A Physics Formulary

Revised October 16, 2013

A supplement to the **NRL Plasma Formulary**[1] (a.k.a. Book's Book). This is a handy collection of important formulas which I have learned over the years. This is intended as a memory aide, not a knowledge substitute. In other words, don't use a formula here if you don't understand it. An expert plasma physics graduate student will have derived all of the formulas here by the time he or she passes their Ph.D. qualifying exam. [This document is not yet complete, I welcome any suggestions at the email address below. Places marked with ?? need more work or double-checking.]

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A folded booklet version of this document (2 pages per side, print it double-sided and fold in half) is available at: `http://w3.pppl.gov/~hammett/hpf/hpf-booklet.pdf`

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References

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[2] H.J. and K.H. Fishbeck, *Formulas, Facts, and Constants*, (Springer-Verlag, 1987). An interesting and comprehensive collection of useful formulas.

[3] Charles F. Stevens, *The Six Core Theories of Modern Physics*, (MIT Press, 1995). A nice and brief summary of the essential concepts in mathematics, classical mechanics, E&M, quantum mechanics, statistical physics, special relativity, and quantum field theory.

[4] T. Padmanabhan, *Theoretical Astrophysics, Volume I: Astrophysical Processes* (Cambridge University Press, 2000). (This is a nice one-volume review of classical mechanics, E&M, relativity, statistical physics, plasma physics, and a few other topics.)

[5] Alan C. Tribble, *Princeton Guide to Advanced Physics*, Princeton Univ. Press, 1996.

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[8] "Fluid Models of Phase Mixing, Landau Damping, and Nonlinear Gyrokinetic Dynamics," G. W. Hammett, W. Dorland and F. W. Perkins, Physics of Fluids B, **4** (2052) 1992. "Fluid Models for Landau Damping with Application to the Ion-Temperature-Gradient Instability," G.W. Hammett and F.W. Perkins, Phys. Rev. Lett. **64**, 3019 (1990).

[9] Per Helander, Dietar J. Sigmar, *Collisional Transport in Magnetized Plasmas*, Cambridge University Press, 2002.

[10] The values quoted here for the high Z_i Lorentz plasma limit differ slightly from Braginskii's original paper, see Epperlein and Haines, Phys. Fluids **29**, (1986).

[11] "A drift ordered short mean free path description for magnetized plasma allowing strong spatial anisotropy", P.J. Catto and A.N. Simakov, Phys. Plasmas **11**, 90 (2004).

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[13] ?? Should add brief annotated list of various plasma textbooks here: F. F. Chen, Hazeltine and Waelbroeck, R.O. Dendy, R. White. R. Kulsrud, J. Wesson *Tokamaks*, Lieberman and Lichtenberg, Krall & Trivelpiece, Sturrock, Schmidt, Nicholson, Miyamoto, Ichimaru, Boyd and Sanderson, Freidberg, Gurnett and Bhattacharjee, Spitzer, Rose and Clark, Stix, Birdsall & Langdon, Goedbloed & Poedts, Helander and Sigmar, F.F. Chen and J. P. Chang, *Lecture Notes on Principles of Plasma Processing*, 2003.

Chapter 1

Introduction

The books by Stevens,[3] Padmanabhan,[4] and Tribble[5] are particularly nice, concise summaries of advanced physics at the graduate student level.

For a brief review of complex analysis, try Tribble's book or Appendix C "Pedestrian's guide to Complex Variables," in Nicholson's *Introduction to Plasma Theory*. (If that starts too deep for you, check out the references he sites. I like Sokolnikoff and Redheffer, *Mathematics of Physics and Modern Engineering*.)

Summaries of E&M and classical mechanics are in K. Miyamoto, *Plasma Physics for Nuclear Fusion* (MIT, 1980).

Unless otherwise indicated, most of the formulas here are in cgs, not SI (MKS).

Acknowledgements: Thanks to my many physics and mathematics teachers over the years, including Prof. Tom Stix, Paul Bamberg, George Carrier, John Krommes, and many others. Useful corrections and suggestions for this document have been made by Tim Stoltzfus-Dueck, Nino Pereira, and ...

Chapter 2

Mathematics

2.1. Basic Equations

Quadratic Equation:

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Factorials:

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

$$1! = 1 \qquad 0! = 1$$

$$(2n+1)!! = (2n+1)(2n-1)(2n-3)\cdots(5)(3)(1) = \frac{(2n+1)!}{2^n n!}$$

$$(2n)!! = (2n)(2n-2)(2n-4)\cdots(4)(2) = n!2^n$$

The number of permutations (where order matters) of k objects selected from a set of n objects, is

$$\frac{n!}{(n-k)!} = n(n-1)(n-2)\ldots(n-k+1).$$

The number of combinations (where order doesn't matter) of k objects selected from a set of n objects is (this is sometimes called "n choose k"):

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}$$

The binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) x^k y^{n-k}$$

Chapter 7

Astrophysics

1 parsec (pc) $= 3.086 \times 10^{16} m = 3.262$ lyr
1 light year (ly) $= 9.461 \times 10^{15} m$
1 Julian year $= 365.25$ days $= 3.156 \times 10^{7} s$

Approx. a hundred, thousand, million (10^{11}) stars per galaxy.
Approx. a hundred, thousand, million (10^{11}) galaxies in the visible universe.
Approx. 1 supernova explosion per galaxy per century.
Age of the universe: 14 billion years.

?? Could add a length/mass scale object plot, starting with largest scale at the size of the (visible) universe, clusters, groups, elllitical and spiral clusters, AGN/MBH, globular clusters, red giants, stars, white dwarfs, neutron stars, jupiter, earth, etc. (like Padmanabhan Table 1.1 or elsewhere), and continuing down to molecules, atoms, nucleons...

Add a phase diagram plot like Fig. 1.1 of Padmanabhan?

Could add a time history plot: big bang, first 3 minutes, light element fusion, recombination, first stars, reionization, galactic formation, age of the solar system, earth, ...

Stellar structure, stellar life cycle...

Chapter 6

Quantum Mechanics

6.1. The essential quantum mechanic

Schrödinger's Equation:

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi = \left(\frac{p^2}{2m}+V\right)\Psi = \left(-\frac{\hbar^2}{2m}\nabla^2+V\right)\Psi$$

Plane waves (with momentum $p=\hbar k$ and energy $E=\hbar\omega$):

$$\Psi \propto e^{i(kx-\omega t)} = e^{i(px-Et)/\hbar}$$

Commutators: $[x,p] = xp - px = i\hbar$

$$\frac{d}{dt}\langle A\rangle = \left\langle\frac{dA}{dt}\right\rangle + \frac{i}{\hbar}\langle[H,A]\rangle$$

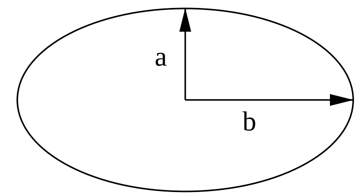
Heisenberg Uncertainty Principle $\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}||\langle\psi|[A,B]|\psi\rangle||^2$.

"Natural units" uses 3 fundamental units: action (or angular momentum) ($\hbar$), velocity c, and energy eV. The 3 fundamental units of cgs are length, mass, and time, and "action" has units of [momentum]×[length]. . In natural units, $\hbar = c = 1$, and all physical units are reported in "eV".

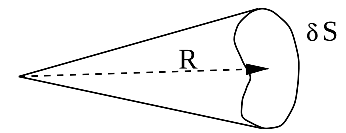
?? Could add: Harmonic oscillator, Variational methods, Bound-state non-degenerate perturbation theory, degenerate perturbation theory, time-dependent perturbation, scattering theory, Born approximation, angular momentum and spin, atomic energy levels.

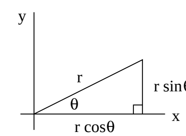
Geometry

Ellipse Area $=\pi ab$.
Circle Area $=\pi r^2$, Circumference $=2\pi r$.
Sphere Volume $=\frac{4}{3}\pi r^3$, Area $=4\pi r^2$.



Solid Angle: $\delta\Omega = \frac{\delta S}{R^2}$ $\quad \int_{\partial V} d\Omega = 4\pi$





Trig identities:

$$\sin^2 x + \cos^2 x = 1$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{1}{\cot x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \cos(x+y) = \cos x\cos y - \sin x \sin y$$

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \sin(x+y) = \sin x\cos y + \cos x \sin y$$

Exponential identities:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2i}$$

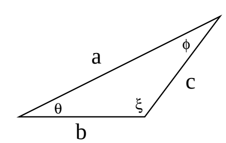
$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} \qquad \cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$

$$\cosh^2 x - \sinh^2 x = 1$$

For an arbitrary triangle:

$$a^2+b^2-2ab\cos\theta = c^2$$

$$\frac{\sin\theta}{c} = \frac{\sin\phi}{b} = \frac{\sin\xi}{a}$$



Differentiation

$$dg(u) = g'(u)du$$
$$d(fg) = fdg + gdf$$
$$d\left(\frac{f}{g}\right) = \frac{gdf - fdg}{g^2}$$
$$d\sin x = \cos x dx \qquad d\tan x = \sec^2 x dx$$
$$d\cos x = -\sin x dx \qquad d\cot x = -\csc^2 x dx$$
$$d\sec x = \tan x \sec x dx \qquad d\csc x = -\cot x \csc x dx$$
$$d\arcsin x = \frac{dx}{\sqrt{1-x^2}} \qquad d\arccos x = \frac{-dx}{\sqrt{1-x^2}}$$
$$d\arctan x = \frac{dx}{1+x^2} \qquad d\,\mathrm{arcsec}\, x = \frac{dx}{x\sqrt{x^2-1}}$$
$$d\log x = \frac{dx}{x}$$

Taylor Series (with remainder):

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(X)}{(n+1)!}(x-a)^{n+1}$$

Infinite Series:

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$
$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots + x^n$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$$
$$\text{for } -1 < x < 1\text{: } \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots$$
$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$$
$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} - \cdots$$
$$\text{for } |x| > |y|\text{: } (x+y)^\alpha = x^\alpha + \frac{\alpha}{1!}x^{\alpha-1}y + \frac{\alpha(\alpha-1)}{2!}x^{\alpha-2}y^2 + \cdots$$

5.6. Stochasticity, Turbulence, and Transport

5.7. Tokamak Equilibrium

5.8. Common Plasma Physics Parameters

"Safety factor" (better, "inverse rotational transform" or "winding ratio"):

$$q = \frac{2\pi}{\iota} = \frac{d\Psi_{tor}}{d\Psi pol} \approx \frac{r}{R}\frac{B_\phi}{B_\theta}$$

Magnetic shear

$$\hat{s} = \frac{r}{q}\frac{dq}{dr}$$

Random walk diffusion coefficient

$$D = \frac{1}{2}\frac{(\Delta x)^2}{\Delta t}$$

$$D_{classical} = \nu_{ei}\rho_e^2$$

Turbulent mixing length estimate

$$D_{ml} = \frac{\gamma}{k_\perp^2}$$

Bohm

$$D_{Bohm} = \frac{1}{16}\frac{cT_e}{eB}$$

Gyro-reduced Bohm ($\Delta x \sim 1/k \sim \rho$, $\Delta t \sim 1/\gamma \sim 1/\omega_*$ evaluated at $k_\perp \rho \sim 1$):

$$D_{gB} = \frac{cT_e}{eB}\frac{\rho_s}{L_n} = c_s\rho_s\frac{\rho_s}{L_n}$$

Reaction rates are of the form $\Gamma = n_\alpha n_\beta \langle \sigma v \rangle/(1+\delta_{ij})$, where the δ_{ij} corrects for the case of self-collisions.

?? The form of 1.5D transport equations in general geometry.

?? 0-D scaling relations for reactor design studies: Troyon beta limit $\beta \propto I/(aB)$, global energy scaling, Greenwald density limit, pedestal scalings, H-mode power thresholds. shaping effects, bootstrap fraction. Trubnikoff's ECE cyclotron power losses.

5.4. MHD/One-Fluid Equations

The standard ordering assumptions to derive simple MHD are: slow time scales compared to the gyrofrequency and large spatial scales compared to the gyroradius (similar to the drift equations), $\omega/\Omega_{ci} \sim \rho_i/L \sim \epsilon \ll 1$. $m_e/m_i \ll 1$ is used and quasineutrality is assumed (this orders out high-frequency electron plasma oscillations), and $v_A/c \ll 1$ is assumed (the displacement current is ignored to order out light waves). MHD allows flows $u \sim c\vec{E} \times \vec{B}/B^2 \sim v_{ti}$ and $\beta \sim 1$, though subsidiary orderings can be made later. Switch from two-fluid variables to one-fluid variables: mass density $\rho = \sum_\alpha n_\alpha m_\alpha$, mass-weighted flow velocity $\rho\vec{u} = \sum_\alpha n_\alpha m_\alpha \vec{v}_\alpha$, current density $\vec{j} = \sum n_\alpha q_\alpha u_\alpha$, and define pressure relative to $\vec{u}$, $\overleftrightarrow{\Pi} = \sum_\alpha m_\alpha n_\alpha \langle(\vec{v}-\vec{u})(\vec{v}-\vec{u})\rangle_\alpha \approx p\overleftrightarrow{1}$

Conservation of Mass $$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$$

Momentum conservation, force balance $$\rho\frac{d\vec{u}}{dt} = -\nabla p + \frac{\vec{j}\times\vec{B}}{c}$$

Energy conservation, adiabatic pressure $$\frac{dp}{dt} = -\Gamma p \nabla\cdot\vec{u}$$

Generalized Ohm's Law (FLR but $m_e \to 0$) $$\vec{E} + \frac{\vec{u}\times\vec{B}}{c} = \eta\vec{j} - \frac{\nabla p_e}{ne} + \frac{\vec{j}\times\vec{B}}{nec}$$

Magnetostatic Maxwell's Eqs: $$\frac{\partial\vec{B}}{\partial t} = -c\nabla\times\vec{E} \qquad \nabla\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

Other Maxwell's equations: $\nabla\cdot\vec{B} = 0$ is only an initial condition, and $\nabla\cdot\vec{E} = 4\pi\sigma$ is used only to verify quasineutrality assumption. The last term of the generalzied Ohm's law is the Hall term, and the last two terms of the Ohm's law are usually ρ_i/L smaller than the first two terms and are neglected in standard MHD. Extensions of simple MHD are sometimes made to keep a CGL pressure tensor or a full pressure tensor, $\nabla p \to \nabla\cdot\overleftrightarrow{\Pi}$, using equations of state or Braginskii transport coefficients from the previous section.

There are **three main waves in MHD**. Linearizing the MHD equations for a uniform plasma with a straight magnetic field and an adiabatic equation of state $\delta p = c_s^2\delta\rho$, the general dispersion relation is

$$(\omega^2 - k_\parallel^2 v_A^2)(\omega^4 - \omega^2 k^2(c_s^2 + v_A^2) + k^2 k_\parallel^2 c_s^2 v_A^2) = 0$$

where the Alfvén speed v_A is given by $v_A^2 = B^2/(4\pi\rho)$, and the sound speed c_s is given by $c_s^2 = \Gamma p/\rho = \Gamma(T_i + T_e)/m_i$. Approximate formulas that interpolate for arbitary β are: the **shear Alfvén wave** $\omega^2 = k_\parallel^2 v_A^2$, the **fast magnetosonic (compressional Alfvén) wave** $\omega^2 = k^2(v_A^2 + c_s^2)$, and the **slow magnetosonic wave, a.k.a. the slow mode (at high beta sometimes called the pseudo-Alfvén wave, and at low beta it becomes an ion acoustic wave)** $\omega^2 = k_z^2 v_A^2 c_s^2/(v_A^2 + c_s^2)$. (There is also the lesser known entropy mode, but this is eliminated by using an adiabatic equation of state instead of the time-dependent pressure equation. In ideal MHD the entropy mode is zero frequency and has $\delta\rho \neq 0$ but $\delta p = 0$ (i.e., force balance is maintained by opposite density and temperature gradients).)

?? δW Energy principle, Grad-Shafranov Equation, MHD equilibria in general geometry.

5.5. Waves

cold-plasma dielectric tensor? quasilinear theory?

Vector & Tensor Operators (in simple Cartesian geometry)

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$
$$d\vec{A} = d\vec{r}\cdot\nabla\vec{A}$$
$$d\vec{A} = (dx, dy, dx)\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}(A_x, A_y, A_z)$$

Here I am using the notation that a row vector times a column vector is a dot product, while a column vector times a row vector is a tensor product. I.e., $\nabla\vec{A}$ is a tensor product, while $\vec{B}\cdot\nabla\vec{A}$ is a vector (the gradient of $\vec{A}$ in the direction of $\vec{B}$).

Einstein summation convention: there is an implied sum over repeated indices. This simplifies working with tensors represented as their indexed matrix elements. Let x_i for $i = 1, 2, 3$ represent the x,y,z coordinates, and A_i the component of $\vec{A}$ in the i'th direction.

$$\vec{A}\cdot\vec{B} = A_i B_i$$
$$(\vec{B}\cdot\nabla\vec{A})_i = B_j\frac{\partial A_i}{\partial x_j}$$

Tensor notation (for simple Cartesian geometry, ignoring contravariant vs. covariant representations and upper vs. lower indices): Writing two vectors next to each other (without a dot that would indicate a dot product or inner product) is called a tensor product (or outer product) and results in a second-rank tensor: $\vec{A}\vec{B} = A_i B_j$ (sometimes this is called a dyad; the tensor product is sometimes denoted by $\vec{A}\otimes\vec{B}$ or $\vec{A}\vec{B}^{\mathrm{T}}$, where $\vec{A}$ is a column vector and $\vec{B}^{\mathrm{T}}$ is a row vector). Tensors are $\approx$ matrices:

$$\overleftrightarrow{T}\cdot\vec{A} = T_{ij}A_j \qquad \overleftrightarrow{T}\cdot\overleftrightarrow{P} = T_{ij}P_{jk}$$
$$\vec{A}\cdot\overleftrightarrow{T} = A_j T_{ji} \qquad \overleftrightarrow{T}:\overleftrightarrow{P} = T_{ij}P_{ij} \qquad (\vec{A}\vec{B}):(\vec{C}\vec{D}) = \vec{C}\cdot\vec{A}\vec{B}\cdot\vec{D},$$

$\overleftrightarrow{T}:\overleftrightarrow{P}$ involves contraction with respect to two indices and is called a colon product (or a "double dot product"). It is a generalization of a scalar inner product from vectors to matrices. The Frobenius matrix norm $||\overleftrightarrow{T}|| = \left(\overleftrightarrow{T}:\overleftrightarrow{T}\right)^{1/2}$.

$$\nabla\psi \text{ is a vector} = \frac{\partial}{\partial x_i}\psi$$
$$\nabla\vec{A} \text{ is a tensor} = \frac{\partial}{\partial x_i}A_j$$
$$\nabla\cdot\overleftrightarrow{T} \text{ is a vector} = \frac{\partial}{\partial x_i}T_{ij}$$
$$\nabla\cdot(\vec{A}\cdot\overleftrightarrow{T}) = \frac{\partial}{\partial x_j}(A_i T_{ij}) = A_i\frac{\partial}{\partial x_j}T_{ij} + \frac{\partial A_i}{\partial x_j}T_{ij} = \vec{A}\cdot\left(\nabla\cdot\overleftrightarrow{T}^t\right) + \left(\nabla\vec{A}\right):\overleftrightarrow{T}$$

where $\overleftrightarrow{T}^t$ is the transpose of $\overleftrightarrow{T}$. The unit tensor $\overleftrightarrow{1} = \mathbf{I}$ = identity matrix = Kronecker delta

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Levi-Civita symbol:

$$(\vec{A}\times\vec{B})_i = \varepsilon_{ijk}A_jB_k \qquad \text{where}$$

$$\epsilon_{ijk} = \begin{cases} 1 \text{ if } i\neq j\neq k \text{ cyclic permutation of 1, 2, 3} \\ -1 \text{ if } i\neq j\neq k \text{ cyclic permutation of 1, 3, 2} \\ 0 \text{ if } i=j \text{ or } j=k \text{ or } i=k \end{cases}$$

$$(\nabla\times A)_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}A_k$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \qquad \text{is equivalent to}$$

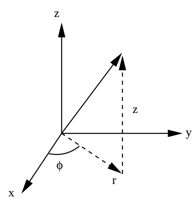
$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

This can be used to prove

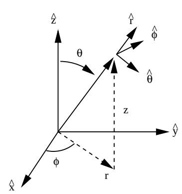
$$\frac{\partial A_j}{\partial x_i}v_j - \frac{\partial A_i}{\partial x_j}v_j = (\mathbf{v}\times\mathbf{B})_i$$

where $\mathbf{B} = \nabla\times\mathbf{A}$. That is, $(\nabla\mathbf{A})\cdot\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{A} = \mathbf{v}\times\mathbf{B}$.

Cylindrical, Spherical, and General Geometry



Cylindrical geometry: $d\vec{r} = \hat{r}dr + \hat{\phi}rd\phi + \hat{z}dz$.



Spherical geometry: $d\vec{r} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\phi}r\sin\theta d\phi$.

Add something here about vector operators in general curvilinear coordinates, Jacobians, coordinate transformations, etc??

$d\vec{S}$ is a vector that is “normal” to the surface, $|d\vec{S}|$ measures the area.

where the magnetization current is given by $\vec{j}_M = \nabla\times\vec{M} = -\nabla\times(cn_s\langle\mu_s\rangle\hat{b}) = -\nabla\times((c/B)p_\perp\hat{b})$, and $\langle\mu_s\rangle$ is the mean magnetic moment for species s.

The NRL/Braginskii expressions for $\vec{\vec{W}}$ can be applied to arbitrary, non-straight, $\vec{B}$ fields, as long as one properly identifies $\hat{z}$ with $\hat{b} = \vec{B}/B$. There is a potential ambiguity in the NRL expressions: the proper relation is $\partial v_z/\partial x_z = (\hat{b}\cdot\nabla\mathbf{v})\cdot\hat{b}$, and **not** $\partial v_z/\partial x_z = \hat{b}\cdot\nabla(\mathbf{v}\cdot\hat{b})$. For example, $W_{zz} = \hat{b}\cdot\vec{\vec{W}}\cdot\hat{b} = 2\hat{b}\cdot(\nabla\vec{v})\cdot\hat{b} - (2/3)\nabla\cdot\vec{v}$. More generally, $\mathbf{W} = \nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}} - (2/3)\mathbf{1}\nabla\cdot\mathbf{v}$. Note that $\vec{\vec{P}}$ and $\vec{\vec{W}}$ are traceless ($W_{xx} + W_{yy} + W_{zz} = 0$) and symmetric. In the strong B limit ($\Omega_c\tau \gg 1$, where τ is the collision time), Braginskii's stress tensor becomes diagonal to lowest order, $\vec{\vec{P}} = -\eta_0[W_{zz}\hat{b}\hat{b} - (W_{zz}/2)(\vec{\vec{1}} - \hat{b}\hat{b})]$. Even without strong collisions, in the strong B limit ($\omega/\Omega_c \ll 1$, $\rho/L \ll 1$) the rapid gyration of particles means that $f(\vec{v})$ to lowest order must be isotropic perpendicular to $\vec{B}$, so the pressure tensor must be diagonal, yielding the CGL (Chew-Goldberger-Low) pressure tensor $\vec{\vec{\Pi}} = p_\parallel\hat{b}\hat{b} + p_\perp(\vec{\vec{1}} - \hat{b}\hat{b})$. The CGL "double adiabatic" equations of state (neglecting heat flows and collisions):

$$\frac{d}{dt}\left(\frac{p_\perp}{nB}\right) = 0 \qquad \text{(from } \mu \text{ conservation)}$$

$$\frac{d}{dt}\left(T_\parallel\left(\frac{B}{n}\right)^2\right) = 0 \qquad \text{(if the magnetic field and plasma move together, } T_\parallel \text{ changes only due to compression parallel to } \vec{B})$$

The fluid equations are often simplified further (such as in simple MHD) by assuming isotropic pressure and neglecting heat flows and collisional energy exchange between species:

$$\frac{\partial p}{\partial t} + \vec{v}\cdot\nabla p = -\Gamma p\nabla\cdot\vec{v} \qquad \text{or} \qquad \frac{d}{dt}\left(\frac{p}{n^\Gamma}\right) = 0$$

i.e., an adiabatic equation of state where a fluid element compresses or decompresses as an ideal gas with $p = Cn^\Gamma$ (C is constant as the fluid element moves, but may differ between fluid elements because of the spatial variation of the initial temperature, so the above form $d/dt(p/n^\Gamma) = 0$ is more general). $\Gamma = 5/3$ in 3-D, or $\Gamma = (2+d)/d$ with d = # of degrees of freedom in general. While this equation of state corresponds to zero heat flux (which may be appropriate for waves that propagate faster than particles, $\omega/k \gg v_t$), choosing $\Gamma = 1$ allows one to consider the opposite limit of a heat flux so rapid that the temperature is uniform (this isothermal closure may be appropriate for phenomena with $\omega/k \ll v_t$). For some phenomena, an even simpler closure of $p = 0$ (the cold-plasma approximation) is made. Intermediate cases where $\omega/k \sim v_t$ gives rise to Landau damping. Approximate fluid models of Landau damping use closures for higher moments that correspond to characteristic damping rates of order $v_t|k|$, the phase-mixing rate.[8]

Equations of state summary: adiabatic $p \propto n^{5/3}$, isothermal $p \propto n$, cold-plasma $p = 0$.

Braginskii's equations are derived for a specific ordering and there are corrections that can become important in some regimes. For example, see papers by Catto and Simakov[11] circa 2002-2005. Mikhailovskii and Tsypin[12] have terms like

$$\nabla\cdot\vec{\vec{\Pi}} \sim c_1\nabla\vec{u} + c_2\nabla\vec{q}$$

where c_1 is Braginskii-type terms and c_2 are Mikhailovskii's new heat flux terms?

Spitzer's resolution of the Fluid-Particle paradox: The fluid flow velocity is the sum of the particle guiding center drifts plus a diamagnetic velocity (a.k.a. magnetization current). I.e., the current from a particular species is

$$\vec{j} = \vec{j}_{E\times B} + \vec{j}_{\nabla B} + \vec{j}_{curv} + \vec{j}_{pol} + \ldots + \vec{j}_M$$

Integration

$$\int f\,dg = fg - \int g\,df$$

$$\int \frac{dx}{x} = \log|x|$$

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{1}{2}$$

Gamma Function: $\Gamma(x) = (x-1)! = \int_0^\infty t^{x-1}e^{-t}dt$

Stirling's approx.: $n! \sim \sqrt{2\pi n}\,n^n e^{-n}\,(1 + \frac{1}{12n} + \ldots)$

uniform approx. good for $n = 0$: $n! \sim \sqrt{2\pi n + 1}\,n^n e^{-n}$

error $\le 1\%$ for integer $n \ge 0$, max error $\le 4\%$ for $n \sim 0.1$

Generalized Maxwellian Moments for complex α, β; Real $\alpha > 0$:

$$G_n = \int_{-\infty}^{+\infty} x^n e^{-\alpha x^2} e^{-\beta x}\,dx$$

$$G_0 = \sqrt{\frac{\pi}{\alpha}}\,e^{\beta^2/(4\alpha)} \qquad G_{2n} = (-1)^n\frac{\partial^n G_0}{\partial\alpha^n} \qquad G_{2n+1} = (-1)^{2n+1}\frac{\partial^{2n+1}G_0}{\partial\beta^{2n+1}}$$

In particular, for a Maxwellian distribution function:

$$f_{\mathrm{M}} = \left(\frac{1}{\sqrt{2\pi}v_t}\right)^3 \exp\left[-(v_x^2 + v_y^2 + v_z^2)/(2v_t^2)\right]$$

$$v_t^2 = \frac{T}{m} \qquad \langle v_x^{2n}\rangle = \int d^3v\,v_x^{2n} f_{\mathrm{M}} = v_t^{2n}(2n-1)!!$$

So that $\langle E\rangle = \frac{1}{2}m\langle v_x^2 + v_y^2 + v_z^2\rangle = \frac{3}{2}T$. I.e., the average energy per degree of freedom is $\frac{1}{2}T$.

"Normal" distribution function:

$$f(x|\bar{x},\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\bar{x}}{\sigma}\right)^2\right]$$

"Error" function:

$$\Phi(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}dt = \int_{-y\sqrt{2}\sigma}^{+y\sqrt{2}\sigma} dx\,f(x,0,\sigma)$$

$$\Phi(0) = 0 \qquad \Phi(\pm\infty) = \pm 1 \qquad \Phi(1\sigma/\sqrt{2}\sigma) = 0.68 \qquad \Phi(2\sigma/\sqrt{2}\sigma) = 0.95$$

2.2. Complex Analysis

$f(z)$ is *analytic* in some region if its derivative df/dz exists (i.e., is independent of the direction of dz in the complex plane). The terms *holomorphic*, *monogenic*, and *regular* are also used. More formally, f is holomorphic if f satisfies the Cauchy-Riemann equations (where u and v are real-valued functions):

$$f(z) = u(z) + iv(z) \qquad z = x + iy$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Equivalently, f is holomorphic if $d(f dz) = 0$ in modern differential geometry notation. If f is holomorphic, then it satisfies

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Cauchy's integral formula: For $z \in$ region D, and $f(z)$ holomorphic everywhere in D, then the n'th derivative of f is related to the following integral around the boundary of D (going counter-clock wise around the contour D):

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$$

The leads to the formula for contour integrals:

$$\oint_C f(\zeta)d\zeta = 2\pi i \times \text{(sum of the residues inside the contour C)}$$

If $f(z)$ has a pole of order n at $z = a$, then its residue is defined as

$$\text{residue} = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\left((z-a)^n f(z)\right)$$

Fourier Transforms:

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}F(\omega)d\omega$$

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}f(t)dt$$

Convolution theorem: $\int_{-\infty}^{\infty} e^{-i\omega t}F(\omega)G(\omega)d\omega = \int_{-\infty}^{\infty} g(t-t')f(t')dt'$

Fourier transform of a Gaussian is a Gaussian: $f(t) = e^{-at^2} \to F(\omega) = \frac{1}{\sqrt{2a}}e^{-\omega^2/(4a)}$

Common forms of Dirac delta function: $\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega t}$

$$\delta(t) = \lim_{L\to\infty}\frac{\sin Lt}{\pi t} \qquad \delta(t) = \lim_{\epsilon\to 0^+}\frac{\epsilon}{\pi(\epsilon^2+t^2)}$$

$$\lim_{\epsilon\to 0^+}\frac{1}{x-a\mp i\epsilon} = P.V.\frac{1}{x-a} \pm i\pi\delta(x-a)$$

Electron-ion collisions cause pitch angle-scattering only, giving rise to resistivity (electrons lose momentum to ions), and electron-electron collisions cause f_e to approach a Mawellian (preserving the electron energy, in the ion rest frame). Ion-ion collisions cause f_i to approach a Maxwellian (preserving the ion energy). T_i and T_e equilibrate only at the very slow ν_{ie} rate. $\nu \sim 1/v^3$ so energetic particles are less collisional.

5.3. Braginskii Fluid Equations

The summary of Braginskii in the NRL is supplemented here. Braginskii uses the Landau collision operator for Coulomb collisions between ionized particles (thus ignoring atomic processes, collisions with neutrals, external sources or sinks of particles or energy). (Note, the NRL reverses the definition of $\vec{\vec{\Pi}}$ and $\vec{\vec{P}}$ relative to Braginskii's original notation.)

$$n_\alpha = \int d^3v f_\alpha \qquad n_\alpha\vec{u}_\alpha = n_\alpha\langle\vec{v}\rangle_\alpha = \int d^3v f_\alpha\vec{v}$$

Pressure tensor $\vec{\vec{\Pi}}_\alpha = p_\alpha\vec{\vec{1}} + \vec{\vec{P}}_\alpha = n_\alpha m_\alpha\langle\delta\vec{v}\delta\vec{v}\rangle_\alpha = n_\alpha m_\alpha\langle(\vec{v}-\langle\vec{v}\rangle_\alpha)(\vec{v}-\langle\vec{v}\rangle_\alpha)\rangle_\alpha$

Heat flux $\vec{q}_\alpha = n_\alpha\frac{1}{2}m_\alpha\langle|\delta\vec{v}|^2\delta\vec{v}\rangle_\alpha$

Friction / Collisional drag rate $\vec{R}_\alpha = \int d^3v\, m_\alpha\delta\vec{v}\; C_\alpha$ & heating $Q = \int d^3v\frac{1}{2}m_\alpha|\delta\vec{v}|^2\; C_\alpha$

Defining $p = nT$ gives $\langle m|\delta\vec{v}|^2/2\rangle = (3/2)T$, i.e. $T/2$ of energy per degree of freedom (dimensions or modes among which energy can be shared). p is the isotropic part of the pressure tensor, so $\vec{\vec{P}}$ must be traceless. Braginskii used a Chapman-Enskog-like approach to calculate the closures in the collisional limit. The NRL has summaries of Braginskii for $\Omega_c\tau \gg 1$ or $\ll 1$, though Braginskii has more general expressions. The NRL expressions are for a hydrogen-electron plasma, while Braginskii gives expressions for a plasma with arbitrary ion charge Z_i and for multiple ion species, $n_e = \sum_i n_i Z_i$. To generalize the NRL formulas for arbitrary Z_i, the electron and ion collision times and various coefficients are modified in the following way:

$$\tau_e = \frac{3\sqrt{m_e}\,T_e^{3/2}}{4\sqrt{2\pi}n_i Z_i^2 e^4\Lambda} \qquad \tau_i = \frac{3\sqrt{m_i}\,T_i^{3/2}}{4\sqrt{\pi}n_i Z_i^4 e^4\Lambda}$$

Z_i dependence of various transport coefficients (Braginskii, Table 1)

Z_i	$\sigma_\parallel$	First term of $\vec{R}_T$ and $\vec{q}^e_u$	$\kappa^e_\parallel$	$\kappa^e_\perp$
1	1.96	0.71	3.16	4.66
2	2.27	0.9	4.9	4.0
3	2.50	1.0	6.1	3.7
4	2.63	1.1	6.9	3.6
∞	3.40 [10]	1.5	13.6 [10]	3.2

I.e., the equation for $\sigma_\parallel$ is $\sigma_\parallel = 1.96\sigma_\perp$ for $Z_i = 1$, and $\sigma_\parallel = 2.63\sigma_\perp$ for $Z_i = 4$. Spitzer's result for resistivity is identical to Braginskii's. Spitzer's result for the energy equilibration rate reduces to Braginskii's result for $m_\alpha/m_\beta \ll 1$.

integrating over all $\vec{\xi}$ and taking the limit $\Delta t \to 0$ gives the generic **Fokker-Planck equation:**

$$\left(\frac{\partial f}{\partial t}\right)_{\rm Coll} = C(f) = -\frac{\partial}{\partial v_i}\left[f(\vec{v},t)\frac{\langle \Delta v_i \rangle}{\Delta t}\right] + \frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}\left[f(\vec{v},t)\frac{\langle \Delta v_i \Delta v_j \rangle}{2\Delta t}\right]$$

$$= -\frac{\partial}{\partial v_i}\left[f(\vec{v},t)\dot{\vec{v}}_i\right] + \frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}\left[f(\vec{v},t)D_{ij}\right] = -\frac{\partial J_i}{\partial v_i}$$

Where $\langle \Delta v_i \rangle = \int d^3\xi P_{\Delta t}(\vec{v},\vec{\xi})\xi_i$, and similarly for $\langle \Delta v_i \Delta v_j \rangle$. For finite size time steps, the diffusion tensor should be given by

$$D_{ij} = \frac{\langle (\Delta v_i - \langle \Delta v_i \rangle)(\Delta v_j - \langle \Delta v_j \rangle) \rangle}{(2\Delta t)}$$

(assuming I did the multi-dimensional generalization of this right??). $\vec{J} = \sum_\beta \vec{J}^{\alpha\beta}$ is given in the NRL formulary and is the flux in velocity space of species α due to collisions with species β. Because of the analogy with electrostatics noted by Rosenbluth, the Rosenbluth potentials in the NRL can also be written as

$$\nabla_v^2 H = -(1 + \frac{m_\alpha}{m_\beta})4\pi f_\beta \qquad \nabla_v^2 G = \frac{2}{1 + (m_\alpha/m_\beta)}H$$

If f_β is Maxwellian, then the collision operator simplifies to the form at the top of NRL p. 36 (this ignores the back-reaction of f_β due to collisions with the non-Maxwellian f_α). A useful I.D.:

$$\frac{\partial}{\partial \vec{v}} \cdot \left[\frac{1}{2v^3}(v^2\vec{\vec{1}} - \vec{v}\vec{v})\right] = -\frac{\vec{v}}{v^3}$$

Coulomb logarithm: The NRL formulary gives a recipe for a general Coulomb logarithm $\ln \Lambda_{\alpha\beta} = \ln(r_{\max}/r_{\min})$ for a test particle α colliding with field particles β with relative velocity $\bar{u} = |\mathbf{v}_\alpha - \mathbf{v}_\beta|$. Note that the symmetry $\ln \lambda_{\alpha\beta} = \ln \lambda_{\beta\alpha}$ is important in proving various conservation properties of the collision operator, and the $\ln \lambda_{\alpha\beta}$ factor should be kept inside the $v' = v_\beta$ integral in the Landau form of the collision operator on p.35 of the 2002 NRL formulary, if the dependence of $\ln \lambda_{\alpha\beta}$ on the relative velocity is retained. (Since the collision operator is only accurate to $\sim 1/\ln \Lambda$, often this can be neglected, but the symmetry $\ln \lambda_{\alpha\beta} = \ln \lambda_{\beta\alpha}$ should still be preserved.) The NRL's recipe says that the maximum impact parameter is cut off by Debye shielding, $r_{\max} = (4\pi \sum_\gamma n_\gamma e_\gamma^2/kT_\gamma)^{-1/2}$, "where the summation extends over all species γ for which $\bar{u}^2 < v_{T_\gamma}^2$." (where $\bar{u}$ is the relative velocity). An obvious question is what happens for suprathermal particles that are even faster than thermal electrons, do they not experience Debye shielding at all? The answer is that they are still shielded, but only on a longer spatial scale on which their transit frequency is of order the plasma frequency. Thus a possible generalization of this recipe is to replace $r_{\max}^{-2} = \sum_\gamma \omega_{p\gamma}^2/v_{t\gamma}^2 \to \sum_\gamma (\omega_{p\gamma}^2 + \Omega_{c\gamma})^2/(v_{t\gamma}^2 + \bar{u}^2)$, keeping a sum over all species. The Coulomb logarithm is usually derived for the standard weakly-coupled plasma regime where is it very large. A more general approximation is to replace $\ln \Lambda_{\alpha\beta} = \ln(r_{\max}/r_{\min}) \to \ln((1 + r_{\max}^2/r_{\min}^2)^{1/2})$. This will give approximately the correct collisional relaxation rates, but in this regime small angle collisions no longer dominate so the diffusive approximation is no longer rigorous.

Qualitative collision rates:

$$\begin{array}{ccccccc} \nu_{ei} & : & \nu_{ee} & : & \nu_{ii} & : & \nu_{ie} \\ Z_{eff} & : & 1 & : & \sqrt{m_e/m_i} & : & m_e/m_i \end{array}$$

2.3. Differential Equations

ODE's, WKB methods, PDE's. Green's functions.
Dirichlet boundary conditions take the form $f(x = x_0) = C$.
Neumann boundary conditions take the form $df/dx|_{x=x_0} = C$.
Three main classes of partial differential equations:

- Hyperbolic (wave-like with characteristics): $u_t = u_x$, or $u_{tt} = u_{xx}$.
- Parabolic (diffusion-like): $u_t = u_{xx}$.
- Elliptic (Poisson-like): $u_{xx} + u_{yy} = 0$

Generalized Langevin equation, Green's function solution.
Special Functions.

2.4. Linear Algebra and Matrices

2.5. Numerical Methods

ODE's: First order explicit and implicit, Second order Runge-Kutta or Predictor-Corrector Schemes, Adams-Bashforth, Leapfrog, Backward differentiation formulas (BDF) for stiff equations. Numerical stability, phase errors of various schemes
PDE's: Diffusion equations and implicit methods. Convection equations and upwind differencing and limiter methods. Tri-diagonal matrix solver

Finite Fourier Transforms, Convolution equations. Dealiased pseudospectral methods by the (2/3) rule.

Modern higher-order upwind algorithms for hyperbolic conservation laws: Total Variation Diminishing flux-limited algorithms, WENO.

2.6. The Greek Alphabet

Alpha	A	α	Nu	N	ν
Beta	B	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	O	o
Delta	Δ	δ	Pi	Π	π, ϖ
Epsilon	E	ϵ, ε	Rho	P	ρ, ϱ
Zeta	Z	ζ	Sigma	Σ	σ, ς
Eta	H	η	Tau	T	τ
Theta	Θ	θ, ϑ	Upsilon	Υ	υ
Iota	I	ι	Phi	Φ	ϕ, φ
Kappa	K	κ	Chi	X	χ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	M	μ	Omega	Ω	ω

Math and LaTeX dictionary

$\aleph$	`\aleph`	aleph
$\Re$	`\Re`	real part
$\Im$	`\Im`	imaginary part
∞	`\infty`	infinity
$\forall$	`\forall`	for all
$\exists$	`\exists`	there exists
$\mathbb{R}$	`\mathbb R`	the set of all real numbers
$\mathbb{C}, \mathbb{Z}, \mathbb{Q}$		the set of all complex numbers, integers, or rationals
$\{\ldots\}$		lists the elements of a set
$\in$	`\in`	element of
$\subset$	`\subset`	subset
$\cap$	`\cap`	intersection
$\cup$	`\cup`	union
$(a, b]$		interval with open and closed ends $\{x : a < x \leq b\}$
$\iff$ or iff		if and only if

Examples of mathematical notation: Let $f(\vec{x}, t)$ be a function that maps an m-dimensional vector $\vec{x}$ and a real valued t to a real number. The notation for this is, $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$.

A trick for replacing these $2N$ ODE's with a single PDE is to use the Klimontovich-Dupree equation for $f_*(\vec{x}, \vec{v}, t) = \sum_i \delta(\vec{x} - \vec{x}_i(t))\delta(\vec{v} - \vec{v}_i(t))$,

$$\frac{\partial f_*}{\partial t} + \vec{v} \cdot \frac{\partial f_*}{\partial \vec{x}} + \vec{a} \cdot \frac{\partial f_*}{\partial \vec{v}} = 0$$

The **Vlasov equation** for f is identical to this equation for f_*, except that f is considered to be a smooth density of particles in phase-space (and so has been course-grained, averaging over a finite volume, or f is considered as a statistical probability function from an ensemble average). This smooth f (which produces a smooth electric field) thus ignores the effects of collisions between discrete particles (where the electric field blows up if any two particular particles get too close). Collisions must be reintroduced via a collision operator on the right-hand side (or will arise from next order corrections in the coarse-graining/averaging procedure as in the BBGKY hierarchy), leading to the **Boltzmann equation**:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{a} \cdot \frac{\partial f}{\partial \vec{v}} = C(f)$$

Another approach: Multidimensional Conservation Laws.

Let $f(x_1, x_2, \ldots, x_N, t)$ be a distribution for an N-dimensional phase space, where the equations of motion are $dx_i/dt = \dot{x}_i = u_i$. Then particle conservation can be expressed as:

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}(\dot{x}_i f) = -\sum_i \frac{\partial}{\partial x_i}(u_i f) = -\vec{\nabla} \cdot (\vec{u} f)$$

Breaking up the phase-space in to the canonical positions $\vec{q} = (x_1, x_2, \ldots, x_{N/2})$ and the canonical momenta $\vec{p} = (x_{N/2+1}, \ldots, x_N)$, then the phase-space conservation law for $f(\vec{p}, \vec{q})$ can be rewritten as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{q}} \cdot \left(\dot{\vec{q}} f\right) + \frac{\partial}{\partial \vec{p}} \cdot \left(\dot{\vec{p}} f\right) = 0.$$

Using the Hamiltonian equations of motion one can then show Liouville's theorem

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \dot{\vec{q}} \cdot \frac{\partial f}{\partial \vec{q}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} = 0,$$

i.e., f is constant along trajectories in phase space (conservation of phase-space).

Equilibrium solutions (if f a function only of constants of the motion, Boltzmann thermodynamic equilibrium...).

2-stream instability, Landau damping.

5.2. Fokker-Planck Collision Operator and Coulomb Scattering

General expression for probabilistic transitions. Let $f(\vec{v}, t)$ be the density of particles (or the probability distribution for a single particle) at velocity $\vec{v}$ at time t. If $P_{\Delta t}(\vec{v}, \vec{\xi})$ is the probability of a particle initially at $\vec{v}$ taking a step to $\vec{v} + \vec{\xi}$, then

$$f(\vec{v}, t) = \int d^3\xi f(\vec{v} - \vec{\xi}, t - \Delta t) P_{\Delta t}(\vec{v} - \vec{\xi}, \vec{\xi})$$

This is also known as a Markov process. Doing a Taylor-series expansion for small ξ

$$f(\vec{v} - \vec{\xi}, t - \Delta t) P_{\Delta t}(\vec{v} - \vec{\xi}, \vec{\xi}) \approx f(\vec{v}, t - \Delta t) P_{\Delta t}(\vec{v}, \vec{\xi}) + \xi_i \frac{\partial}{\partial v_i} f(\vec{v}, t - \Delta t) P_{\Delta t}(\vec{v}, \vec{\xi})$$
$$+ \frac{1}{2} \xi_i \xi_j \frac{\partial}{\partial v_i} \frac{\partial}{\partial v_j} f(\vec{v}, t - \Delta t) P_{\Delta t}(\vec{v}, \vec{\xi})$$

Chapter 5

Plasma Physics

Fundamental phenomena: electron plasma oscillations, Debye shielding, gyroradius, gyrofrequency, collisions, plasma skin depth.

Debye shielding from Boltzmann response in thermodynamic equilibrium: $f \propto \exp(-H/T) \propto \exp(-(mv^2/2 + q\Phi)/T \to n \propto \exp(-q\Phi/T)$

Plasma Parameter $\Lambda = n\lambda_D^3$ = # of particles in a Debye sphere. $\Lambda \gg 1$ defines the usual plasma state. Nearest neighbor interactions weak: (potential energy of nearest neighbors)/(kinetic energy) $\sim 1/\Lambda^{2/3}$. Collective interactions strong (quasineutrality, Debye-shielding length is short, 2-stream instability, frozen-in field lines, Alfvén and other plasma waves).

Fundamental length scales (evaluated for $\Lambda \sim 10^6$):

90° impact parameter	:	average interparticle spacing	:	Debye shielding length	:	mean free path
b	:	$n^{-1/3}$	:	λ_D	:	λ_{mfp}
Λ^{-1}	:	$\Lambda^{-1/3}$	:	1	:	$\Lambda/\log\Lambda$
10^{-6}	:	10^{-2}	:	1	:	10^5

b is the "distance of closest-approach" for a single $90°$ collision (though it turns out that the net scattering rate is enhanced by a factor of $\log\Lambda$ due to many small-angle scatters.) $\lambda_{mfp} \sim v/\nu$ is the mean free path between collisions.

Time scales: Collision frequency is weak: $\nu/\omega_{pe} \sim log\Lambda/\Lambda$.

?? Guiding center drift equations (Lagrangian formulation).
Laser-plasma interactions. Figure-8 orbits.

5.1. Fundamental Kinetic Theory

Classical (non-quantum) non-relativistic Lorentz equation of Motion for the i'th particle:

$$\frac{d\vec{x}_i}{dt} = \vec{v}_i$$

$$m_i\frac{d\vec{v}_i}{dt} = \vec{F}_i = m_i\vec{a}_i = e_i\left(\vec{E}(\vec{x}_i) + \frac{\vec{v}_i \times \vec{B}(\vec{x}_i)}{c}\right)$$

Chapter 3

Classical Mechanics

Classical (non-quantum, non-relativistic) Lorentz equation of motion for a particle in an electric and magnetic field:

$$\frac{d\vec{x}}{dt} = \vec{v}$$

$$m\frac{d\vec{v}}{dt} = \vec{F} = m\vec{a} = e\left(\vec{E}(\vec{x}) + \frac{\vec{v} \times \vec{B}(\vec{x})}{c}\right)$$

Lagrangian formulation for generalized coordinates q_i:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

$$L(q_i, \dot{q}_i) = \frac{1}{2}mv^2 + \frac{e}{c}\vec{v}\cdot\vec{A} - e\phi$$

where $\dot{q}_i = dq_i/dt$. The Hamiltonian formulation uses the generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

To obtain the Hamiltonian

$$H(p_i, q_i) = -L + \sum_i p_i\dot{q}_i$$

$$= \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi$$

And the Hamiltonian equations of motion are:

$$\dot{\vec{q}} = \frac{\partial H}{\partial \vec{p}}, \qquad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}$$

The meaning of all this?

$$K = \text{KineticEnergy}$$

$$U = \text{PotentialEnergy}$$

$$L = K - U, \qquad H = K + U$$

Note that $L = L(\vec{q}, \dot{\vec{q}})$ while $H = H(\vec{q}, \vec{p})$, so that $\partial/\partial q_i$ in the two different approaches (Lagrangian and Hamiltonian) holds different independent variables fixed because $\vec{p} \neq \dot{\vec{q}}$.

The time evolution of any function defined on phase space (and time) $f(\vec{q}, \vec{p}, t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \equiv \frac{\partial f}{\partial t} + \{f, H\}$$

which serves to define the Poisson bracket $\{f, H\}$.

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Chapter 4

Electricity & Magnetism

"To convert any expression from SI to cgs units, make the replacements, $\mathbf{B} \to \mathbf{B}/c$, $\epsilon_0 \to 1/(4\pi)$, $\mu_0 \to 4\pi/c^2$. The inverse transformation is more complicated, and is described in Jackson (1975)"[9] and in the NRL formulary.