Investigate stability of the iterative semi-implicit algorithm, demonstrating that it is symplectic for an arbitrary number of iterations.

GWH 6/29/2007

This gives the details of the proof at the end of Sec. 2.3 of "An iterative semi-implicit scheme with robust damping", N.F. Loureiro and G.W. Hammett, submitted to the J. of Comp. Phys., http://www.arxiv.org/physics/...

```
> restart; kernelopts(version);
```

*Maple 10.06, IBM INTEL NT, Oct 2 2006 Build ID 255401* (1)

```
> # remind ourselves how matrices work in maple :
  Matrix([[1, 2], [3, 4]]);
```

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{2}$$

Start with the iteration matrix B, defined by Eq.39:

```
> B := Matrix([[dt^2·f·g/(4·(L+1)), dt·g·L/(2·(L+1))], [dt·f/(2·(L+1)), L/(L+1)]]);
```

$$B := \begin{bmatrix} \frac{dt^2 f g}{4 L + 4} & \frac{dt\, g\, L}{2 L + 2} \\ \frac{dt\, f}{2 L + 2} & \frac{L}{L + 1} \end{bmatrix} \tag{3}$$

```
> B[1, 2];
```

$$\frac{dt\, g\, L}{2 L + 2} \tag{4}$$

```
> with(LinearAlgebra) :
> (v, U) := Eigenvectors(B);
```

$$v, U := \begin{bmatrix} 0 \\ \frac{1}{4} \quad \frac{4 L + dt^2 f g}{L + 1} \end{bmatrix}, \begin{bmatrix} -\frac{2 L}{dt\, f} & \frac{1}{2}\, dt\, g \\ 1 & 1 \end{bmatrix} \tag{5}$$

```
> # v is the list of eigenvalues, and the columns of U are the corresponding eigenvectors
  # check the eigenvector with a zero eigenvalue
  # (caution: maple changes the order of the eigenvalues sometimes)
> zeigenindex := 1;
  e2 := Matrix([[U[1, zeigenindex]], [U[2, zeigenindex]]]);
```

*zeigenindex* := 1

(6)

$$e2 := \begin{bmatrix} -\frac{2\,L}{dt\,f} \\ 1 \end{bmatrix} \tag{6}$$

> $e3 := B.e2;$

$$e3 := \begin{bmatrix} -\frac{2\,dt\,g\,L}{4\,L+4} + \frac{dt\,g\,L}{2\,L+2} \\ -\frac{2\,L}{2\,L+2} + \frac{L}{L+1} \end{bmatrix} \tag{7}$$

> $simplify(e3);$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

> $Dia := Matrix([[v[1], 0], [0, v[2]]]);$

$$Dia := \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{4}\,\frac{4\,L+dt^2\,f\,g}{L+1} \end{bmatrix} \tag{9}$$

> $e10 := Dia^p;$

$$e10 := \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{4}\,\frac{4\,L+dt^2\,f\,g}{L+1} \end{bmatrix}^p \tag{10}$$

> $simplify(e10);$

$$\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{4}\,\frac{4\,L+dt^2\,f\,g}{L+1} \end{bmatrix}^p \tag{11}$$

Since this didn't work to simplify the matrix power, do it directly:

> $DiaP := Matrix([[v[1]^p, 0], [0, v[2]^p]]);$

$$DiaP := \begin{bmatrix} 0 & 0 \\ 0 & \left(\frac{1}{4}\,\frac{4\,L+dt^2\,f\,g}{L+1}\right)^p \end{bmatrix} \tag{12}$$

> # *Be careful when using this maple script. Maple seems* **to** *switch the order*
> # *of the two eigenvalues* and *eigenvectors each time it is executed. The following*
> # *procedure works no matter which order they are* **in**.
> #
> # *Subsume DiaP into a single term* :
> $DiaP := \frac{DiaP \cdot ratio}{\left(\frac{1}{4} \cdot \frac{4 \cdot L + dt^2 \cdot f \cdot g}{L+1}\right)^p};$

$$DiaP := \begin{bmatrix} 0 & 0 \\ 0 & ratio \end{bmatrix} \tag{13}$$

> $U$;

$$\begin{bmatrix} -\frac{2\,L}{dt\,f} & \frac{1}{2}\,dt\,g \\ 1 & 1 \end{bmatrix} \tag{14}$$

> $Uinv := U^{-1}$;

$$Uinv := \begin{bmatrix} -\frac{2\,dt\,f}{4\,L + dt^2\,f\,g} & \frac{dt^2\,g\,f}{4\,L + dt^2\,f\,g} \\ \frac{2\,dt\,f}{4\,L + dt^2\,f\,g} & \frac{4\,L}{4\,L + dt^2\,f\,g} \end{bmatrix} \tag{15}$$

> $U.Dia.Uinv - B$;

$$\begin{bmatrix} \frac{1}{4}\,\frac{dt^2\,g\,f}{L+1} - \frac{dt^2\,f\,g}{4\,L+4} & \frac{1}{2}\,\frac{dt\,g\,L}{L+1} - \frac{dt\,g\,L}{2\,L+2} \\ \frac{1}{2}\,\frac{dt\,f}{L+1} - \frac{dt\,f}{2\,L+2} & 0 \end{bmatrix} \tag{16}$$

> $simplify(\%)$;

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{17}$$

Thus we have verified that B can be written as U D $U^{-1}$.

Now calculate $B^p$:

> $Bp := U.\,DiaP\,.\,Uinv$;

$$Bp := \begin{bmatrix} \frac{dt^2\,g\,ratio\,f}{4\,L + dt^2\,f\,g} & \frac{2\,dt\,g\,ratio\,L}{4\,L + dt^2\,f\,g} \\ \frac{2\,ratio\,dt\,f}{4\,L + dt^2\,f\,g} & \frac{4\,ratio\,L}{4\,L + dt^2\,f\,g} \end{bmatrix} \tag{18}$$

> $Bp := simplify(Bp)$;

$$Bp := \begin{bmatrix} \frac{dt^2\,g\,ratio\,f}{4\,L + dt^2\,f\,g} & \frac{2\,dt\,g\,ratio\,L}{4\,L + dt^2\,f\,g} \\ \frac{2\,ratio\,dt\,f}{4\,L + dt^2\,f\,g} & \frac{4\,ratio\,L}{4\,L + dt^2\,f\,g} \end{bmatrix} \tag{19}$$

Next calculate the matrix corresponding to the exact Crank-Nicolson (CN) solution. The exact CN solution is of the form:

$L\, y_{n+1} = R\, y_n$

which we will rearrange as

$y_{n+1} = CN\, y_n$

where $CN = L^{-1} R$. The L and R matrices are:

> $e19 := Matrix\left(\left[\left[1, -\frac{g \cdot dt}{2}\right], \left[-\frac{f \cdot dt}{2}, 1\right]\right]\right);$

$$e19 := \begin{bmatrix} 1 & -\frac{1}{2}\, dt\, g \\ -\frac{1}{2}\, f\, dt & 1 \end{bmatrix} \tag{20}$$

> $e20 := Matrix\left(\left[\left[1, \frac{g \cdot dt}{2}\right], \left[\frac{f \cdot dt}{2}, 1\right]\right]\right);$

$$e20 := \begin{bmatrix} 1 & \frac{1}{2}\, dt\, g \\ \frac{1}{2}\, f\, dt & 1 \end{bmatrix} \tag{21}$$

> $CN := e19^{-1}.e20;$

$$CN := \begin{bmatrix} -\frac{4}{-4 + dt^2 f g} - \frac{dt^2 g f}{-4 + dt^2 f g} & -\frac{4\, dt\, g}{-4 + dt^2 f g} \\ -\frac{4\, f\, dt}{-4 + dt^2 f g} & -\frac{4}{-4 + dt^2 f g} - \frac{dt^2 g f}{-4 + dt^2 f g} \end{bmatrix} \tag{22}$$

> $CN := simplify(CN);$

$$CN := \begin{bmatrix} -\frac{4 + dt^2 f g}{-4 + dt^2 f g} & -\frac{4\, dt\, g}{-4 + dt^2 f g} \\ -\frac{4\, f\, dt}{-4 + dt^2 f g} & -\frac{4 + dt^2 f g}{-4 + dt^2 f g} \end{bmatrix} \tag{23}$$

The matrix corresponding to the solution from the first iteration (calculating p=1), as given by Eq. 34, is:

> $A0 := Matrix\left(\left[\left[1 + \frac{f \cdot g \cdot dt^2}{2 \cdot L + 2}, \frac{dt \cdot g \cdot \left(f \cdot g \cdot dt^2 + 4 \cdot L + 4\right)}{4 \cdot (L + 1)}\right], \left[\frac{f \cdot dt}{L + 1}, 1 + \frac{f \cdot g \cdot dt^2}{2 \cdot L + 2}\right]\right]\right);$

(24)

$$A0 := \begin{bmatrix} 1 + \frac{f\,g\,dt^2}{2\,L+2} & \frac{dt\,g\,(dt^2 f\,g + 4\,L + 4)}{4\,L+4} \\ \frac{dt\,f}{L+1} & 1 + \frac{f\,g\,dt^2}{2\,L+2} \end{bmatrix} \tag{24}$$

Try directly proving the result now, without changing to the basis of B:

```
> Ap := simplify(CN + Bp.(A0-CN));
```

$$Ap := [\frac{1}{2}\,\frac{-8\,L-8-2\,L\,dt^2 f\,g-2\,dt^2 f\,g + ratio\,dt^4 f^2 g^2 + 4\,ratio\,L\,dt^2 f\,g}{(L+1)\,(-4+dt^2 f\,g)}, \frac{1}{4}\,\frac{(-16\,L-16 + ratio\,dt^4 f^2 g^2 + 4\,ratio\,L\,dt^2 f\,g)\,dt\,g}{(L+1)\,(-4+dt^2 f\,g)}], [\frac{(-4\,L-4 + ratio\,dt^2 f\,g + 4\,ratio\,L)\,f\,dt}{(L+1)\,(-4+dt^2 f\,g)}, \frac{1}{2}\,\frac{-8\,L-8-2\,L\,dt^2 f\,g-2\,dt^2 f\,g + ratio\,dt^4 f^2 g^2 + 4\,ratio\,L\,dt^2 f\,g}{(L+1)\,(-4+dt^2 f\,g)}] \tag{25}$$

```
> vAp := Eigenvalues(Ap);
```

$$vAp := [\frac{1}{2}\,\frac{1}{-4\,L + L\,dt^2 f\,g - 4 + dt^2 f\,g}\,\Big(-8 + 4\,ratio\,L\,dt^2 f\,g - 2\,L\,dt^2 f\,g - 2\,dt^2 f\,g - 8\,L + ratio\,dt^4 f^2 g^2 + \big(64\,dt^2 f\,g\,L^2 + 128\,L\,dt^2 f\,g - 4\,dt^6 f^3 g^3\,L\,ratio - 16\,dt^4 f^2 g^2\,ratio\,L^2 + 64\,dt^2 f\,g - 4\,dt^6 f^3 g^3\,ratio - 32\,L\,ratio\,dt^4 f^2 g^2 - 16\,ratio\,dt^4 f^2 g^2 + dt^8 f^4 g^4\,ratio^2 + 8\,dt^6 f^3 g^3\,ratio^2\,L - 64\,ratio\,L^2\,dt^2 f\,g - 64\,ratio\,L\,dt^2 f\,g + 16\,dt^4 f^2 g^2\,ratio^2\,L^2\big)^{1/2}\Big)], [-\frac{1}{2}\,\frac{1}{-4\,L + L\,dt^2 f\,g - 4 + dt^2 f\,g}\,\Big(8 - 4\,ratio\,L\,dt^2 f\,g + 2\,L\,dt^2 f\,g + 2\,dt^2 f\,g + 8\,L - ratio\,dt^4 f^2 g^2 + \big(64\,dt^2 f\,g\,L^2 + 128\,L\,dt^2 f\,g - 4\,dt^6 f^3 g^3\,L\,ratio - 16\,dt^4 f^2 g^2\,ratio\,L^2 + 64\,dt^2 f\,g - 4\,dt^6 f^3 g^3\,ratio - 32\,L\,ratio\,dt^4 f^2 g^2 - 16\,ratio\,dt^4 f^2 g^2 + dt^8 f^4 g^4\,ratio^2 + 8\,dt^6 f^3 g^3\,ratio^2\,L - 64\,ratio\,L^2\,dt^2 f\,g - 64\,ratio\,L\,dt^2 f\,g + 16\,dt^4 f^2 g^2\,ratio^2\,L^2\big)^{1/2}\Big)] \tag{26}$$

```
> vAp := simplify(vAp);
```

$$vAp := [\frac{1}{2}\,\frac{1}{-4\,L + L\,dt^2 f\,g - 4 + dt^2 f\,g}\,\Big(-8 + 4\,ratio\,L\,dt^2 f\,g - 2\,L\,dt^2 f\,g - 2\,dt^2 f\,g - 8\,L + ratio\,dt^4 f^2 \tag{27}$$

$$g^2 + \left(dt^2 f g \left(64 L^2 + 128 L - 4 L\, ratio\, dt^4 f^2 g^2 - 16\, ratio\, L^2 dt^2 f g + 64 - 4\, ratio\, dt^4 f^2 g^2 - 32\, ratio\, L\, dt^2 f g - 16\, ratio\, dt^2 f g + ratio^2 dt^6 f^3 g^3 + 8\, ratio^2 dt^4 f^2 g^2 L - 64\, ratio\, L^2 - 64\, ratio\, L + 16\, ratio^2 L^2 dt^2 f g\right)^{1/2}\right)], [ \frac{1}{2} \frac{1}{-4 L + L\, dt^2 f g - 4 + dt^2 f g} \left(-8 + 4\, ratio\, L\, dt^2 f g - 2 L\, dt^2 f g - 2\, dt^2 f g - 8 L + ratio\, dt^4 f^2 g^2 - \left(dt^2 f g \left(64 L^2 + 128 L - 4 L\, ratio\, dt^4 f^2 g^2 - 16\, ratio\, L^2 dt^2 f g + 64 - 4\, ratio\, dt^4 f^2 g^2 - 32\, ratio\, L\, dt^2 f g - 16\, ratio\, dt^2 f g + ratio^2 dt^6 f^3 g^3 + 8\, ratio^2 dt^4 f^2 g^2 L - 64\, ratio\, L^2 - 64\, ratio\, L + 16\, ratio^2 L^2 dt^2 f g\right)^{1/2}\right)]$$

> # Try **to** simplify this some **by** switching **to** normalized variables.
# **use** f·g =−ω$^2$, and *switch* **to** *normalized dt given* **by** *dtn* = *dt*·|omega| :
$e29 := simplify\left(subs\left(f = -\frac{\omega^2}{g}, dt = \frac{dtn}{|\omega|}, vAp\right)\right)$ assuming ω :: *real*;

$$e29 := [\frac{1}{2} \frac{1}{4 L + L\, dtn^2 + 4 + dtn^2} \left(8 + 4\, ratio\, L\, dtn^2 - 2 L\, dtn^2 - 2\, dtn^2 + 8 L - ratio\, dtn^4 - \left(dtn^2 \left(-64 L^2 - 128 L + 4 L\, ratio\, dtn^4 - 16\, ratio\, L^2 dtn^2 - 64 + 4\, ratio\, dtn^4 - 32\, ratio\, L\, dtn^2 - 16\, ratio\, dtn^2 + ratio^2 dtn^6 - 8\, ratio^2 dtn^4 L + 64\, ratio\, L^2 + 64\, ratio\, L + 16\, ratio^2 L^2 dtn^2\right)^{1/2}\right)], [\frac{1}{2} \frac{1}{4 L + L\, dtn^2 + 4 + dtn^2} \left(8 + 4\, ratio\, L\, dtn^2 - 2 L\, dtn^2 - 2\, dtn^2 + 8 L - ratio\, dtn^4 + \left(dtn^2 \left(-64 L^2 - 128 L + 4 L\, ratio\, dtn^4 - 16\, ratio\, L^2 dtn^2 - 64 + 4\, ratio\, dtn^4 - 32\, ratio\, L\, dtn^2 - 16\, ratio\, dtn^2 + ratio^2 dtn^6 - 8\, ratio^2 dtn^4 L + 64\, ratio\, L^2 + 64\, ratio\, L + 16\, ratio^2 L^2 dtn^2\right)^{1/2}\right)] \quad (28)$$

> $e30 := simplify(e29)$ assuming $dtn > 0, L > 0, 4 \cdot L > dtn^2$;

$$e30 := [-\frac{1}{2} \frac{1}{4 L + L\, dtn^2 + 4 + dtn^2} \left(-8 - 4\, ratio\, L\, dtn^2 + 2 L\, dtn^2 + 2\, dtn^2 - 8 L + ratio\, dtn^4 + dtn \left(-64 L^2 - 128 L + 4 L\, ratio\, dtn^4 - 16\, ratio\, L^2 dtn^2 - 64 \quad (29)$$

$$+ 4\, ratio\, dtn^4 - 32\, ratio\, L\, dtn^2 - 16\, ratio\, dtn^2 + ratio^2\, dtn^6 - 8\, ratio^2\, dtn^4\, L + 64\, ratio\, L^2$$
$$+ 64\, ratio\, L + 16\, ratio^2\, L^2\, dtn^2)^{1/2})], [$$
$$\frac{1}{2}\, \frac{1}{4\, L + L\, dtn^2 + 4 + dtn^2}\, (8 + 4\, ratio\, L\, dtn^2 - 2\, L\, dtn^2 - 2\, dtn^2 + 8\, L - ratio\, dtn^4$$
$$+ dtn\, (-64\, L^2 - 128\, L + 4\, L\, ratio\, dtn^4 - 16\, ratio\, L^2\, dtn^2 - 64$$
$$+ 4\, ratio\, dtn^4 - 32\, ratio\, L\, dtn^2 - 16\, ratio\, dtn^2 + ratio^2\, dtn^6 - 8\, ratio^2\, dtn^4\, L + 64\, ratio\, L^2$$
$$+ 64\, ratio\, L + 16\, ratio^2\, L^2\, dtn^2)^{1/2})]$$

For simplicity, select one of the eigenvalues for simplificiation. Repeat this calculation checking the other eigenvalue. (For this problem, I'm pretty sure the eigenvalues come in complex conjugate pairs, so that showing that one of the eigenvalues satisfies |lambda|=1 is sufficient. But it is easy enough to check both eigenvalues.)

Select each one here:

```
> #e31 := e30[1];
  e31 := e30[2];
```

$$e31 := \frac{1}{2}\, \frac{1}{4\, L + L\, dtn^2 + 4 + dtn^2}\, (8 + 4\, ratio\, L\, dtn^2 - 2\, L\, dtn^2 - 2\, dtn^2 + 8\, L - ratio\, dtn^4 \tag{30}$$
$$+ dtn\, (-64\, L^2 - 128\, L + 4\, L\, ratio\, dtn^4 - 16\, ratio\, L^2\, dtn^2 - 64$$
$$+ 4\, ratio\, dtn^4 - 32\, ratio\, L\, dtn^2 - 16\, ratio\, dtn^2 + ratio^2\, dtn^6 - 8\, ratio^2\, dtn^4\, L + 64\, ratio\, L^2$$
$$+ 64\, ratio\, L + 16\, ratio^2\, L^2\, dtn^2)^{1/2})$$

Maple needs more help recognizing that the argument of the square root is always negative (if the semi-implicit operator is sufficiently large), and so the sqrt(-arg) can be written as I*sqrt(arg). Select out just the argument of the square root (depending on the order Maple has put the terms, one needs to change the argument to the "op" command below):

```
> e31b1 := op([2, 7, 2, 1], e31);
```

$$e31b1 := -64\, L^2 - 128\, L + 4\, L\, ratio\, dtn^4 - 16\, ratio\, L^2\, dtn^2 - 64 \tag{31}$$
$$+ 4\, ratio\, dtn^4 - 32\, ratio\, L\, dtn^2 - 16\, ratio\, dtn^2 + ratio^2\, dtn^6 - 8\, ratio^2\, dtn^4\, L + 64\, ratio\, L^2$$
$$+ 64\, ratio\, L + 16\, ratio^2\, L^2\, dtn^2$$

```
> e31b2 := simplify(e31b1);
```

$$e31b2 := -64\, L^2 - 128\, L + 4\, L\, ratio\, dtn^4 - 16\, ratio\, L^2\, dtn^2 - 64 \tag{32}$$
$$+ 4\, ratio\, dtn^4 - 32\, ratio\, L\, dtn^2 - 16\, ratio\, dtn^2 + ratio^2\, dtn^6 - 8\, ratio^2\, dtn^4\, L + 64\, ratio\, L^2$$
$$+ 64\, ratio\, L + 16\, ratio^2\, L^2\, dtn^2$$

```
> e31b3 := factor(e31b2);
```

$$e31b3 := (4\, ratio\, L\, dtn^2 - ratio\, dtn^4 + 16 + 16\, L)\, (4\, ratio\, L - ratio\, dtn^2 - 4 - 4\, L) \tag{33}$$

It is straightforward to prove that the above (the argument of the square root in the eigenvalue) is negative (that the first factor is always positive and the second factor always negative) for arbitrary 0<ratio<1, as long as 4*L>dtn^2. I.e., a sufficient condition for the argument to be negative is omega_hat^2 > omega^2.

However, it is actually a little more general than that. I have been able to show that the condition that the first iteration be symplectic

4 + (omega_hat^2 - omega^2) dt^2 > 0

is also a sufficient condition for all of the following iterations to be symplectic. This is a little more complicated to prove, and one must consider the possibility of -1<ratio<0 as a special case, but with a few pages of checking the various cases, one can show that this is true.

Now knowing that the argument of the square root is always negative, we will make some substitutions to isolate it and then simplify it:

> $e32 := factor(e31);$

$$e32 := \frac{1}{2}\,\frac{1}{(4+dtn^2)\,(L+1)}\,\Big(8+4\,ratio\,L\,dtn^2-2\,L\,dtn^2-2\,dtn^2+8\,L-ratio\,dtn^4 + dtn\sqrt{(4\,ratio\,L\,dtn^2-ratio\,dtn^4+16+16\,L)\,(4\,ratio\,L-ratio\,dtn^2-4-4\,L)}\Big) \tag{34}$$

> $e32b1 := subs\big((4\cdot dtn^2\cdot ratio\cdot L-ratio\cdot dtn^4+16+16\cdot L)\cdot(4\cdot ratio\cdot L-ratio\cdot dtn^2-4-4\cdot L)=-arg,\ e32\big);$

$$e32b1 := \frac{1}{2}\,\frac{8+4\,ratio\,L\,dtn^2-2\,L\,dtn^2-2\,dtn^2+8\,L-ratio\,dtn^4+dtn\sqrt{-arg}}{(4+dtn^2)\,(L+1)} \tag{35}$$

> $e33 := simplify(e32b1)$ assuming $arg > 0;$

$$e33 := \frac{1}{2}\,\frac{8+4\,ratio\,L\,dtn^2-2\,L\,dtn^2-2\,dtn^2+8\,L-ratio\,dtn^4+\mathrm{I}\,dtn\sqrt{arg}}{(4+dtn^2)\,(L+1)} \tag{36}$$

Calculate the squared magnitude of the eigenvalue |lambda|^2 :

> $e34 := e33\cdot conjugate(e33);$

$$e34 := \frac{1}{4}\,\frac{1}{(4+dtn^2)\,(L+1)}\,\left(\left(8+4\,ratio\,L\,dtn^2-2\,L\,dtn^2-2\,dtn^2+8\,L-ratio\,dtn^4 + \mathrm{I}\,dtn\sqrt{arg}\right)\overline{\frac{8+4\,ratio\,L\,dtn^2-2\,L\,dtn^2-2\,dtn^2+8\,L-ratio\,dtn^4+\mathrm{I}\,dtn\sqrt{arg}}{(4+dtn^2)\,(L+1)}}\right) \tag{37}$$

> $e35 := simplify(e34)$ assuming $dtn :: real,\ L :: real,\ ratio :: real,\ arg > 0;$

$$e35 := \frac{1}{4}\,\frac{1}{(4+dtn^2)^2\,(L+1)^2}\,\Big(\big(-8-4\,ratio\,L\,dtn^2+2\,L\,dtn^2+2\,dtn^2-8\,L + ratio\,dtn^4-\mathrm{I}\,dtn\sqrt{arg}\big)\big(-8-4\,ratio\,L\,dtn^2+2\,L\,dtn^2+2\,dtn^2-8\,L+ratio\,dtn^4 + \mathrm{I}\,dtn\sqrt{arg}\big)\Big) \tag{38}$$

> $e36 := simplify(expand(e35));$

(39)

$$e36 := \frac{1}{4}\,\frac{1}{(4+dtn^2)^2\,(L+1)^2}\,\big(64+128\,L+4\,L^2\,dtn^4+64\,L^2-64\,L\,dtn^2-16\,ratio\,dtn^4 + 8\,L\,dtn^4-32\,L^2\,dtn^2+4\,dtn^6\,ratio+ratio^2\,dtn^8+64\,ratio\,L\,dtn^2 + 16\,ratio^2\,L^2\,dtn^4-16\,ratio\,L^2\,dtn^4-32\,L\,ratio\,dtn^4+64\,ratio\,L^2\,dtn^2-8\,ratio^2\,L\,dtn^6 + 4\,L\,dtn^6\,ratio+dtn^2\,arg-32\,dtn^2+4\,dtn^4\big) \tag{39}$$

```
> e37 := subs(arg = -(4·dtn^2·ratio·L - ratio·dtn^4 + 16 + 16·L)·(4·ratio·L - ratio·dtn^2 - 4 - 4·L),
      e36);
```

$$e37 := \frac{1}{4}\,\frac{1}{(4+dtn^2)^2\,(L+1)^2}\,\big(64+128\,L+4\,L^2\,dtn^4+64\,L^2-64\,L\,dtn^2-16\,ratio\,dtn^4 + 8\,L\,dtn^4-32\,L^2\,dtn^2+4\,dtn^6\,ratio+ratio^2\,dtn^8+64\,ratio\,L\,dtn^2 + 16\,ratio^2\,L^2\,dtn^4-16\,ratio\,L^2\,dtn^4-32\,L\,ratio\,dtn^4+64\,ratio\,L^2\,dtn^2-8\,ratio^2\,L\,dtn^6 + 4\,L\,dtn^6\,ratio-dtn^2\,(4\,ratio\,L\,dtn^2-ratio\,dtn^4+16+16\,L)\,(4\,ratio\,L-ratio\,dtn^2-4 -4\,L)-32\,dtn^2+4\,dtn^4\big) \tag{40}$$

```
> simplify(e37);
```

$$1 \tag{41}$$

Q.E.D.

This completes the proof that the eigenvalues for an arbitrary number of iterations satisfy |lambda_i| = 1, so the algorithm is symplectic.