\omega_dA_\parallel,\end{aligned} \tag{23}$$

where the $J_1\nabla B$ term has cancelled exactly.

Finally, the phase space conservation term can be rewritten, omitting the finite β component for consistency with the treatment of the curvature drift

$$\frac{\partial F}{\partial v_\parallel}Bv_\parallel(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E=-\frac{\partial F_0}{\partial v_\parallel}v_\parallel\frac{c}{B^2}\mathbf{B}\times\nabla B\cdot\nabla J_0\phi \tag{24}$$

$$=-\frac{e}{T}\left[\frac{\partial}{\partial v_\parallel}(F_0BJ_0v_\parallel)-F_0BJ_0\right]i\omega_d\phi. \tag{25}$$

Combining all the above terms, and defining $\nabla_\parallel=\hat{\mathbf{b}}\cdot\nabla$, the electromagnetic gyrokinetic equation can be written in the following cumbersome but useful form:

$$\begin{aligned}&\frac{\partial}{\partial t}FB+B\nabla_\parallel Fv_\parallel+\mathbf{v}_\phi'\cdot\nabla(FBJ_0)+2FBJ_0i\omega_d\frac{e\phi'}{T}+FBJ_1\frac{\alpha}{2}i\omega_d\frac{e\phi'}{T}+\frac{e}{cT}v_\parallel FBJ_0i\omega_dA_\parallel+\frac{1}{v_t^2}i\omega_d[FB(v_\parallel^2+\mu B)]\\&\quad-\frac{e}{mc}\frac{\partial}{\partial v_\parallel}\left(F_0BJ_0\frac{\partial A_\parallel}{\partial t}\right)-\frac{e}{m}\nabla_\parallel\left(\frac{\partial F_0}{\partial v_\parallel}BJ_0\phi\right)+\frac{e}{m}J_0\phi\frac{\partial F_0}{\partial v_\parallel}B\left(1-\frac{\mu B}{v_t^2}\right)\nabla_\parallel\ln B+\frac{e}{m}\frac{\partial F_0}{\partial v_\parallel}J_{0A_\parallel}J_{0\phi}\\&\quad\times(\hat{\mathbf{b}}\times\nabla A_\parallel)\cdot\nabla\phi-\mu B^2\frac{\partial F}{\partial v_\parallel}\nabla_\parallel\ln B-\frac{\partial F_0}{\partial v_\parallel}BJ_0\frac{e\mu B}{cT}i\omega_dA_\parallel-\frac{\partial}{\partial v_\parallel}(FBJ_0v_\parallel)\frac{e}{T}i\omega_d\phi=0.\end{aligned} \tag{26}$$

Nearly all terms with velocity space dependence are now grouped on the same side of spatial and temporal operators so that moments may easily be taken. The exception is the $v_\parallel$ term which appears in $\phi'=\phi-(v_\parallel/c)A_\parallel$ and $\mathbf{v}_\phi'=(c/B)\hat{\mathbf{b}}\times\nabla\phi'$. However, $v_\parallel$ commutes with J_0, J_1 and all spatial operators, and may be easily moved to the appropriate place inside velocity space integrals. The collision operator $C(F)$ has been omitted here. Collisions are considered in Sec. III D.

Equation (26) contains terms through $\mathcal{O}(\varepsilon^2)$ in the gyrokinetic ordering. Assuming a time independent equilibrium distribution F_0 with gradients that scale as $1/L$, only two first-order terms remain. These terms represent free streaming along the equilibrium field, and the lowest order mirror force. To first order, the equation can be written

$$B\nabla_\parallel F_0v_\parallel-\mu B^2\frac{\partial F_0}{\partial v_\parallel}\nabla_\parallel\ln B=0, \tag{27}$$

a condition which is satisfied exactly by the equilibrium Maxwellian

$$F_0=F_M=\frac{n_0}{(2\pi v_t^2)^{3/2}}e^{-v_\parallel^2/2v_t^2-\mu B/v_t^2}.$$

This leaves only second-order terms in the equation.

We furthermore divide the first-order distribution F_1 into two parts, $F_1\doteq\tilde{f}+F_{1nc}$. Here F_{1nc} is defined to be an equilibrium part of the distribution with no time dependence and gradients which scale as $1/L$. It is further defined to be an exact solution of the equation

$$B\nabla_\parallel F_{1nc}v_\parallel+\frac{1}{v_t^2}i\omega_d[F_0B(v_\parallel^2+\mu B)]-\mu B^2\frac{\partial F_{1nc}}{\partial v_\parallel}\nabla_\parallel\ln B=0. \tag{28}$$

Note that the F_{1nc} contribution to all other terms is $\mathcal{O}(\varepsilon^3)$ or higher and can be neglected. This removes all terms with no time dependence, and leaves us with an evolution equation for the fluctuating first-order distribution $\tilde{f}$, containing only second-order terms which are either linear or quadratic in the fluctuating quantities $\tilde{f}$, ϕ, and $A_\parallel$

$$\frac{\partial}{\partial t}\tilde{f}B+B\nabla_\parallel\tilde{f}v_\parallel+\mathbf{v}_\phi\cdot\nabla[(F_0+\tilde{f})BJ_0]-\mathbf{v}_{A_\parallel}\cdot\nabla\left[(F_0+\tilde{f})B\frac{v_\parallel}{c}J_0\right]+2F_0BJ_0i\omega_d\frac{e\phi}{T}-F_0B\frac{v_\parallel}{c}J_0i\omega_d\frac{eA_\parallel}{T}+F_0BJ_1\frac{\alpha}{2}i\omega_d\frac{e\phi}{T}$$
$$-F_0B\frac{v_\parallel}{c}J_1\frac{\alpha}{2}i\omega_d\frac{eA_\parallel}{T}+\frac{i\omega_d}{v_t^2}[\tilde{f}B(v_\parallel^2+\mu B)]-\frac{e}{mc}\frac{\partial F_0}{\partial v_\parallel}BJ_0\frac{\partial A_\parallel}{\partial t}-\frac{e}{m}\nabla_\parallel\left(\frac{\partial F_0}{\partial v_\parallel}BJ_0\phi\right)+\frac{e}{m}J_0\phi\frac{\partial F_0}{\partial v_\parallel}B\left(1-\frac{\mu B}{v_t^2}\right)\nabla_\parallel\ln B$$
$$+\frac{e}{m}\frac{\partial F_0}{\partial v_\parallel}J_{0A_\parallel}J_{0\phi}(\hat{\mathbf{b}}\times\nabla A_\parallel)\cdot\nabla\phi-\mu B^2\frac{\partial\tilde{f}}{\partial v_\parallel}\nabla_\parallel\ln B-\frac{\partial F_0}{\partial v_\parallel}BJ_0\frac{\mu B}{c}i\omega_d\frac{eA_\parallel}{T}-\frac{\partial}{\partial v_\parallel}(F_0BJ_0v_\parallel)i\omega_d\frac{e\phi}{T}=0. \tag{29}$$

Terms containing ϕ and $A_\parallel$ have been separated by defining $\mathbf{v}_\phi=(c/B)\hat{\mathbf{b}}\times\nabla\phi$ and $\mathbf{v}_{A_\parallel}=(c/B)\hat{\mathbf{b}}\times\nabla A_\parallel$. Nonlinear terms enter through $\mathbf{v}_\phi\cdot\nabla[\tilde{f}BJ_0]$, $\mathbf{v}_{A_\parallel}\cdot\nabla[\tilde{f}B(v_\parallel/c)J_0]$, and $e/m(\partial F_0/\partial v_\parallel)J_{0A_\parallel}J_{0\phi}(\hat{\mathbf{b}}\times\nabla A_\parallel)\cdot\nabla\phi$.

It is also possible to derive Eq. (29) starting with the conservative form of the gyrokinetic equation. Making sure to include the second-order part of $\langle\delta\mathbf{B}_\perp\rangle$ from Eq. (9), it is possible to prove Liouville's theorem to the required order

$$\frac{\partial B^*}{\partial t}+\nabla\cdot[B^*\dot{\mathbf{X}}]+\frac{\partial}{\partial v_\parallel}[B^*\dot{v}_\parallel]=0, \tag{30}$$

where $B^*=B+(mc/e)v_\parallel\hat{\mathbf{b}}\cdot\nabla\times\hat{\mathbf{b}}$ contains the parallel velocity correction. The gyrokinetic equation can then be written

$$\frac{\partial}{\partial t}FB^*+\nabla\cdot[FB^*\dot{\mathbf{X}}]+\frac{\partial}{\partial v_\parallel}[FB^*\dot{v}_\parallel]=0. \tag{31}$$

Again working within the context of the low β approximation $\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})=(1/B^2)\mathbf{B}\times\nabla B$, and rearranging terms, one finds Eq. (29) to second order as expected.

A further check on Eq. (29) is to calculate the linear nonadiabatic response in the local limit. Dividing the distribution into adiabatic and nonadiabatic pieces, $\tilde{f}=g-F_0J_0e\phi/T_0$, linearizing, transforming, and taking the $\nabla_\parallel\ln B=0$ limit, we find the expected nonadiabatic distribution

$$g=F_0\frac{\omega-\omega_*^T}{\omega-k_\parallel v_\parallel-\omega_{dv}}\frac{e}{T}J_0\left(\phi-\frac{v_\parallel}{c}A_\parallel\right), \tag{32}$$

where $\omega_*^T=\omega_*[1+\eta(v_\parallel^2/2v_t^2+\mu B/v_t^2-3/2)]$, $\omega_{dv}=\omega_d(v_\parallel^2+\mu B)/v_t^2$, and we have introduced the diamagnetic frequency $i\omega_*\doteq-(cT_0/eBn_0)\nabla n_0\cdot\hat{\mathbf{b}}\times\nabla$, and the ratio of scale lengths $\eta=L_n/L_T$.

A. The moment equations

Fluid moment equations can now be derived by taking velocity space moments of Eq. (29). In this section a careful distinction is made between equilibrium and fluctuating components, and equilibrium quantities are written with a subscript 0. Both $v_t=\sqrt{T_0/m}$ and $\rho_i=v_t/\Omega$ are defined in terms of equilibrium quantities. It should also be noted that because all terms in Eq. (29) are $\mathcal{O}(\varepsilon^2)$, only their lowest-order components need be kept, e.g., $T\to T_0$.

Velocity space moments are often defined in terms of the total distribution function F. Here we again separate F into equilibrium and fluctuating components $F=F_0+\tilde{f}$, noting that F_{1nc} and its moments do not enter the equations to the required order and can be neglected. Velocity space moments of

$$F_0=F_M=\frac{n_0}{(2\pi v_t^2)^{3/2}}e^{-v_\parallel^2/2v_t^2-\mu B/v_t^2},$$

are all well defined. We define the following moments of the fluctuating distribution:

$$\tilde{n}=\int\tilde{f}\,d^3v,\quad n_0\tilde{u}_\parallel=\int\tilde{f}v_\parallel\,d^3v,$$

$$\tilde{p}_\parallel=m\int\tilde{f}v_\parallel^2\,d^3v,\quad \tilde{p}_\perp=m\int\tilde{f}B\mu\,d^3v,$$

$$\tilde{q}_\parallel=-3mv_t^2n_0\tilde{u}_\parallel+m\int\tilde{f}v_\parallel^3\,d^3v,$$

$$\tilde{q}_\perp=-mv_t^2n_0\tilde{u}_\parallel+m\int\tilde{f}B\mu\,v_\parallel\,d^3v,$$

$$\tilde{r}_{\parallel,\parallel}=m\int\tilde{f}v_\parallel^4\,d^3v,\quad \tilde{r}_{\parallel,\perp}=m\int\tilde{f}B\mu\,v_\parallel^2\,d^3v,$$

$$\tilde{r}_{\perp,\perp}=m\int\tilde{f}B^2\mu^2\,d^3v,$$

$$\tilde{s}_{\perp,\perp}=-2mv_t^4n_0\tilde{u}_\parallel+m\int\tilde{f}B^2\mu^2v_\parallel\,d^3v,$$

$$\tilde{s}_{\parallel,\parallel}=-15mv_t^4n_0\tilde{u}_\parallel+m\int\tilde{f}v_\parallel^5\,d^3v,$$

$$\tilde{s}_{\parallel,\perp}=-3mv_t^4n_0\tilde{u}_\parallel+m\int\tilde{f}B\mu\,v_\parallel^3\,d^3v,$$

where $d^3v=2\pi dv_\parallel B\,d\mu$. The definitions of the q and s moments above have been chosen for consistency of notation with Beer.[30] Each moment is coupled to higher moments through the terms in Eq. (29) which contain factors of $v_\parallel$ or μ, including terms due to parallel free streaming, toroidal drift, FLR effects, and the mirror force. This moment hierarchy is truncated using closures described in the following sections in order to generate a useful set of equations.

Taking integrals of Eq. (29) of the form $2\pi\int dv_\parallel\,d\mu\,v_\parallel^j\mu^k$, and defining the shorthand $\langle A\rangle\doteq2\pi\int A\,dv_\parallel\,B\,d\mu$ yields the following set of moment equations:

$$\frac{\partial \tilde{n}}{\partial t}+B\nabla_\parallel \frac{n_0\tilde{u}_\parallel}{B}+\mathbf{v}_\phi\cdot\nabla\langle FJ_0\rangle-\frac{1}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\langle Fv_\parallel J_0\rangle + \left\langle F_0\left(2J_0+J_1\frac{\alpha}{2}\right)\right\rangle i\omega_d\frac{e\phi}{T_0}+\frac{1}{T_0}i\omega_d(\tilde{p}_\parallel+\tilde{p}_\perp)=0, \tag{33}$$

$$n_0\frac{\partial \tilde{u}_\parallel}{\partial t}+B\nabla_\parallel\frac{\tilde{p}_\parallel}{mB}+\mathbf{v}_\phi\cdot\nabla\langle Fv_\parallel J_0\rangle-\frac{1}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\langle Fv_\parallel^2J_0\rangle - \left\langle F_0v_\parallel^2\left(J_0+J_1\frac{\alpha}{2}\right)\right\rangle i\omega_d\frac{eA_\parallel}{cT_0}+\frac{1}{T_0}i\omega_d \times(\tilde{q}_\parallel+\tilde{q}_\perp+4p_0\tilde{u}_\parallel)+\langle F_0J_0\rangle\frac{e}{mc}\frac{\partial A_\parallel}{\partial t}+\frac{e}{m}\nabla_\parallel\langle F_0J_0\rangle\phi - \frac{e}{m}\phi\left\langle F_0J_0\left(1-\frac{\mu B}{v_t^2}\right)\right\rangle\nabla_\parallel\ln B-\frac{e}{mB}\langle F_0J_{0A_\parallel}J_{0\phi}\rangle\hat{\mathbf{b}} \times\nabla A_\parallel\cdot\nabla\phi+\frac{\tilde{p}_\perp}{m}\nabla_\parallel\ln B+\langle F_0\mu BJ_0\rangle i\omega_d\frac{eA_\parallel}{cT_0}=0, \tag{34}$$

$$\frac{\partial \tilde{p}_\parallel}{\partial t}+B\nabla_\parallel\frac{\tilde{q}_\parallel+3p_0\tilde{u}_\parallel}{B}+m\mathbf{v}_\phi\cdot\nabla\langle Fv_\parallel^2J_0\rangle-\frac{m}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\langle Fv_\parallel^3J_0\rangle + m\left\langle F_0v_\parallel^2\left(2J_0+J_1\frac{\alpha}{2}\right)\right\rangle i\omega_d\frac{e\phi}{T_0}+\frac{1}{v_t^2}i\omega_d(\tilde{r}_{\parallel,\parallel}+\tilde{r}_{\parallel,\perp}) + 2(\tilde{q}_\perp+p_0\tilde{u}_\parallel)\nabla_\parallel\ln B+2m\langle F_0v_\parallel^2J_0\rangle i\omega_d\frac{e\phi}{T_0}=0, \tag{35}$$

$$\frac{\partial \tilde{p}_\perp}{\partial t}+B^2\nabla_\parallel\left[\frac{1}{B^2}(\tilde{q}_\perp+p_0\tilde{u}_\parallel)\right]+mB\mathbf{v}_\phi\cdot\nabla\langle F\mu J_0\rangle - \frac{mB}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\langle F\mu v_\parallel J_0\rangle+mB\left\langle F_0\mu\left(2J_0+J_1\frac{\alpha}{2}\right)\right\rangle \times i\omega_d\frac{e\phi}{T_0}+\frac{1}{v_t^2}i\omega_d(\tilde{r}_{\parallel,\perp}+\tilde{r}_{\perp,\perp})=0, \tag{36}$$

$$\frac{\partial}{\partial t}(\tilde{q}_\parallel+3p_0\tilde{u}_\parallel)+B\nabla_\parallel\frac{\tilde{r}_{\parallel,\parallel}}{B}+m\mathbf{v}_\phi\cdot\nabla\langle Fv_\parallel^3J_0\rangle-\frac{m}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\langle Fv_\parallel^4J_0\rangle-m\left\langle F_0v_\parallel^4\left(J_0+J_1\frac{\alpha}{2}\right)\right\rangle i\omega_d\frac{eA_\parallel}{cT_0} + \frac{1}{v_t^2}i\omega_d(\tilde{s}_{\parallel,\parallel}+\tilde{s}_{\parallel,\perp}+18mv_t^4n_0\tilde{u}_\parallel)+\frac{3e}{c}\langle F_0v_\parallel^2J_0\rangle\frac{\partial A_\parallel}{\partial t}+3e\nabla_\parallel\langle F_0v_\parallel^2J_0\rangle\phi-3e\left\langle F_0v_\parallel^2\left(1-\frac{\mu B}{v_t^2}\right)J_0\right\rangle\phi\nabla_\parallel\ln B - \frac{3e}{B}\langle F_0v_\parallel^2J_{0A_\parallel}J_{0\phi}\rangle\hat{\mathbf{b}}\times\nabla A_\parallel\cdot\nabla\phi+3\tilde{r}_{\parallel,\perp}\nabla_\parallel\ln B+3mB\langle F_0\mu v_\parallel^2J_0\rangle i\omega_d\frac{eA_\parallel}{cT}=0, \tag{37}$$

$$\frac{\partial}{\partial t}(\tilde{q}_\perp+p_0\tilde{u}_\parallel)+B^2\nabla_\parallel\frac{\tilde{r}_{\parallel,\perp}}{B^2}+mB\mathbf{v}_\phi\cdot\nabla\langle Fv_\parallel\mu J_0\rangle-\frac{mB}{c}\mathbf{v}_{A_\parallel}\cdot\nabla\langle Fv_\parallel^2\mu J_0\rangle-mB\left\langle F_0v_\parallel^2\mu\left(J_0+J_1\frac{\alpha}{2}\right)\right\rangle i\omega_d\frac{eA_\parallel}{ct_0} + \frac{1}{v_t^2}i\omega_d(\tilde{s}_{\parallel,\perp}+\tilde{s}_{\perp,\perp}+5mv_t^4n_0\tilde{u}_\parallel)+\frac{eB}{c}\langle F_0\mu J_0\rangle\frac{\partial A_\parallel}{\partial t}+eB\nabla_\parallel\langle F_0\mu J_0\rangle\phi-eB\left\langle F_0\mu\left(1-\frac{\mu B}{v_t^2}\right)J_0\right\rangle\phi\nabla_\parallel\ln B - e\langle F_0\mu J_{0A_\parallel}J_{0\phi}\rangle\hat{\mathbf{b}}\times\nabla A_\parallel\cdot\nabla\phi+\tilde{r}_{\perp,\perp}\nabla_\parallel\ln B+mB^2\langle F_0\mu^2J_0\rangle i\omega_d\frac{eA_\parallel}{cT_0}=0. \tag{38}$$

B. Finite Larmor radius terms

Closures are developed for the finite Larmor radius terms appearing in Eqs. (33)–(38), using the techniques of Dorland[39] as adapted to the toroidal case by Beer.[30] We choose to evolve ion moments in guiding center space rather than real space in order to better describe both linear and nonlinear FLR effects, including the Bakshi–Linsker effect.[40] Nonetheless, our FLR terms, when expanded, contain higher velocity space moments and these must be carefully closed to properly model kinetic behavior.

Turning first to the Maxwellian FLR terms, we must close terms of the forms $\langle F_0v_\parallel^{2i}\mu^jJ_0\rangle$ and $\langle F_0v_\parallel^{2i}\mu^jJ_1\alpha\rangle$, where $i=0,1,2$ and $j=0,1,2$. Note that purely Maxwellian FLR terms with odd powers of $v_\parallel$ vanish identically, as F_M is even in $v_\parallel$, while J_0 and $J_1\alpha$ are independent of $v_\parallel$.

The FLR closures are chosen in careful consideration of the entire system of equations. It is the combination of J_0 terms from the $\mathbf{E}\times\mathbf{B}$ and $\mathbf{v}_{A_\parallel}$ terms with the J_0 terms in Poisson's equation and Ampere's Law which motivates the basic approximation $\langle J_0\rangle\approx\langle J_0^2\rangle^{1/2}\approx\Gamma_0(b)^{1/2}$, where $b=k_\perp^2\rho_i^2$. Following and extending Ref. 39, we choose

$$\langle F_0J_0\rangle=n_0\Gamma_0^{1/2}, \tag{39}$$

$$\langle F_0J_0v_\parallel^2\rangle=n_0v_t^2\Gamma_0^{1/2}, \tag{40}$$

$$\langle F_0J_0\mu\rangle=\frac{n_0v_t^2}{B}\frac{\partial}{\partial b}(b\Gamma_0^{1/2})=\frac{v_t^2}{2B}(2\Gamma_0^{1/2}+\hat{\nabla}_\perp^2), \tag{41}$$

$$\langle F_0J_0v_\parallel^4\rangle=3n_0v_t^4\Gamma_0^{1/2}, \tag{42}$$

$$\langle F_0J_0v_\parallel^2\mu\rangle=\frac{n_0v_t^4}{B}\frac{\partial}{\partial b}(b\Gamma_0^{1/2})=\frac{v_t^4}{2B}(2\Gamma_0^{1/2}+\hat{\nabla}_\perp^2), \tag{43}$$

$$\langle F_0 J_0 \mu^2\rangle \approx \frac{v_t^4}{B^2}\left[b\frac{\partial^2}{\partial b^2}(b\Gamma_0^{1/2}) + 2b\frac{\partial}{\partial b}(b\Gamma_0^{1/2})\right] = \frac{v_t^4}{B^2}(2\Gamma_0^{1/2} + \hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2). \tag{44}$$

The modified Laplacian operators $\hat{\nabla}_\perp^2$ and $\hat{\hat{\nabla}}_\perp^2$ are defined as follows:

$$\frac{1}{2}\hat{\nabla}_\perp^2 \Phi = b\frac{\partial \Gamma_0^{1/2}}{\partial b}\phi, \tag{45}$$

$$\hat{\hat{\nabla}}_\perp^2 \Phi = b\frac{\partial^2}{\partial b^2}(b\Gamma_0^{1/2})\phi, \tag{46}$$

where the notation $\Phi = \Gamma_0^{1/2}\phi$ has been introduced for the gyroaveraged electrostatic potential. The analogous notation $\mathcal{A}_\parallel = \Gamma_0^{1/2} A_\parallel$ is used for the gyroaveraged magnetic potential.

The J_1 terms are evaluated following Ref. 30, using the following trick:

$$\langle F J_1 \alpha\rangle \approx -\left.\frac{\partial}{\partial \zeta}\right|_{\zeta=1}\langle F J_0(\zeta\alpha)\rangle. \tag{47}$$

Again using $\langle F J_0\rangle \approx \Gamma_0^{1/2}$ yields

$$\langle F_0 J_1 \alpha\rangle \approx -\left.\frac{\partial}{\partial \zeta}\right|_{\zeta=1}\Gamma_0^{1/2}(\zeta^2 b) = -2b\frac{\partial \Gamma_0^{1/2}}{\partial b} = -\hat{\nabla}_\perp^2, \tag{48}$$

$$\langle F_0 J_1 v_\parallel^2 \alpha\rangle \approx -2v_t^2 b\frac{\partial \Gamma_0^{1/2}}{\partial b} = -v_t^2\hat{\nabla}_\perp^2, \tag{49}$$

$$\langle F_0 J_1 \mu\alpha\rangle \approx -\left.\frac{\partial}{\partial \zeta}\right|_{\zeta=1}\frac{v_t^2}{B}\frac{\partial}{\partial T_\perp}[T_\perp\langle F_0 J_0(\zeta\alpha)\rangle] = -2\frac{v_t^2}{B}\frac{\partial}{\partial b}\left(b^2\frac{\partial \Gamma_0^{1/2}}{\partial b}\right) = -2\frac{v_t^2}{B}\hat{\hat{\nabla}}_\perp^2, \tag{50}$$

$$\langle F_0 J_1 v_\parallel^4 \alpha\rangle \approx -6v_t^4 b\frac{\partial \Gamma_0^{1/2}}{\partial b} = -3v_t^4\hat{\nabla}_\perp^2, \tag{51}$$

$$\langle F_0 J_1 v_\parallel^2 \mu\alpha\rangle \approx -2\frac{v_t^4}{B}\hat{\hat{\nabla}}_\perp^2. \tag{52}$$

The Maxwellian terms which contain more than one factor of J_0 are closed analogously

$$\langle F_0 J_{0A_\parallel} J_{0\phi}\rangle = n_0\Gamma_{0A_\parallel}^{1/2}\Gamma_{0\phi}^{1/2}, \tag{53}$$

$$\langle F_0 v_\parallel^2 J_{0A_\parallel} J_{0\phi}\rangle = n_0 v_t^2\Gamma_{0A_\parallel}^{1/2}\Gamma_{0\phi}^{1/2}, \tag{54}$$

$$\langle F_0 \mu J_{0A_\parallel} J_{0\phi}\rangle = \frac{v_t^2}{2B}[(2\Gamma_0^{1/2} + \hat{\nabla}_\perp^2)_{A_\parallel} + (2\Gamma_0^{1/2} + \hat{\nabla}_\perp^2)_\phi], \tag{55}$$

where the subscript ϕ or $A_\parallel$ again designates the field on which the operator acts. These closures can be thought of in terms of separate expansions of the two Bessel function operators, through first order in b, so that no cross term enters.

The $\mathbf{v}_\phi\cdot\nabla\langle F J_0 \cdots\rangle$ and $\mathbf{v}_{A_\parallel}\cdot\nabla\langle F J_0 \cdots\rangle$ terms introduce two additional complications. These terms contain both the Maxwellian and the perturbed distribution, and the gyroaveraging terms are acted on by a perpendicular gradient operator, requiring that gradients of both fluctuating and equilibrium quantities be kept. In considering these terms, we redefine $b \doteq (1/\Omega)\sqrt{(T_\perp/m)}$, in terms of the total perpendicular temperature $T_\perp$, which contains both an equilibrium part (T_0, as the equilibrium is assumed isotropic) and a fluctuating part, $\tilde{T}_\perp = (\tilde{p}_\perp - T_0\tilde{n})/n_0$. The gradient of b is then calculated as follows:

$$\nabla b = \frac{b}{T_0}(\nabla T_0 + \nabla\tilde{T}_\perp) - \frac{2b}{B}\nabla B. \tag{56}$$

Closing these FLR terms analogously to Eqs. (39)–(44) leads to, for example,

$$\mathbf{v}_\phi\cdot\nabla\langle J_0 F\rangle = \mathbf{v}_\phi\cdot\nabla[n\Gamma_0^{1/2}(b)], \tag{57}$$

where n is the total density, $n_0 + \tilde{n}$. Introducing the diamagnetic frequency $i\omega_* \doteq -(cT_0/eBn_0)\nabla n_0\cdot\hat{\mathbf{b}}\times\nabla$, and ratio $\eta_i = L_n/L_T$, where L_T is the scale length of the equilibrium temperature, this leads to three linear terms

$$\mathbf{v}_\phi\cdot\nabla\langle J_0 F\rangle = -n_0 i\omega_*\Gamma_0^{1/2}\frac{e\phi}{T_0} - \frac{n_0}{2}\eta_i\hat{\nabla}_\perp^2 i\omega_*\frac{e\Phi}{T_0} + n_0\hat{\nabla}_\perp^2 i\omega_d\frac{e\phi}{T_0} + N_L. \tag{58}$$

Nonlinear terms arise both from $\tilde{n}$ and b, and can be written

$$N_L = \mathbf{v}_\Phi\cdot\nabla\tilde{n} + \frac{n_0}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{T}_\perp. \tag{59}$$

To account for the $\mathbf{v}_{A_\parallel}\cdot\nabla$ and $\mathbf{v}_\phi\cdot\nabla$ terms with higher powers of $v_\parallel$ and μ, we note that the linear terms from Eq. (58) can be generalized as follows:

$$\mathbf{v}_\phi\cdot\nabla n_0 g(b) = -n_0 g(b) i\omega_*\frac{e\phi}{T_0} - n_0\eta_i b\frac{\partial g}{\partial b}i\omega_*\frac{e\phi}{T_0} + 2n_0 b\frac{\partial g}{\partial b}i\omega_d\frac{e\phi}{T_0}. \tag{60}$$

The treatment of the nonlinear terms is somewhat more subtle, as these can involve higher moments which are not evolved. Following Ref. 39, and introducing the notation $N_L(x)$ for the nonlinear terms generated by $\mathbf{v}_\phi\cdot\nabla\langle F J_0 x\rangle$

$$N_L(v_\parallel) = n_0\mathbf{v}_\Phi\cdot\nabla\tilde{u}_\parallel + \frac{1}{2T_0}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{q}_\perp, \tag{61}$$

$$N_L(mv_\parallel^2) = \mathbf{v}_\Phi\cdot\nabla\tilde{p}_\parallel + \frac{n_0}{2}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{T}_\perp, \tag{62}$$

$$N_L(mB\mu) = \mathbf{v}_\Phi\cdot\nabla\tilde{p}_\perp + \tfrac{1}{2}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{q}_\perp, \tag{63}$$

$$N_L(mv_\parallel^3) = \mathbf{v}_\Phi\cdot\nabla\tilde{q}_\parallel + 3p_0\mathbf{v}_\Phi\cdot\nabla\tilde{u}_\parallel + \tfrac{3}{2}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{q}_\perp, \tag{64}$$

$$N_L(mBv_\parallel\mu) = \mathbf{v}_\Phi\cdot\nabla\tilde{q}_\perp + p_0\mathbf{v}_\Phi\cdot\nabla\tilde{u}_\parallel + \frac{p_0}{2}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{u}_\parallel + \tfrac{1}{2}[\hat{\nabla}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{q}_\perp + [\hat{\hat{\nabla}}_\perp^2\mathbf{v}_\Phi]\cdot\nabla\tilde{q}_\perp. \tag{65}$$

The $\mathbf{v}_{\mathcal{A}}\cdot\nabla\cdots$ nonlinear terms are closed identically to the above with the substitution $\Phi\rightarrow\mathcal{A}_{\parallel}$. However, the $\mathbf{v}_{\mathcal{A}}\cdot\nabla\cdots$ terms in the $\tilde{q}_{\parallel}$ and $\tilde{q}_{\perp}$ equations contain higher moments which are closed using results from the next section.

To simplify the equations, we introduce the following normalization. Time, parallel lengths, and perpendicular lengths are normalized to v_t/L_n, L_n and ρ_i, respectively:

$$(\hat{t},\hat{k}_{\parallel},\hat{k}_{\perp})=\left(\frac{tv_t}{L_n},k_{\parallel}L_n,k_{\perp}\rho_i\right), \tag{66}$$

and the fluctuating quantities are normalized as follows:

$$(\hat{\phi},\hat{A}_{\parallel},\hat{n},\hat{u},\hat{p},\hat{q},\hat{r},\hat{s})=\frac{L_n}{\rho_i}\left(\frac{e\phi}{T_0},\frac{A_{\parallel}}{\rho_i B},\frac{\tilde{n}}{n_0},\frac{\tilde{u}}{v_t},\frac{\tilde{p}}{n_0 m v_t^2},\frac{\tilde{q}}{n_0 m v_t^3},\frac{\tilde{r}}{n_0 m v_t^4},\frac{\tilde{s}}{n_0 m v_t^5}\right). \tag{67}$$

Normalized quantities appear on the left. The caret designating a normalized quantity is dropped for simplicity of notation. Note that these normalizations mesh with the gyrokinetic ordering such that all characteristic drift scales are $\mathcal{O}(1)$. Because β is formally taken to be $\mathcal{O}(1)$, all shear Alfvén scales are $\mathcal{O}(1)$ as well.

C. Closures

Closures must be introduced for the highest moments, r and s, in order to have a complete and useful set of gyrofluid equations. The terms requiring closure divide naturally into three categories, the parallel terms $\nabla_{\parallel}r_{\parallel,\parallel}$ and $\nabla_{\parallel}r_{\parallel,\perp}$, the toroidal terms $\omega_d(r_{\parallel,\parallel}+r_{\parallel,\perp})$, $\omega_d(r_{\parallel,\perp}+r_{\perp,\perp})$, $\omega_d(s_{\parallel,\parallel}+s_{\parallel,\perp})$, and $\omega_d(s_{\parallel,\perp}+s_{\perp,\perp})$, and the mirroring terms $r_{\parallel,\parallel}\nabla_{\parallel}\ln B$, $r_{\parallel,\perp}\nabla_{\parallel}\ln B$, and $r_{\perp,\perp}\nabla_{\parallel}\ln B$. Following the work of Beer,[30] we separately treat each group of terms, making closure approximations that accurately model the physical processes that each set of terms represents.

1. Parallel Landau closures

Closures which provide an accurate model of linear Landau damping are chosen for the parallel terms, $\tilde{\nabla}_{\parallel}r_{\parallel,\parallel}$ and $\tilde{\nabla}_{\parallel}r_{\parallel,\perp}$, where we have introduced the notation $\tilde{\nabla}_{\parallel}=\nabla_{\parallel}-\mathbf{v}_{A_{\parallel}}\cdot\nabla=\nabla_{\parallel}-\hat{\mathbf{b}}\times\nabla\mathcal{A}_{\parallel}\cdot\nabla$. Landau damping along the magnetic field occurs due to the velocity dependence of the $k_{\parallel}v_{\parallel}$ term in the kinetic equation. Components with different $k_{\parallel}$ stream along the field at different velocities, causing moments of F to phase mix away.

As an illustration, consider the one-dimensional kinetic equation

$$\frac{\partial f}{\partial t}+v_{\parallel}\frac{\partial f}{\partial z}=\delta(t)f_0(z,v), \tag{68}$$

where f_0 provides the initial condition. The solution to this simple equation $f(z,v,t)=f_0(z-vt,v)H(t)$, provides Green's functions which can be used to solve more general problems with additional source terms, such as the electric field $-(e/m)E_{\parallel}(\partial F_M/\partial v)$. Consider an initial condition with a small single harmonic perturbation $f_0=(n_0+n_1e^{ikz})F_M(v)$. The general solution is just $[n_0+n_1e^{ik(z-vt)}]$, which simply oscillates in time at $\omega=kv$ and does not damp. However, upon taking velocity space moments, the velocity integration introduces mixing of the phases as follows:

$$n(z,t)=\int f\,dv=n_0+n_1\frac{e^{ikz}}{\sqrt{2\pi v_t^2}}\int dv\,e^{-ikvt}e^{-v^2/(2v_t^2)}. \tag{69}$$

The perturbed density $n_1=n_{1(t=0)}e^{-k^2v_t^2t^2/2}$ decays with a Gaussian time dependence. This decay due to linear Landau damping is not captured by a simple fluid model with a finite number of moments, and hence it must be accounted for in the fluid closure if it is to be included in a fluid model.

A number of different ''Landau closures'' which model linear Landau damping in fluid models have been developed.[41–46] Here the four moment model of Refs. 13, 30, and 44 is employed. This closure accurately models linear kinetic response functions, conserves energy, and takes a simple, frequency independent form in Fourier space, allowing for easy implementation in nonlinear initial value simulations. The introduction of electromagnetic effects does not significantly alter the process of deriving linear Landau closures. Response functions are simply written in terms of the total $E_{\parallel}$ rather than ϕ. In Ref. 47, Landau closures are derived for the general electromagnetic case with both parallel and perpendicular magnetic fluctuations. Here we consider only perpendicular fluctuations, hence the magnitude of the fluctuating field $\tilde{B}$ is zero to first order in the perturbation. The general response functions and closures are given in Sec. IV of Ref. 47. Here we take the $B_1=0$ limit of that result, for the case in which the equilibrium distribution is isotropic. In this limit the result is identical to the earlier result of Ref. 12

$$r_{\parallel,\parallel}=3(2p_{\parallel}-n)+c_{\parallel}T_{\parallel}-\sqrt{2}D_{\parallel}\frac{i|k_{\parallel}|q_{\parallel}}{k_{\parallel}}, \tag{70}$$

$$r_{\parallel,\perp}=p_{\parallel}+p_{\perp}-n-\sqrt{2}D_{\perp}\frac{i|k_{\parallel}|q_{\perp}}{k_{\parallel}}, \tag{71}$$

where $c_{\parallel}=(32-9\pi)/(3\pi-8)$, $D_{\parallel}=2\sqrt{\pi}/(3\pi-8)$, and $D_{\perp}=\sqrt{\pi}/2$. Note that here and elsewhere the dissipative terms in the closure $(\sim|k_{\parallel}|/k_{\parallel})$ are written in their Fourier space form for conciseness. In configuration space these terms are convolution integrals.

Because the dissipative part of the closure above (the $|k_{\parallel}|/k_{\parallel}$ terms) is written in terms of moments with no equilibrium component, the fluctuating field makes no contribution to the linear Landau closures. Hence the linear Landau closure is equally accurate in the electrostatic and electromagnetic cases. However, there is an additional nonlinear Landau damping term due to $A_\parallel$ which is discussed in Sec. VII of Ref. 47. This and other nonlinear Landau damping mechanisms are not accounted for in the fluid closures given here. As discussed in Ref. 39, nonlinear phase-mixing may be important at large amplitudes or large $k_\perp \rho$, and extensions to include some nonlinear phase mixing effects are of interest for future work.

2. Toroidal closures

The velocity dependence of the ∇B and curvature drifts also introduces phase mixing. This process is modeled using toroidal closures of Beer,[30] which include dissipative pieces proportional to $|\omega_d|/\omega_d$.

Beer's closures include both Maxwellian parts and dissipative pieces derived by careful fitting with all parts of the kinetic toroidal response function, and can be written in the following form:

$$r_{\parallel,\parallel}+r_{\parallel,\perp}=7p_\parallel+p_\perp-4n-2i\frac{|\omega_d|}{\omega_d}(\nu_1 T_\parallel+\nu_2 T_\perp), \quad (72)$$

$$r_{\parallel,\perp}+r_{\perp,\perp}=p_\parallel+5p_\perp-3n-2i\frac{|\omega_d|}{\omega_d}(\nu_3 T_\parallel+\nu_4 T_\perp), \quad (73)$$

$$s_{\parallel,\parallel}+s_{\parallel,\perp}=-i\frac{|\omega_d|}{\omega_d}(\nu_5 u_\parallel+\nu_6 q_\parallel+\nu_7 q_\perp), \quad (74)$$

$$s_{\parallel,\perp}+s_{\perp,\perp}=-i\frac{|\omega_d|}{\omega_d}(\nu_8 u_\parallel+\nu_9 q_\parallel+\nu_{10} q_\perp), \quad (75)$$

where the complex coefficients take the form $\nu=\nu_r+i\nu_i|\omega_d|/\omega_d$. The coefficients chosen are, in the form (ν_r,ν_i), $\nu_1=(2.019,-1.620)$, $\nu_2=(0.433,1.018)$, $\nu_3=(-0.256,1.487)$, $\nu_4=(-0.070,-1.382)$, $\nu_5=(-8.927,12.649)$, $\nu_6=(8.094,12.638)$, $\nu_7=(13.720,5.139)$, $\nu_8=(3.368,-8.110)$, $\nu_9=(1.974,-1.984)$, $\nu_{10}=(8.269,2.06)$. As shown in Figs. 2.1 and 2.2 of Ref. 30, these closures provide a good fit to the linear toroidal response functions, including a reasonable model of the toroidal branch cut at $\omega/\omega_d=-k_\parallel^2 v_t^2/4\omega_d^2$.

As noted in Ref. 30, this set of toroidal closures accurately models the fast linear collisionless damping of zonal flows for $t<qR/v_{ti}\sqrt{\epsilon}$, but does not account for the residual undamped component of the zonal flow noted by Rosenbluth and Hinton.[48] Efforts to incorporate this residual flow and other neoclassical effects into a new toroidal closure are ongoing.[49]

3. Mirroring closures

The mirroring terms $r_{\parallel,\parallel}\nabla_\parallel \ln B$, $r_{\parallel,\perp}\nabla_\parallel \ln B$, and $r_{\perp,\perp}\nabla_\parallel \ln B$ incorporate trapped ion effects and magnetic pumping. However, they do not introduce new dissipative processes and hence they are closed with simple Maxwellian closures, again following Ref. 30

$$r_{\parallel,\parallel}=6p_\parallel-3n, \quad (76)$$

$$r_{\parallel,\perp}=p_\parallel+p_\perp-n, \quad (77)$$

$$r_{\perp,\perp}=4p_\perp-2n. \quad (78)$$

Taken together, the closure approximations provide models of linear and nonlinear FLR effects, as well as parallel phase mixing, drift resonance, and trapped particle effects. The accuracy of these closures is tested extensively through linear benchmarks with kinetic theory given in Sec. VI and Ref. 32.

D. Ion collisions

Ion–ion collisions are modeled with a simple particle, momentum and energy conserving Bohm–Gross–Krook (BGK) operator[50]

$$C(F_j)=-\sum_k \nu_{jk}(F_j-F_{Mjk}), \quad (79)$$

where j and k are species indices, and ν_{jk} is the collision rate of species j with species k. Collisions cause the distribution to relax to a shifted Maxwellian with the appropriate total (equilibrium+fluctuating) momentum and energy. Upon linearizing, the single species operator can be written

$$C(F)=-\nu_{ii}\left\{F_1-\left[n+u_\parallel\frac{v_\parallel}{v_t}+T\left(\frac{v^2}{2v_t^2}-\frac{3}{2}\right)\right]F_0\right\}, \quad (80)$$

where n, $u_\parallel$, and $T=(T_\parallel+2T_\perp)/3$ are normalized fluctuating moments, and $v^2=v_\parallel^2+4B^2\mu^2$.

Ion–electron collisions are negligible due to the smallness of the electron–ion mass ratio. Electron–ion collisions are considered in Sec. IV C.

E. Final gyrofluid equations

Incorporating the parallel, toroidal, and mirror term closures defined above, and including moments of the ion–ion collision operator, yields the final set of single species electromagnetic gyrofluid equations.

The derivation in the previous sections has focused on a single ion species for simplicity. In general, tokamak plasmas contain multiple ion species, as well as electrons. In some cases, such as the deuterium–tritium plasmas used in fusion experiments, the bulk plasma may contain more than one dominant ion species. In addition, impurity ions are expected to play an important role, especially near the plasma edge.

The extension to multiple species is reasonably straightforward. A separate set of gyrofluid equations is solved for each species j, noting that charge e, mass m, and the equilibrium moments (n_0, T_0) and scale lengths are functions of the species j.

Here each species is normalized to its own n_0, v_t, etc., but one ion species is chosen as a reference. The reference species is designated with the subscript i, and the following dimensionless constants are introduced, $\tau_j=T_{0j}/T_{0i}$, $v_j=v_{tj}/v_{ti}$, and $\hat{\rho}_j=\rho_j/\rho_i$. Z is the ratio of the species charge to the unit charge, $Z=e_j/|e|$, and the reference species i is assumed to have $Z=1$. η_j is the usual ratio of scale lengths $\eta_j=L_{nj}/L_{Tj}$. The basic macroscopic length is taken to be the electron density scale length L_{ne}, and the following normalized scale length is defined for each ion species, $\hat{L}_{nj} = L_{nj}/L_{ne}$.

The multispecies equations are then written as follows:

$$\frac{dn}{dt} + v_j B \tilde{\nabla}_\parallel \frac{u_\parallel}{B} + \left(\frac{1}{2}\hat{\nabla}_\perp^2 \mathbf{v}_\Phi\right)\cdot\nabla T_\perp - v_j\left(\frac{1}{2}\hat{\nabla}_\perp^2 \mathbf{v}_{\mathcal{A}}\right)\cdot\nabla q_\perp - \left(1+\frac{\eta_j}{2}\hat{\nabla}_\perp^2\right)\frac{i\omega_*}{\hat{L}_{nj}}\Phi + \left(2+\frac{1}{2}\hat{\nabla}_\perp^2\right) i\omega_d\Phi + i\omega_d\hat{\rho}_j v_j(p_\parallel + p_\perp) = 0, \tag{81}$$

$$\frac{du_\parallel}{dt} + v_j B\tilde{\nabla}_\parallel \frac{p_\parallel}{B} + \left(\frac{1}{2}\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right)\cdot\nabla q_\perp - v_j\left(\frac{1}{2}\hat{\nabla}_\perp^2\mathbf{v}_{\mathcal{A}}\right)\cdot\nabla T_\perp + \frac{v_j Z}{\tau_j}\tilde{\nabla}_\parallel\Phi + \frac{v_j Z}{\tau_j}\frac{\partial\mathcal{A}_\parallel}{\partial t} + \left[1+\eta_j\left(1+\frac{\hat{\nabla}_\perp^2}{2}\right)\right]\frac{i\omega_*}{\hat{L}_{nj}} v_j\mathcal{A}_\parallel + \left(p_\perp + \frac{Z}{\tau_j}\frac{\hat{\nabla}_\perp^2\Phi}{2}\right) v_j\nabla_\parallel \ln B + i\omega_d\hat{\rho}_j v_j(q_\parallel + q_\perp + 4u_\parallel) = 0, \tag{82}$$

$$\frac{dp_\parallel}{dt} + v_j B\tilde{\nabla}_\parallel\frac{q_\parallel + 3u_\parallel}{B} + \left(\frac{1}{2}\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right)\cdot\nabla T_\perp + 2v_j(q_\perp + u_\parallel)\nabla_\parallel \ln B - \left[1+\eta_j\left(1+\frac{\hat{\nabla}_\perp^2}{2}\right)\right]\frac{i\omega_*}{\hat{L}_{nj}}\Phi + \left(4+\frac{\hat{\nabla}_\perp^2}{2}\right)i\omega_d\Phi + i\omega_d\hat{\rho}_j v_j(7p_\parallel + p_\perp - 4n) + 2|\omega_d\hat{\rho}_j v_j|(\nu_1 T_\parallel + \nu_2 T_\perp) = -\tfrac{2}{3}\nu_s(p_\parallel - p_\perp), \tag{83}$$

$$\frac{dp_\perp}{dt} + v_j B^2\tilde{\nabla}_\parallel\frac{q_\perp + u_\parallel}{B^2} + \left[\frac{1}{2}\hat{\nabla}_\perp^2\mathbf{v}_\Phi\right]\cdot\nabla p_\perp + [\hat{\hat{\nabla}}_\perp^2\mathbf{v}_\Phi]\cdot\nabla T_\perp - v_j[\tfrac{1}{2}\hat{\nabla}_\perp^2\mathbf{v}_{\mathcal{A}}]\cdot\nabla(q_\perp + u_\parallel) - \left[1+\frac{\hat{\nabla}_\perp^2}{2} + \eta_j\left(1+\frac{\hat{\nabla}_\perp^2}{2}+\hat{\hat{\nabla}}_\perp^2\right)\right]\frac{i\omega_*}{\hat{L}_{nj}}\Phi + (3+\tfrac{3}{2}\hat{\nabla}_\perp^2 + \hat{\hat{\nabla}}_\perp^2)i\omega_d\Phi + i\omega_d\hat{\rho}_j v_j(5p_\perp + p_\parallel - 3n) + 2|\omega_d\hat{\rho}_j v_j|(\nu_3 T_\parallel + \nu_4 T_\perp) = \tfrac{1}{3}\nu_s(p_\parallel - p_\perp), \tag{84}$$

$$\frac{dq_\parallel}{dt} + (3+c_\parallel)v_j\tilde{\nabla}_\parallel T_\parallel + \sqrt{2}D_\parallel v_j|k_\parallel|q_\parallel + i\omega_d\hat{\rho}_j v_j(-3q_\parallel - 3q_\perp + 6u_\parallel) + (3+c_\parallel)\,\eta_j\frac{i\omega_*}{\hat{L}_{nj}}v_j\mathcal{A}_\parallel + |\omega_d\hat{\rho}_j v_j|(\nu_5 u_\parallel + \nu_6 q_\parallel + \nu_7 q_\perp) = -\nu_s q_\parallel, \tag{85}$$

$$\frac{dq_\perp}{dt} + v_j\tilde{\nabla}_\parallel T_\perp + \sqrt{2}D_\perp v_j|k_\parallel|q_\perp + \left(\frac{1}{2}\hat{\nabla}_\perp^2 v_\Phi\right)\cdot\nabla u_\parallel + (\hat{\hat{\nabla}}_\perp^2\mathbf{v}_\Phi)\cdot\nabla q_\perp - v_j(\hat{\hat{\nabla}}_\perp^2 - \tfrac{1}{2}\hat{\nabla}_\perp^2)\mathbf{v}_{\mathcal{A}}\cdot\nabla T_\perp + \left[\eta_j(1+\hat{\hat{\nabla}}_\perp^2)(1+\eta_j)\frac{\hat{\nabla}_\perp^2}{2}\right]\frac{i\omega_*}{\hat{L}_{nj}}v_j\mathcal{A}_\parallel + \frac{Zv_j}{\tau_j}\frac{\hat{\nabla}_\perp^2}{2}\left(\frac{d\mathcal{A}_\parallel}{dt} + \tilde{\nabla}_\parallel\Phi - i\omega_d\hat{\rho}_j v_j\mathcal{A}_\parallel\right) + i\omega_d\hat{\rho}_j v_j(-q_\parallel - q_\perp + u_\parallel) + \left[p_\perp - p_\parallel + \frac{Z}{\tau_j}\left(\hat{\hat{\nabla}}_\perp^2\Phi - \frac{\hat{\nabla}_\perp^2\Phi}{2}\right)\right]v_j\nabla_\parallel \ln B + |\omega_d\hat{\rho}_j v_j|(\nu_8 u_\parallel + \nu_9 q_\parallel + \nu_{10}q_\perp) = -\nu_s q_\perp, \tag{86}$$

where $d/dt = (\partial/\partial t) + \mathbf{v}_\Phi\cdot\nabla$.

We emphasize that in the above equations, the fluid quantities $(n, u_\parallel, p_\parallel, p_\perp, \ldots)$ are all for species j, and an implied j subscript as been dropped. For clarity, we explicitly write here their relation to the physical quantities

$$(n, u_\parallel, p_\parallel, p_\perp, T_\parallel, T_\perp, q_\parallel, q_\perp) = \frac{L_{ne}}{\rho_i}\left(\frac{n_{1j}}{n_{0j}}, \frac{u_{1j}}{v_{tj}}, \frac{p_{\parallel 1j}}{n_{0j}m_j v_{tj}^2}, \frac{p_{\perp 1j}}{n_{0j}m_j v_{tj}^2}, \frac{T_{\parallel 1j}}{T_{0j}}, \frac{T_{\perp 1j}}{T_{0j}}, \frac{q_{\parallel 1j}}{n_{0j}m_j v_{tj}^3}, \frac{q_{\perp 1j}}{n_{0j}m_j v_{tj}^3}\right), \tag{87}$$

where 0 subscripts refer to equilibrium values and 1 subscripts refer to unnormalized fluctuating values. The gyroaveraging operators $\Gamma_0^{1/2}$, $\hat{\nabla}_\perp^2$, and $\hat{\hat{\nabla}}_\perp^2$, which act only on the fields ϕ and $A_\parallel$, are also species dependent through their implied argument b_j, which in Fourier space is $k_\perp^2\rho_j^2$. Since Φ and $\mathcal{A}$ are gyroaveraged quantities, they also implicitly depend on species j, via

$$(\Phi, \mathcal{A}_\parallel) = \frac{L_{ne}}{\rho_i}\frac{e}{T_{0i}}\left(\Gamma_0^{1/2}\phi, \frac{v_t}{c}\Gamma_0^{1/2}A_\parallel\right). \tag{88}$$

IV. THE ELECTRON LANDAU FLUID EQUATIONS

Electron dynamics can be described by the full set of gyrofluid equations in the previous section. However, the full set of ion and electron equations will then contain widely separated spatial and temporal scales which make fully resolved explicit numerical simulation challenging.

For many problems, a reduced set of equations, describing fluctuations on a smaller range of scales is appropriate. Here we will develop a reduced electron model appropriate for the description of fluctuations on the ion drift and shear Alfvén scales often associated with tokamak plasma microturbulence. An analytic expansion in the electron to ion mass ratio is constructed, allowing electron dynamics on the characteristic ion and Alfvén scales to be treated explicitly, while the fast electron transit time scale and the small spatial scales associated with the electron gyroradius and the electron skin depth are removed from the set of fluid equations to be solved numerically. (This is not to suggest that microturbulence on ρ_e and c/ω_{pe} scales does not exist or is not important in some phenomena. When these scales are important, the full electron gyrofluid equations, or another appropriate physics model should be used.)

The resulting electromagnetic electron Landau fluid equations include the effects of electron temperature and density gradients, electron $\mathbf{E}\times\mathbf{B}$ motion, Landau damping, electron–ion collisions, and the parallel electron currents which, along with parallel ion currents, give rise to the parallel magnetic potential. The equations given here focus on the dynamics of the passing electrons. Developing an electromagnetic model of trapped electron dynamics analogous to the electrostatic model of Beer[30] is left as an important piece of future work.

A. Analytic expansion in the electron mass ratio

We invoke an analytic expansion in the electron–ion mass ratio, similar to the technique employed by Kadomtsev and Pogutse.[51] This expansion removes the small electron gyroradius scale and the fast electron transit time scale from the equations, leaving an efficient model appropriate for the study of turbulence on ion and shear Alfvén scales.

A lowest order model, containing no finite electron mass terms, will be derived first. This simple model will then be extended to include higher order dissipative terms in Sec. IV C.

1. Electron FLR and transit

To efficiently study fluctuation scales on the order of the ion gyroradius, we employ a subsidiary formal ordering in the smallness of the electron–ion mass ratio in order to remove electron finite Larmor radius terms. The gyroaveraging operator J_0 can be expanded $1+k_\perp^2\rho_e^2+\cdots$. In the gyrokinetic ordering employed here, $k_\perp\rho_i\sim 1$ the first electron FLR term is $\mathcal{O}(m_e/m_i)$. We introduce the subsidiary ordering parameter $\delta\sim\sqrt{m_e/m_i}$ and note that electron FLR effects first enter at $\mathcal{O}(\delta^2)$.

The small electron mass also implies a fast electron thermal speed ($v_{te}\gg v_{ti}$), and rapid electron streaming along the magnetic field. The speed of this streaming motion along the field introduces a Courant constraint on the size of the time step which can be used in an explicit numerical simulation. Adding electron parallel dynamics to a simulation which previously modeled only ions reduces this time step constraint by a factor of $\sqrt{T_e m_i/T_i m_e}\sim 60$ for a deuterium fusion plasma. This is a severe numerical burden, though perhaps one that it may be possible to contemplate bearing in the near future, as computational power continues to increase. Imposing the mass ratio ordering $\sqrt{m_e/m_i}\sim\delta\ll 1$, allows the fast electron transit motion to be analytically removed.

2. General ordering

The use of the electron–ion mass ratio as an ordering parameter has a long history in plasma physics. It has been invoked in many forms of the magnetohydrodynamic equations as well as in the more detailed equations of Kadomtsev and Pogutse,[51] and in many other fluid and simplified kinetic formulations. In the context of gyrokinetics, the mass ratio expansion has generally been used to justify the neglect of electron FLR terms, and treatment of electron dynamics with the drift kinetic equation. Here we wish to consistently apply the ordering $m_e/m_i\sim\mathcal{O}(\delta^2)$ to all terms in the drift fluid equations.

The fundamental assumption is that the fluctuating scales of interest are those typical of ion thermal, drift and gyro-motion, and those of shear Alfvén waves. Length and time scales associated with electron thermal and gyromotion are taken to be small.

For a typical perpendicular wave number $k_\perp$, we impose the following ordering:

$$k_\perp^{-1}\sim\rho_i\sim c/\omega_{pi}\gg\rho_e, c/\omega_{pe}, \tag{89}$$

where ω_p is the plasma frequency. The lengths on the left are independent of the electron mass, while the two lengths on the right are proportional to $\sqrt{m_e}$. Note that the skin depth c/ω_{pj} can be written as $\rho_j\sqrt{2/\beta_j}$ for the single species case, where the species $\beta_j=8\pi n_{0j}T_{0j}/B^2$. Formally taking $\beta\sim\mathcal{O}(1)$, the above ordering of lengths follows directly from $\sqrt{m_e/m_i}\sim\mathcal{O}(\delta)$.

For a typical fluctuation frequency ω we choose the ordering

$$\omega\sim k_\parallel v_{ti}\sim\omega_*\sim\omega_{Di}\sim\omega_{De}\sim k_\parallel c_s\sim k_\parallel v_A\ll k_\parallel v_{te}\sim\omega_{\mathrm{ETG}}, \tag{90}$$

where ω_D is the curvature and ∇B drift frequency, $c_s=\sqrt{T_{0e}/m_i}$ is the cold ion sound speed, and $v_A=B/\sqrt{4\pi n_0 m_i}$ is the Alfvén speed. We define ω_{ETG} to be a frequency characteristic of the electron temperature gradient (ETG) mode. These short wavelength modes typically have $k_\theta\sim 1/\rho_e$, and hence $\omega_{\mathrm{ETG}}\sim\sqrt{m_i/m_e}\,\omega_*$, where ω_* is the diamagnetic frequency taken with $k_\theta\rho_i\sim 1$. The quantities on the left are independent of m_e while those on the right are proportional to $m_e^{-1/2}$.

The desired time and length scale orderings above follow directly from $m_e/m_i\sim\mathcal{O}(\delta^2)$ and $\beta_j\sim\mathcal{O}(1)$. The constraints on the validity of this expansion are found through inspection of Eqs. (89) and (90). The separation of scales between the shear Alfvén frequency and the electron transit frequency (and equivalently between ρ_i and the electron skin depth) requires $\beta_e\gg 2m_e/m_i$. In fusion relevant plasmas, this condition is generally satisfied everywhere except very near the plasma edge. Another constraint is provided by the condition $\omega_*\ll k_\parallel v_{te}$. Using a typical ballooning $k_\parallel\approx 1/qR$, and $\omega_* \approx k_\theta \rho_i v_{ti}/L_n$, this requires $k_\theta \rho_i \sqrt{m_e/m_i} \ll L_n/qR$. For large $k_\theta \rho_i \approx 1$, this condition can break down in the edge region where q is often large, while L_n can become rather short.

B. Derivation of the electron equations

The formal expansion in mass ratio can now be used to derive a set of equations which describe electron dynamics consistent with the time and space scale orderings described above.

All fluctuations, including those in the electron moments $\tilde{n}_e$, $\tilde{u}_{\parallel_e}$, $\tilde{p}_e$, etc., are taken to occur on the ion/Alfvén scales. It is thus convenient to normalize fluctuating electron moments to the ion quantities, v_{ti} and m_i, so that a consistent ordering is easily maintained.[52] The fluctuating electron moments are normalized as follows:

$$(\hat{n}_e, \hat{u}_e, \hat{p}_e, \hat{q}_e, \hat{r}_e, \hat{s}_e) = \frac{L_n}{\rho_i}\left(\frac{\tilde{n}_e}{n_0}, \frac{\tilde{u}_e}{v_{ti}}, \frac{\tilde{p}_e}{n_0 m_i v_{ti}^2}, \frac{\tilde{q}_e}{n_0 m_i v_{ti}^3}, \frac{\tilde{r}_e}{n_0 m_i v_{ti}^4}, \frac{\tilde{s}_e}{n_0 m_i v_{ti}^5}\right), \tag{91}$$

where the normalized quantities on the left are all $\mathcal{O}(1)$. In the general multiple ion species case, the quantities m_i and v_{ti} above refer to the reference ion species, as in Sec. III E.

This normalization differs from that employed in Sec. III E, where each species' moments are normalized to its own mass and thermal velocity. Lengths, times, and the fields ϕ and $A_\parallel$ are normalized as in the ion equations. The unsubscripted normalized operators are again defined in terms of the $Z=1$ ion charge (e), the reference ion species mass (m_i) and temperature (T_{0i}), and the equilibrium electron density (n_0) and density scale length (L_n)

$$i\hat{\omega}_* = -\frac{L_n}{v_{ti}}\frac{cT_{0i}}{eBn_0}\nabla n_0 \cdot \hat{\mathbf{b}} \times \nabla, \tag{92}$$

$$i\hat{\omega}_d = \frac{L_n}{v_{ti}}\frac{cT_{0i}}{eB^3}\mathbf{B}\times\nabla B \cdot \nabla. \tag{93}$$

The normalized electron density equation is

$$\frac{\partial n_e}{\partial t} + \mathbf{v}_E \cdot \nabla n_e + B\tilde{\nabla}_\parallel \frac{u_{\parallel_e}}{B} - i\omega_*\phi + i\omega_d(2\phi - 2n_e/\tau - T_{\parallel_e} - T_{\perp_e}) = 0, \tag{94}$$

where the carets on normalized quantities have been dropped for conciseness of notation. The notation $\tilde{\nabla}_\parallel = \nabla_\parallel - \hat{\mathbf{b}} \times \nabla A_\parallel \cdot \nabla$ has been employed. Note that no factors of m_e appear in the above equation, and all terms are of the same order.

The momentum equation can be written

$$\frac{m_e}{m_i}\frac{\partial u_{\parallel_e}}{\partial t} + \frac{m_e}{m_i}\mathbf{v}_E \cdot \nabla u_{\parallel_e} + B\tilde{\nabla}_\parallel \frac{p_{\parallel_e}}{B} + (1+\eta_e)i\omega_*\frac{A_\parallel}{\tau} + \frac{m_e}{m_i}i\omega_d(q_{\parallel_e} + q_{\perp_e} + 4u_{\parallel_e}/\tau) - \frac{\partial A_\parallel}{\partial t} - \tilde{\nabla}_\parallel\phi + p_{\perp_e}\nabla_\parallel \ln B = 0. \tag{95}$$

The electron inertia term, which is associated with the electron skin depth, and the curvature and ∇B drift terms are both small by a factor of $m_e/m_i \sim \delta^2$. Neglecting these higher-order terms, and expanding the pressure, noting that $p_{\parallel_e} = T_{\parallel_e} + n_e/\tau$ because of the normalization to ion temperature, the momentum equation can be recast as a time evolution equation for the magnetic potential

$$\frac{\partial A_\parallel}{\partial t} + \tilde{\nabla}_\parallel\phi - \frac{1}{\tau}\tilde{\nabla}_\parallel n_e - \tilde{\nabla}_\parallel T_{\parallel_e} - (1+\eta_e)i\omega_*\frac{A_\parallel}{\tau} + (T_{\parallel_e} - T_{\perp_e})\nabla_\parallel \ln B = 0. \tag{96}$$

The equations for $T_{\parallel_e}$ and $T_{\perp_e}$ needed to complete the above set come from the $q_{\parallel_e}$ and $q_{\perp_e}$ moment equations. The $p_{\parallel_e}$ and $p_{\perp_e}$ moment equations provide information on the next order evolution of the temperature fluctuations.

The $q_{\parallel_e}$ and $q_{\perp_e}$ equations contain the higher moments r_e and s_e which are closed as in Sec. III C. However, the electron closure terms are not in general $\mathcal{O}(1)$. Consider, for example, the Maxwellian closure for the moment $r_{\parallel,\parallel_e}$. This closure is derived by taking the first-order fluctuating part of the generalized Maxwellian result $r_{\parallel,\parallel_e} = 3p_{\parallel_e}^2/m_e n_e$. The factor of $1/m_e$ insures that this term is $\mathcal{O}(\delta^{-2})$. In the normalized units

$$r_{\parallel,\parallel_e} \to 6\frac{m_i}{\tau m_e}T_{\parallel_e} + 3\frac{m_i}{\tau^2 m_e}n_e, \tag{97}$$

and similarly for $r_{\parallel,\perp_e}$ and $r_{\perp,\perp_e}$. The Landau damping portion of the closure is smaller than the Maxwellian part by $\sqrt{m_e/m_i}$, and is neglected here, though it is reconsidered in Sec. IV C.

Before normalizing or substituting in the closures, the $q_{\parallel_e}$ equation can be written to lowest order

$$B\tilde{\nabla}_\parallel \frac{\tilde{r}_{\parallel\parallel_e}}{B} - 3T_{0e}B\tilde{\nabla}_\parallel \frac{\tilde{p}_{\parallel_e}}{m_e B} + 3\eta_e \frac{n_0 T_{0e}^2}{m_e} i\omega_* \frac{eA_\parallel}{cT_{0i}} + 3\left(\tilde{r}_{\parallel\perp_e} - \frac{T_{0i}}{m_e}\tilde{p}_{\perp_e}\right)\nabla_\parallel \ln B = 0, \tag{98}$$

where the d/dt and ω_d terms again drop out, as they are higher order in m_e/m_i.

Substituting the Maxwellian closures, normalizing and simplifying gives

$$\tilde{\nabla}_\parallel T_{\parallel_e} + \eta_e i\omega_* A_\parallel/\tau = 0. \tag{99}$$

The second term on the left is the gradient of the equilibrium temperature T_{0e} along the perturbed field, $\hat{\mathbf{b}} \times \nabla A_\parallel \cdot \nabla T_{0e}$, or equivalently the gradient along the total field $\tilde{\nabla}_\parallel T_{0e}$, as T_{0e} is constant along the equilibrium field. Equation (99) can thus be written in the more physically intuitive form

$$\tilde{\nabla}_\parallel (T_{\parallel_e} + T_{0e}) = \frac{1}{B}(\mathbf{B}_0 + \mathbf{B}_1)\cdot\nabla(T_{\parallel_e} + T_{0e}) = 0. \tag{100}$$

Quite simply, the total temperature is constant along the total magnetic field including fluctuations. This result is expected from our ordering of the velocities $v_{ti}, v_A \ll v_{te}$. The speeds of the microturbulence being evolved are all slow compared to v_{te}, and furthermore the Alfvén speed at which the magnetic field fluctuates is also much less than the electron thermal speed. Hence as the field fluctuates across the equilibrium temperature gradient, the electrons are able to almost instantaneously re-thermalize, leaving no electron temperature gradient along the total field. Note that this condition is quite different from the occasionally employed closures $\nabla_\parallel \tilde{T}_e = 0$, or $\tilde{T}_e = 0$, both of which fail to properly account for the magnetic fluctuations across the equilibrium temperature gradient, and lead to errors when η_e is finite.

Turning now to the $q_{\perp_e}$ moment equation, and again inserting Maxwellian closures, normalizing, and keeping only the dominant terms, the equation becomes

$$\tilde{\nabla}_\parallel T_{\perp_e} + \frac{\eta_e i \omega_* A_\parallel}{\tau} + (T_{\perp_e} - T_{\parallel_e}) \nabla_\parallel \ln B = 0. \tag{101}$$

Again the second term is simply the derivative along the perturbed field of the equilibrium temperature (T_{0e}). A mirror force term enters as well.

Equations (99) and (101) can be recast by defining $T_e = (T_{\parallel_e} + T_{\perp_e})/2$ and $\delta T_e = (T_{\perp_e} - T_{\parallel_e})$. Note that once Eq. (99) has been substituted into the momentum equation, the temperature enters the momentum equation only as a mirroring term $\delta T_e \nabla_\parallel \ln B$, and enters the density equation only as $-i\omega_d T_e$. The equations for T_e and δT_e are

$$\tilde{\nabla}_\parallel T_e + \frac{\eta_e i \omega_* A_\parallel}{\tau} + \frac{\delta T_e}{2} \nabla_\parallel \ln B = 0, \tag{102}$$

$$(\tilde{\nabla}_\parallel + \nabla_\parallel \ln B)\, \delta T_e = 0. \tag{103}$$

In either the small mirror force limit ($\nabla_\parallel \ln B \to 0$) or the high collisionality limit ($\delta T_e \to 0$), the above equations reduce to $\tilde{\nabla}_\parallel T_e = -\eta_e i \omega_* A_\parallel / \tau$. Because the model only describes passing electrons, we employ this simple limit.

The full set of normalized electron equations is then

$$\frac{\partial n_e}{\partial t} + \mathbf{v}_E \cdot \nabla n_e + B \tilde{\nabla}_\parallel \frac{u_{\parallel_e}}{B} - i\omega_* \phi + 2 i \omega_d \left(\phi - \frac{n_e}{\tau} - T_e \right) = 0, \tag{104}$$

$$\frac{\partial A_\parallel}{\partial t} + \tilde{\nabla}_\parallel \phi - \frac{1}{\tau} \tilde{\nabla}_\parallel n_e - \frac{1}{\tau} i \omega_* A_\parallel = 0, \tag{105}$$

$$\tilde{\nabla}_\parallel T_e = - \frac{\eta_e}{\tau} i \omega_* A_\parallel, \tag{106}$$

where the $\omega_d T_e$ term in Eq. (104) is evaluated by numerically inverting Eq. (106). It is assumed that any fluctuating component of T_e which is constant on a field line does not contribute significantly to the $\omega_d T_e$ term.

We emphasize that, provided an appropriate numerical inversion of Eq. (106) can be achieved, no separate closure approximations are required for nonlinear terms. In this lowest order expansion in m_e/m_i, nonlinear closure terms such as those associated with nonlinear Landau damping drop out naturally, and no separate assumptions about the smallness of nonlinear terms are needed.

The above equations provide a simple description of electromagnetic electron dynamics on shear Alfvén and ion scales. While only two moment equations need be solved, the physics content of a full six moment model has been incorporated to lowest order in m_e/m_i.

Though the model is simple, it represents a substantial improvement over the adiabatic electron models ($n_e/\tau = \phi - \langle \phi \rangle_{\text{surface}}$) that have been used to describe the passing electrons in many previous gyrofluid and gyrokinetic particle simulations. In addition to finite-β effects and Alfvén wave dynamics, the above model also incorporates electron $\mathbf{E} \times \mathbf{B}$, curvature, and ∇B drift motion, as well as the $\mathbf{E} \times \mathbf{B}$ nonlinearity and nonlinear terms due to magnetic flutter. The accuracy of this model in describing both finite-β ion drift waves and shear Alfvén waves is gauged in Sec. VI with a series of linear benchmarks.

Furthermore, the numerical challenge of resolving short electron space and time scales has been entirely removed. The electron mass appears nowhere in Eqs. (104)–(106) or in the normalizations [Eq. (91)], and it is apparent that the electron scales, (ρ_e, c/ω_{pe}, $k_\parallel v_{te}$, ω_{ETG}), all of which contain the electron mass, have been successfully removed from the equations which are numerically evolved.

It can be shown, in a proof analogous to that of Cowley,[53] that this electron model preserves magnetic flux surfaces.

C. Electron collisions and Landau damping

One consequence of keeping only the lowest order terms in the mass ratio expansion is the absence of any damping mechanism in the electron channel. It is well known that damping terms, even when linearly small, can significantly impact the nonlinear dynamics of an otherwise dissipationless system. While the gyrofluid system has dissipation through ion collisions and ion Landau damping, it is expected that damping in the electron channel may play an important role as well.

The dominant electron damping mechanisms are expected to be Landau damping and pitch angle scattering collisions with ions. These effects are introduced by extending the mass ratio expansion to include terms of $\mathcal{O}(\sqrt{m_e/m_i})$. Equations (94) and (96) remain unchanged, but additional terms are introduced into the closures for $T_{\perp_e}$ and $T_{\parallel_e}$. The lowest order $T^{(0)}_{\parallel_e} = T^{(0)}_{\perp_e} = T^{(0)}_e$ is given by Eq. (106). The full expressions for the first order corrections $T^{(1)}_{\perp_e}$ and $T^{(1)}_{\parallel_e}$ can be derived from Eqs. (81)–(86). Here the $\omega_*, \omega_d \to 0$ limit is taken, leaving only the correction due to Landau damping along the field

$$\nabla_\parallel T^{(1)}_{\parallel_e} = \sqrt{\frac{\pi}{2\tau} \frac{m_e}{m_i}} |k_\parallel| u_{\parallel_e}, \tag{107}$$

and $T^{(1)}_{\perp_e} = 0$, where the operator $|k_\parallel|$ is again written in its Fourier space form for conciseness. A more precise description of finite electron mass effects is possible either by using the full expressions for $T^{(1)}_{\parallel_e}$ and $T^{(1)}_{\perp_e}$, or by employing the full six moment equation set [Eqs. (81)–(86)] to describe the electrons. As in the general multispecies case, discussed in

Sec. III C 1, nonlinear phase mixing effects are not incorporated in this closure. The range of resonant electron velocities $\Delta v_\parallel \sim \omega/k_\parallel \ll v_{te}$ is fairly small in core plasmas, and one might worry about nonlinear particle trapping causing electron Landau damping to turn off. However, even a small amount of collisions can be important for such narrow resonances. The rate of scattering out of the resonant region is $\nu_{ei} v_e^2/(\Delta v_\parallel)^2$, and for a wide range of core parameters this is large compared to relevant linear or nonlinear rates even though ν_{ei} is small. While one would thus expect linear Landau damping to hold, nonlinear kinetic effects can be subtle, and comparisons between the simulation results using this reduced electron model and fully kinetic calculations including collisions will be interesting future work.

Electron–ion collisions are modeled with a Lorentz pitch angle scattering operator. Adding this operator to the right-hand side of the drift kinetic equation and taking moments leads to the following collision term in the normalized electron momentum equation:

$$-\nu_{ei}\frac{m_e}{m_i}(u_{\parallel_e}-u_{\parallel_i}), \tag{108}$$

where ν_{ei} is the effective scattering rate, normalized to v_{ti}/L_n. Because $\nu_{ei}\sim m_e^{-1/2}$, this term is ordered $\nu_{ei}\sim\delta^{-1}$ so that the collision term enters at $\mathcal{O}(\delta)$. This caveat allows a formally consistent ordering in the mass ratio. It is recognized that the collision term may be smaller than other neglected terms. The collision term is kept to assess the impact of this damping mechanism in the electron channel.

Including the pitch angle scattering model and the first-order temperature correction [Eq. (107)] in Eqs. (94) and (96), in the limit of small mirror force, yields the following set of electron equations:

$$\frac{\partial n_e}{\partial t}+\mathbf{v}_E\cdot\nabla n_e+B\tilde{\nabla}_\parallel\frac{u_{\parallel_e}}{B}-i\omega_*\phi+2i\omega_d\left(\phi-\frac{n_e}{\tau}-T_e\right)=0, \tag{109}$$

$$\frac{\partial A_\parallel}{\partial t}+\tilde{\nabla}_\parallel\phi-\frac{1}{\tau}\tilde{\nabla}_\parallel n_e-\frac{1}{\tau}i\omega_*A_\parallel-\sqrt{\frac{\pi}{2\tau}\frac{m_e}{m_i}}|k_\parallel|u_{\parallel_e}$$
$$=\nu_{ei}\frac{m_e}{m_i}(u_{\parallel_e}-u_{\parallel_i}), \tag{110}$$

$$\tilde{\nabla}_\parallel T_e=-\frac{\eta_e}{\tau}i\omega_*A_\parallel. \tag{111}$$

The $w_d T_{\parallel_e}^{(1)}$ term has been neglected, and the $\nabla_\parallel T_{\parallel_e}^{(1)}$ introduces a two moment Landau damping model.[44] Note that the Landau damping operator ($|k_\parallel|$) acts on an odd moment ($u_\parallel$), which has no equilibrium component, so that there is no linear magnetic flutter contribution to the Landau closure, avoiding a concern expressed by Finn and Gerwin.[54] However, magnetic flutter does introduce an additional nonlinear Landau damping term, as discussed in Ref. 47. The size of this term has been calculated in simulations and found to be small.

This electron model can be viewed as an extension of the equations of Kadomtsev and Pogutse[51] to include toroidal drifts, parallel ion flow, and an improved Landau damping model which properly phase-mixes $\mathbf{E}\times\mathbf{B}$ driven perturbations.

The model can be reduced to the familiar adiabatic response in the appropriate limits. Taking the limits $\beta\to 0$, which implies $A_\parallel\to 0$ from Eq. (114), and $m_e/m_i\to 0$, Eq. (110) reduces to the adiabatic electron response $\nabla_\parallel(\phi-n_e/\tau)=0$; or, with the appropriate choice of constants, $n_e=\tau(\phi-\langle\phi\rangle)$. The adiabatic response can also be derived in the formal limit $k_\parallel\to\infty$.

Upon neglect of the "small scale" effects associated with the ∇p term in the momentum equation (here these are the $\tilde{\nabla}_\parallel n_e$ and $i\omega_* A_\parallel$ terms), and in the limit $m_e/m_i\to 0$, Eq. (110) reduces to the parallel ideal magnetohydrodynamic (MHD) Ohm's Law $E_\parallel=-(\partial A_\parallel/\partial t)-\tilde{\nabla}_\parallel\phi=0$. Including the collisional term gives the parallel resistive MHD Ohm's Law. Adding the $-1/\tau(\tilde{\nabla}_\parallel n_e-i\omega_*A_\parallel)$ terms gives a version of the extended MHD Ohm's Law appropriate for $\omega\ll k_\parallel v_{te}$.

V. POISSON'S EQUATION AND AMPERE'S LAW

The system of equations is completed using the gyrokinetic Poisson's equation and Ampere's Law. In the limit of small Debye length, $k\lambda_D\ll 1$, the gyrokinetic Poisson's equation becomes a quasineutrality constraint[2]

$$n_e=\overline{n_i}-(1-\Gamma_0)\phi, \tag{112}$$

where $\overline{n_i}$ is the gyrophase independent part of the real space ion density. The $(1-\Gamma_0)\phi$ term, often called the polarization density, arises from the gyrophase dependent part of the distribution function, and accounts for the difference between guiding center density and ion particle density.

Following Beer,[30] the transformation from gyrocenter to real space is accomplished with the simple Padé approximation:

$$\overline{n_i}=\frac{1}{1+b/2}n_i-\frac{2b}{(2+b)^2}T_{\perp_i}, \tag{113}$$

where $b=k_\perp^2\rho_i^2$. This approximation is first order accurate in b for both n_i and $T_{\perp_i}$, and it behaves properly ($\overline{n_i}\to 0$) for large b.

Within the gyrokinetic ordering, the parallel Ampere's Law is[33]

$$\nabla_\perp^2 A_\parallel=-\frac{\tau\beta_e}{2}(\overline{u_{\parallel_i}}-u_{\parallel_e}), \tag{114}$$

where $\beta_e=8\pi n_0 T_{0e}/B^2$.

The transformation to real space is again accomplished with a Padé approximation

$$\overline{u_{\parallel_i}}=\frac{1}{1+b/2}u_{\parallel_i}-\frac{2b}{(2+b)^2}q_{\perp_i}. \tag{115}$$

Poisson's equation [Eq. (112)] and Ampere's Law [Eq. (114)], together with six ion moment equations [Eqs. (81)–(86)], the two electron moment equations [Eqs. (109)–(110)], and the T_e condition [Eq. (111)], provide a complete description of the ten unknowns (n_i, $u_{\parallel_i}$, $p_{\parallel_i}$, $p_{\perp_i}$, $q_{\parallel_i}$, $q_{\perp_i}$,

n_e, $u_{\parallel_e}$, T_e, ϕ, and $A_\parallel$). The system is solved by evolving the eight partial differential equations in time, while using Eq. (112) to solve for ϕ, Eq. (114) to solve for $u_{\parallel_e}$, and Eq. (111) to solve for T_e.

It is shown in Sec. II C of Ref. 55 that this system of equations exactly reproduces the kinetic dispersion relation in the local fluid limit ($k_\parallel^2 v_{ti}^2 \ll \omega^2 \ll k_\parallel^2 v_{te}^2$, $|\omega_d| \ll |\omega|$, $k_\perp^2 \rho_i^2 \ll 1$). The ability of the equations to model nonlocal kinetic toroidal drift instabilities is tested in the following section with a series of linear benchmarks.

VI. LINEAR BENCHMARKS WITH KINETIC THEORY

Benchmarking the model against linear kinetic theory is an important step in verifying the accuracy and reliability of both the electromagnetic gyrofluid physics model and the simulation code used to implement the model.

An extensive series of linear benchmarks in the electrostatic case is given in Ref. 30, so we focus here on the impact of finite plasma β. Finite-β effects on the collisionless ion temperature gradient (ITG) instability are benchmarked in toroidal flux tube geometry. In addition, the growth rates and real frequencies of the kinetic ballooning mode (KBM) are benchmarked in toroidal geometry. Both the case with no temperature gradient and the more interesting case with finite ion temperature gradient are investigated. It is shown that the gyrofluid model is able to reproduce the finite growth rates of the KBM below the ideal MHD β-limit in this case.

It is important to note that this set of benchmarks provides a test of the electron physics model, as well as the ion physics model. While a simple adiabatic electron model can produce the correct ITG growth rate in the electrostatic limit, this is not the case for the finite-β ITG and KBM modes considered here, as discussed for example in Sec. II C of Ref. 55. A description of electron ∇B and curvature drift motion and proper consideration of magnetic flutter across equilibrium electron temperature gradients are required to accurately calculate growth rates of both the finite-β ITG and KBM instabilities.

A. The finite-β ITG instability

The toroidal ion temperature gradient (ITG) instability is widely thought to play an important role in core transport. Capturing the finite-β effects on this mode has been a principal motivation for developing an electromagnetic turbulence model.

Linear kinetic theory for the electromagnetic case in nonlocal toroidal geometry is quite involved, and a fairly limited set of codes is available. A code developed by Kim, Horton, and Dong,[16] solves a simplified set of integral equations in ballooning coordinates, using an $s-\alpha$ equilibrium model. Figure 1 shows a benchmark using parameters selected from Fig. 6(a) in Ref. 16. The plot shows linear growth rate vs the safety factor q, at two values of β. Quantitative agreement in the finite-β case is found to be as good as in the electrostatic case. The trend emphasized in Ref. 16, that finite-β effects become more important at higher q, is reproduced by the gyrofluid model.

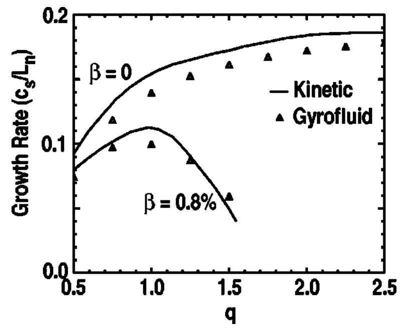

FIG. 1. Linear growth rates of the toroidal ITG mode as a function of the safety factor q, for $\beta=0$ and $\beta=0.8\%$, with $\eta_i=2.5$, $\eta_e=2$, $k_\theta\rho_i=0.5$, $\epsilon_n=0.2$, $s=0.6$, and $\tau=1$. The gyrofluid model is compared to linear kinetic theory in $\hat{s}-\alpha$ geometry, with $\alpha=q^2\beta_e/\epsilon_n[1+\eta_e+\tau(1+\eta_i)]$ chosen to be consistent with β and q.

The structure of the eigenfunctions of ϕ and $A_\parallel$ in ballooning space has also been analyzed. For the parameter set $\beta=0.8\%$, $\eta_i=2.5$, $\eta_e=2$, $k_\theta\rho_i=0.5$, $\epsilon_n=0.2$, $s=0.6$, $q=1.5$, and $\tau=1$, the gyrofluid eigenfunctions have been compared to Fig. 5 of Ref. 16. Good agreement is found in both the shape and parity of the real and imaginary eigenfunctions of ϕ and $A_\parallel$ as well as in the ratio $A_\parallel/\phi \ll 1$. We note that the real part of ϕ has even parity, while the real part of $A_\parallel$ is odd, and in the normalized units, the ratio $\phi_{\max}/A_{\parallel\max} \simeq 15$. The eigenfunctions extend roughly 2π in ballooning angle before becoming negligible. The shape and parity of these eigenfunctions and the ratio $A_\parallel/\phi \ll 1$ are all typical of the finite-β ITG mode.

A second set of toroidal benchmarks employing the widely used GS2 linear gyrokinetic code developed by Kotschenreuther[56] is given in Ref. 32. Good agreement is found in the growth rate and frequency spectra of the finite-β ITG mode.

B. The kinetic ballooning mode

The electromagnetic gyrofluid model also introduces instabilities in the shear Alfvén branch of the dispersion relation not found in the electrostatic case. An example is the kinetic ballooning mode (KBM),[17–22] here defined to be an instability in the shear Alfvén branch of the dispersion relation, analogous to the ideal MHD ballooning mode, with the addition of kinetic effects such as FLR, drift resonance, and Landau damping. The KBM is driven unstable largely by bad curvature effects in the presence of density and/or temperature gradients, though kinetic effects impact the instability threshold and growth rate. [Because the plasma equilibrium is taken to be Maxwellian, there is no fast particle drive, and hence no unstable toroidal Alfvén eigenmode (TAE).] The KBM is expected to play an important role in transport in cases where it is driven unstable below the ideal MHD threshold by the toroidal ion drift resonance. Benchmarks are performed both in the flat temperature gradient case, where the KBM goes unstable exactly at the ideal MHD β_c, and the finite ion temperature gradient case, where the KBM is unstable below β_c.

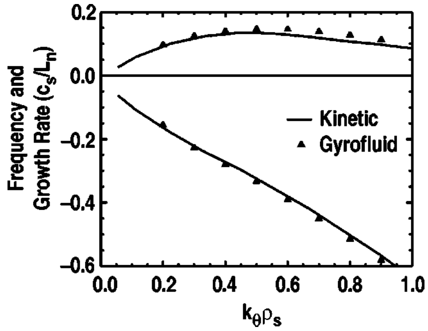

FIG. 2. Linear growth rate (positive) and frequency (negative) spectra of the toroidal kinetic ballooning mode. The gyrofluid model is compared to the kinetic code of Ref. 21, in a simple circular equilibrium at $\beta=6.25\%$. Other parameters are $s=1$, $q=2$, $\tau=1$, $\epsilon_n=0.25$, $\eta_i=\eta_e=0$.

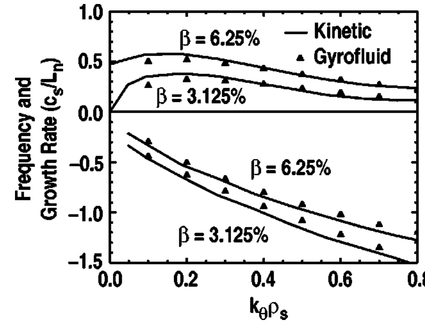

FIG. 3. Frequency (negative) and linear growth rate (positive) spectra for the toroidal kinetic ballooning mode in the presence of a finite ion temperature gradient. Parameters chosen are $\eta_i=2$, $\eta_e=0$, $\epsilon_n=0.25$, $s=1$, $q=2$, and $\tau=1$. The gyrofluid model is compared to a linear kinetic calculation at two values of $\beta=3.125\%,6.25\%$.

1. Benchmarks with zero ion temperature gradient

A set of benchmarks is performed against the kinetic code developed by Hong, Horton, and Choi.[21] It should be noted that this code does not solve the complete kinetic equations, but rather focuses on the coupling between drift and shear Alfvén waves, and neglects ion transit and bounce frequency resonant effects.

Figure 2 shows a comparison with Fig. 1 in Ref. 21 [the figure captions on Figs. 1 and 2 on p. 1593 of this article have been reversed; the figure in the upper right is Fig. 1, while the figure in the lower left is Fig. 2]. Growth rate and frequency spectra are compared in a simple circular geometry at $\beta=6.25\%$. Good agreement is found for the frequency, which is nearly dispersionless with a phase velocity of roughly $-0.6c_s\rho_s/L_n$ in the ion diamagnetic direction. Agreement for the growth rate is also good, though some variance is seen at short wavelengths. A comparison of the growth rate and frequency of the KBM as a function of β is given in Ref. 32, and good agreement is found.

2. Benchmarks with finite ion temperature gradient

The KBM becomes particularly interesting in the presence of finite ion temperature gradient because, as shown by Andersson and Weiland,[57] finite η_i is a necessary and sufficient condition for instability of the shear Alfvén branch below the ideal MHD β limit. Hence this mode may play a significant and direct role in driving transport in plasmas which are ideal MHD stable.

A set of benchmarks is again performed, using parameters and results from Ref. 21. Figure 3 shows frequency and growth rate spectra for the toroidal KBM at two values of $\beta=3.125\%,6.25\%$. Other parameters are identical to Fig. 2, except that $\eta_i=2$. Agreement between the two models is fairly good, with the gyrofluid model correctly accounting for the dramatic increase in growth rates at finite η_i. A comparison of the growth rate of the finite-η_i toroidal KBM is given in Ref. 32, and good agreement is found, with the gyrofluid model accurately reproducing the finite growth rate of the mode both below and above the ideal ballooning β limit.

A final benchmark, Fig. 4, shows the growth rate dependence on the magnetic shear, for two different values of ϵ_n. Again quantitative agreement is reasonably good, with the gyrofluid model successfully reproducing the trends emphasized in Ref. 21.

VII. SUMMARY AND CONCLUSION

A model has been developed to describe electromagnetic microturbulence and transport in long mean-free-path plasmas. The model consists of a set of electromagnetic multispecies gyrofluid and electron Landau fluid equations derived by taking moments of the nonlinear toroidal electromagnetic gyrokinetic equation,[4,33] along with the gyrokinetic Poisson equation and Ampere's Law.

A full hierarchy of six multispecies electromagnetic gyrofluid equations is derived, which can be used to describe both ions and electrons. However, a key result of this paper is the derivation of a reduced set of electron equations, appropriate for the efficient description of the passing electron response to turbulence on scales characteristic of ion drift and kinetic ballooning instabilities. The reduced set of electron equations is derived via an analytic expansion in temporal ($\omega\sim\omega_*,\omega_d,k_\parallel v_{ti},k_\parallel v_A \ll k_\parallel v_{te}$) and spatial ($k_\perp^{-1}\sim\rho_i \gg \rho_e,c/\omega_{pe}$) scales, formally carried out as an expansion in the electron–ion mass ratio, treating plasma β as an order unity quantity. This expansion results in a simple set of electron fluid equations which describe electromagnetic electron dynamics on the typical ion drift and shear Alfvén length and time scales, while analytically removing the numerically

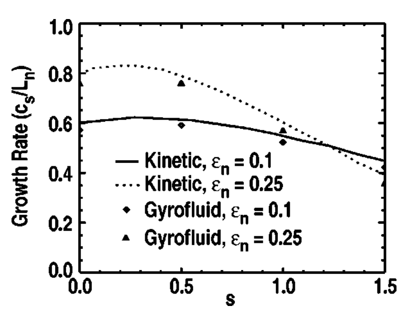

FIG. 4. Linear growth rate of the kinetic ballooning mode vs magnetic shear, at two values of $\epsilon_n=0.1,0.25$, for $\beta=9.375\%$, $k_\theta\rho_i=0.3$, $q=2$, $\tau=1$, $\eta_i=2$, and $\eta_e=0$. The gyrofluid model is compared to linear kinetic theory (Ref. 21).

challenging electron transit time scale as well as the small electron gyroradius and skin depth length scales. While the resulting electron model is simple and relatively straightforward to implement numerically, it describes substantial physics not incorporated in the adiabatic electron models that have been used to describe the passing electrons in many previous gyrofluid and gyrokinetic simulations. In addition to finite-β effects and Alfvén wave dynamics, the model also incorporates electron $\mathbf{E}\times\mathbf{B}$, curvature, and ∇B drift motion, as well as the $\mathbf{E}\times\mathbf{B}$ nonlinearity and nonlinear terms due to magnetic flutter. The use of an electron temperature closure appropriate for $\omega\sim\omega_A \ll k_\parallel v_{te}$ allows for the proper inclusion of the ∇T_e as well as the ∇n_e drive of the kinetic ballooning mode.

In the lowest order form, including no finite electron mass effects, the reduced electron equations lead to an electron response in which the total, equilibrium plus fluctuating, electron temperature is constant along the total magnetic field, $(\mathbf{B_0}+\tilde{\mathbf{B}})\cdot\nabla(T_{0e}+\widetilde{T_{\parallel_e}})=0$. This intuitive result is expected from our ordering of the velocities $\omega/k\sim v_{ti},v_A \ll v_{te}$, implying the characteristic time scale of both plasma and magnetic fluctuations is long compared to the time scale on which the electrons thermalize. It is important to note that this isothermal response is quite different from the occasionally employed closures $\nabla_\parallel \tilde{T}_e=0$, or $\tilde{T}_e=0$, both of which fail to properly account for magnetic fluctuations across the equilibrium temperature gradient. In the finite m_e version of the reduced electron model, formally carried to order $(m_e/m_i)^{1/2}$, models of parallel electron Landau damping as well as electron–ion collisions enter, and the isothermal electron condition is only approximate.

Ion species are described with the full set of six gyrofluid moment equations, truncated with kinetic closures based on Refs. 12, 30, and 44, incorporating both parallel and toroidal kinetic effects. The full set of electromagnetic ion gyrofluid equations include models of parallel Landau damping, ion drift resonance, ion–ion collisions, and linear and nonlinear finite-Larmor-radius (FLR) effects. Magnetic fluctuations enter the ion equations through the inductive electric field, as well as through several linear and nonlinear magnetic flutter terms.

The model has been benchmarked with linear gyrokinetic calculations, and good agreement has been found for the growth rates and real frequencies of both the finite-β toroidal ion temperature gradient (ITG) and kinetic Alfvén ballooning (KBM) instabilities. In particular, the model has been found to accurately reproduce the finite-β stabilization of the toroidal ITG mode. The model is also able to reproduce the behavior described by Refs. 21, 22, and 57, in which the kinetic ballooning mode is driven unstable below the ideal MHD ballooning limit (β_c) by ion drift resonance.

The model with reduced electron equations is intended for use in nonlinear simulations of ion drift/shear Alfvén turbulence and transport. Use of the full six moment model for the electrons should allow simulation of turbulence on the smaller scales characteristic of electron drift instabilities. Considerations associated with the use of linearized kinetic closures in nonlinear simulations have been discussed extensively in the gyrofluid literature.[10–14,30,31,39] An important strength of the reduced electron equation derivation given here is that it requires no separate closure approximations for nonlinear terms in the limit $m_e/m_i\to 0$. One important caveat for the multispecies gyrofluid equations is that the closures employed here reproduce the fast linear collisionless damping of zonal flows for $t< qR/v_{ti}\sqrt{\epsilon}$, but do not account for the residual undamped component of the zonal flow noted by Rosenbluth and Hinton,[48] which can be important at low collisionality. Efforts to incorporate neoclassical effects such as this residual flow into the toroidal closure are ongoing.[49] Nonlinear Landau damping processes may also be important, particularly in weak turbulence/low-collisionality regimes where the rate of scattering out of the resonant region is small. Important avenues for future work include nonlinear benchmarking with gyrokinetic codes to assess the impact of nonlinear kinetic effects, and incorporation of an appropriate model for trapped electrons.

Simulations of nonlinear toroidal microturbulence using the reduced electron model have been carried out, and are described in Refs. 32 and 55.

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