Tests of Limiters for Discontinuous Galerkin Advection Algorithms

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Problem

- Continuum kinetic plasma simulations need to maintain positivity and monotonicity of the distribution function for a physical solution
- This requires the introduction of *limiters* to the simulations
- Here, we investigate the utility of various limiters for the Discontinuous Galerkin (DG) method, a highly parallelizable and efficient technique with many recent developments

Plasma edge: A tricky place

- Plasma edge presents simulation challenges
  - Large density/amplitude variations, large relative banana width, wide range of collisionalities
    - Stick with full-F simulations
    - Need good limiters to ensure positivity, many algorithms produce oscillations at large gradients (Gibb's phenomenon)
  - Small charge imbalances lead to large fields
    - Need to ensure particle conservation exactly
  - Algorithm also needs to minimize artificial dissipation (some is OK as a subgrid model)

Test problem: advection

$$\frac{\partial}{\partial t} f(x,t) = -\frac{\partial}{\partial x}(v f)$$

- Paradigm problem on algorithm subtleties - thousands of papers written on this equation and extensions across many application domains (climate, CFD, architecture, astrophysics, nanotech/MEMS and more)
- Surprisingly tricky to get a robust, efficient, high order accurate solution with desired conservation and monotonicity/positivity properties
- Exact solution for constant v: translation

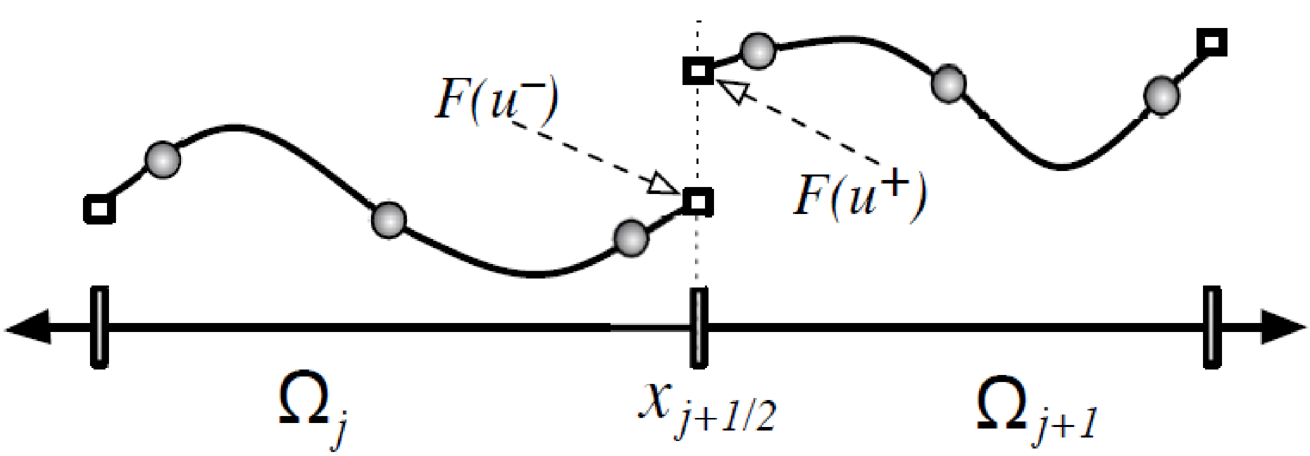
$$f(x,t) = f_0(x - vt)$$

DG Algorithm

- Multiply the equation by a test function and integrate over one cell

$$\int_{\Omega_j} \frac{\partial f}{\partial t} \phi_h dx = \int_{\Omega_j} (vf) \frac{\partial \phi_h}{\partial x} dx - \left[\hat{F}\left(f^-, f^+\right)_x \phi_h(x) \right]_{x^+_{j-\frac{1}{2}}}^{x^-_{j+\frac{1}{2}}}$$

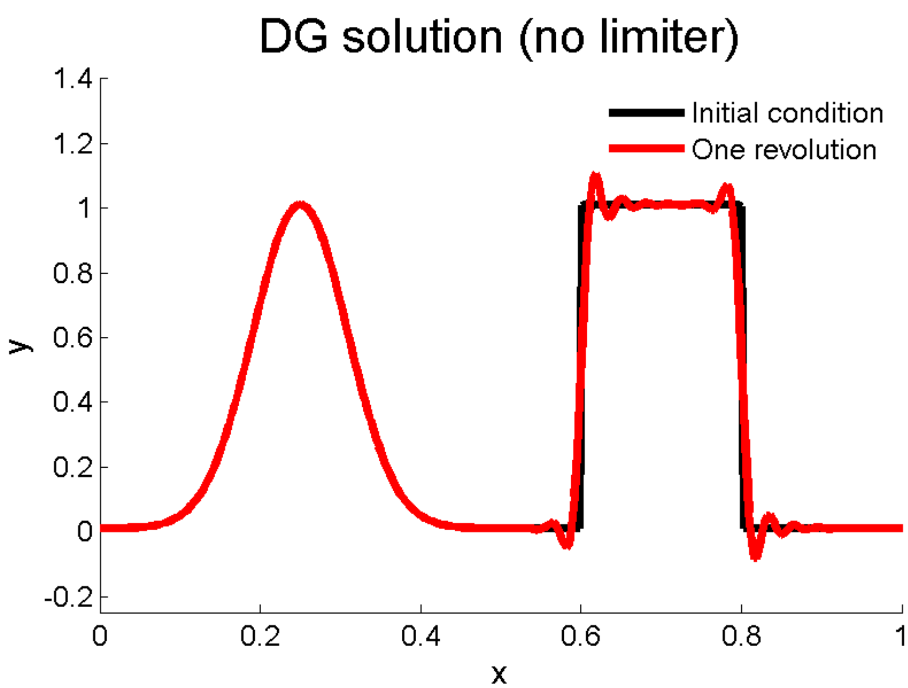
- $\hat{F}\left(f^-, f^+\right)$ is a numerical flux
- Expand f as polynomial: $f_j(x,t) = \sum_{k=0}^{N} f_j^k(t) P^k(x)$
- Pick suitable test function (usually same basis as for f), and numerical flux
- Gives equations for polynomial weight time evolution



(From Nair,Levy, Lauritzen, 2011)

Typical DG solution

- Advection equation simulated for one revolution, Lagrange basis for polynomials, 3rd order polynomials in each cell (5th order accuracy due to DG superconvergence)



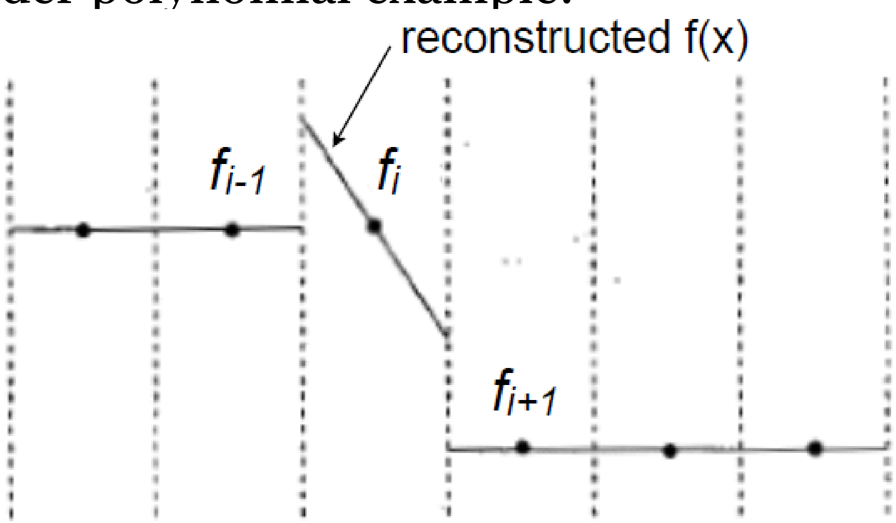
- Negative values, oscillations around the discontinuity typical of high order methods without limiters

- Otherwise good in smooth regions, low dissipation, conservative (all DG)

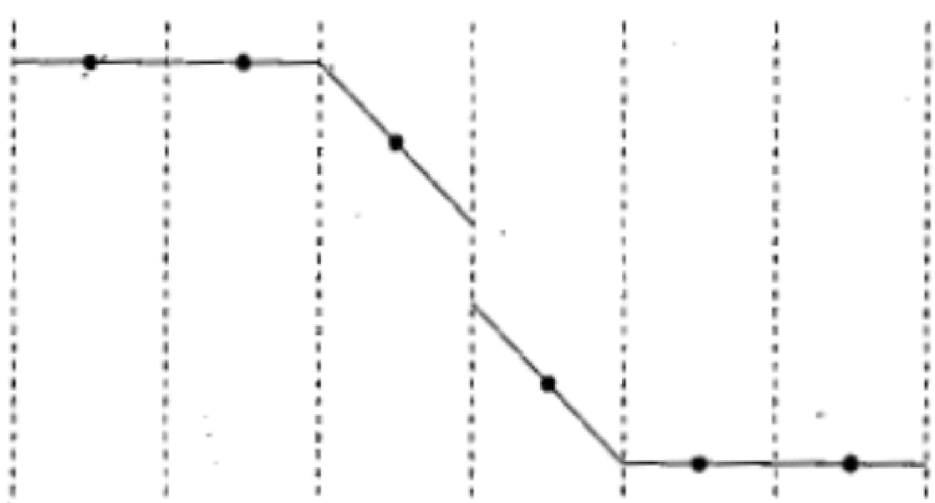
Limiting

- Need to determine restrictions on local polynomial coefficients to keep solution nonoscillatory

1st order polynomial example:



Want:



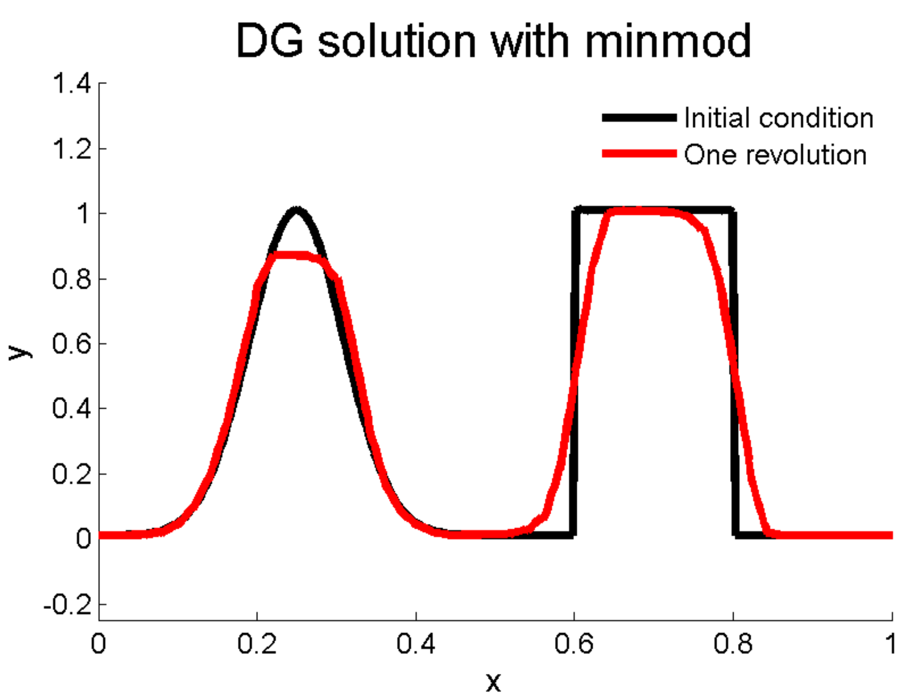
(From R J Leveque, 2002)

Minmod

- Basic limiter for first order reconstruction & building block for advanced limiters

$$\hat{f}_j^1 = minmod\left(f_j^1, \frac{f_{j+1}^0 - f_j^0}{(\Delta x/2)}, \frac{f_j^0 - f_{j-1}^0}{(\Delta x/2)}\right)$$

$$\text{minmod}(a,b,c) = \begin{cases} s \min(|a|,|b|,|c|) & \text{if } s = \text{sign}(a) = \text{sign}(b) = \text{sign}(c) \\ 0 & \text{otherwise .} \end{cases}$$



- Clips extrema, somewhat diffusive

Minmod

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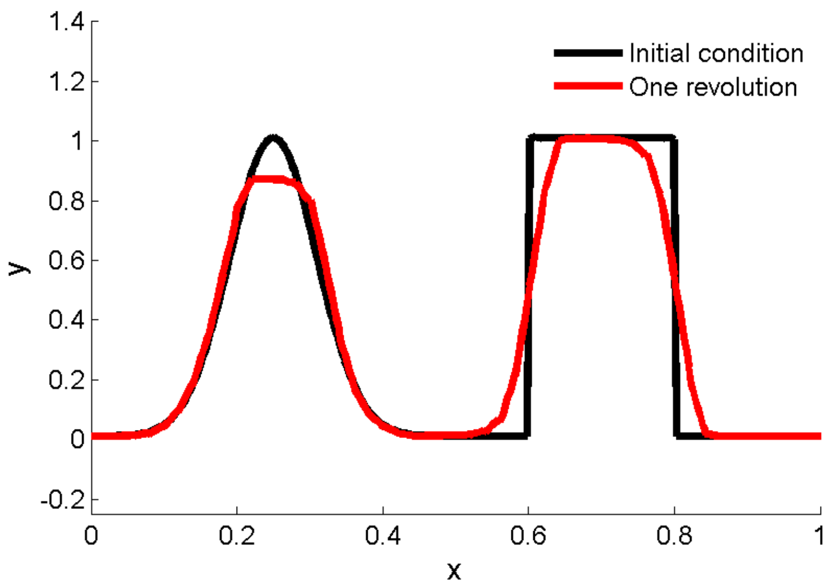
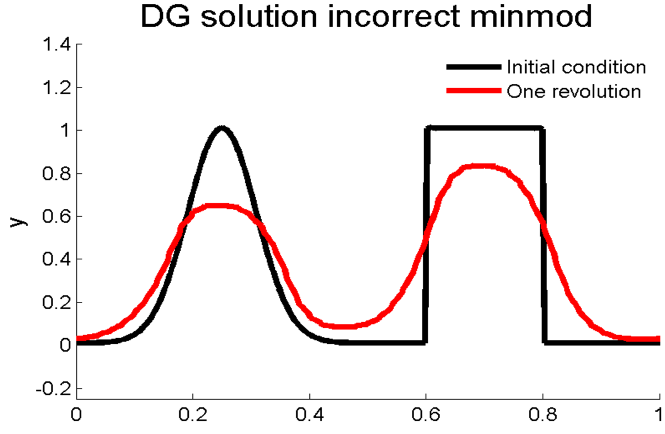
DG solution with minmod



- Implemented incorrectly for DG in some literature (e.g. Nair, Levy, Lauritzen)

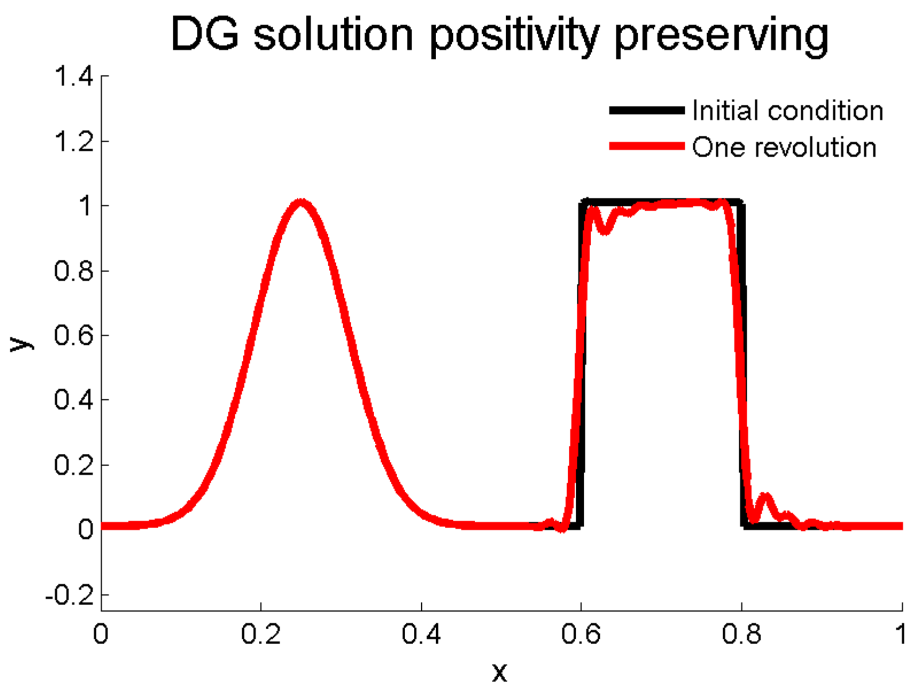
DG solution incorrect minmod



Positivity preservation

- Possible to design limiters that enforce the less restrictive condition of positivity, rather than monotonicity

  E.g. Zhang & Shu maximum principle limiter

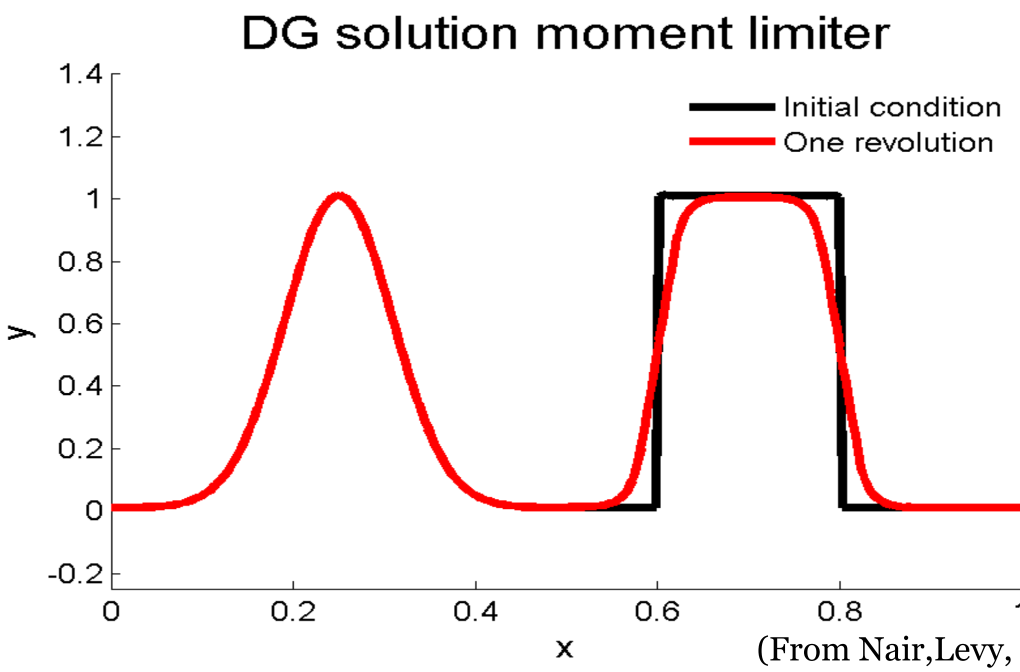


- Idea is to scale polynomials within a cell so that interpolated values used in the DG scheme never exceed [0,Max]

- Preserves positivity and smooth extrema is *not clipped*, but also allows oscillations to be generated in the solution

High order limiters

- DG moment limiter uses a recursive minmod approach – use minmod on highest polynomial moment, recurse to next highest moment if previous one limited by minmod



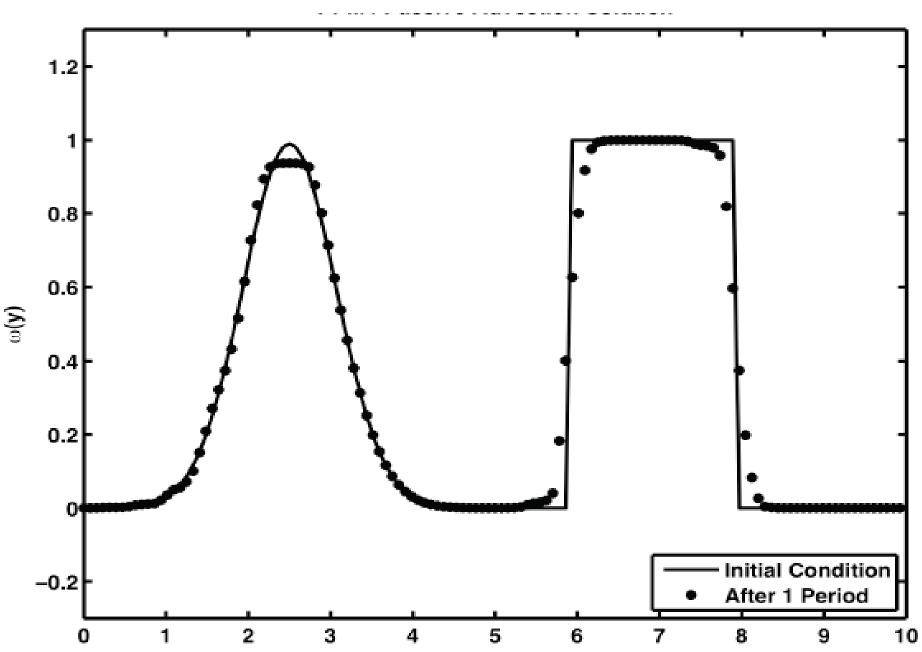
(From Nair,Levy, Lauritzen, 2011)

- Appears to maintain high order accuracy near smooth extrema and no oscillations generated

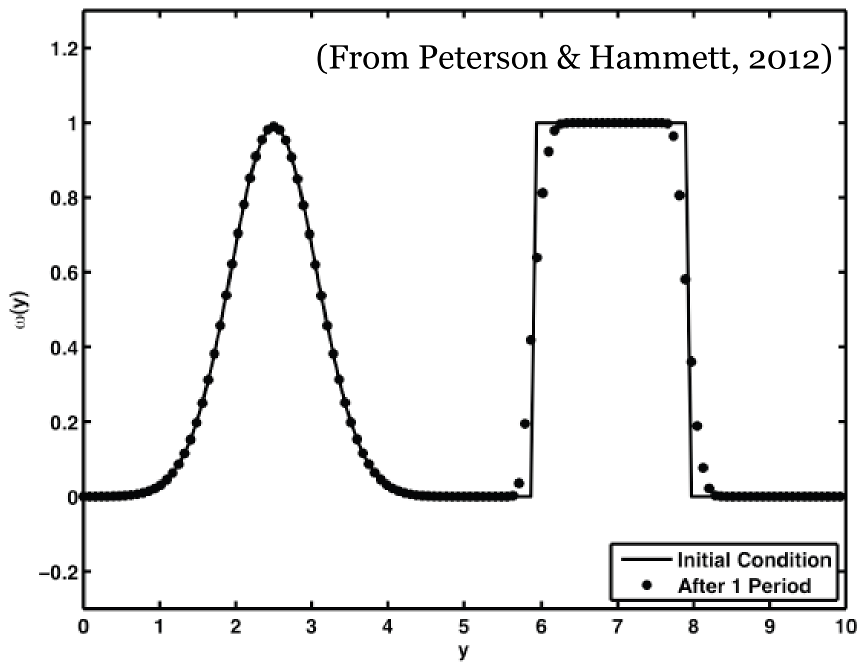
- Expensive to extend this limiter to high dimensions (Krivodonova, 2007)

Non-clipping limiters

Standard PPM4

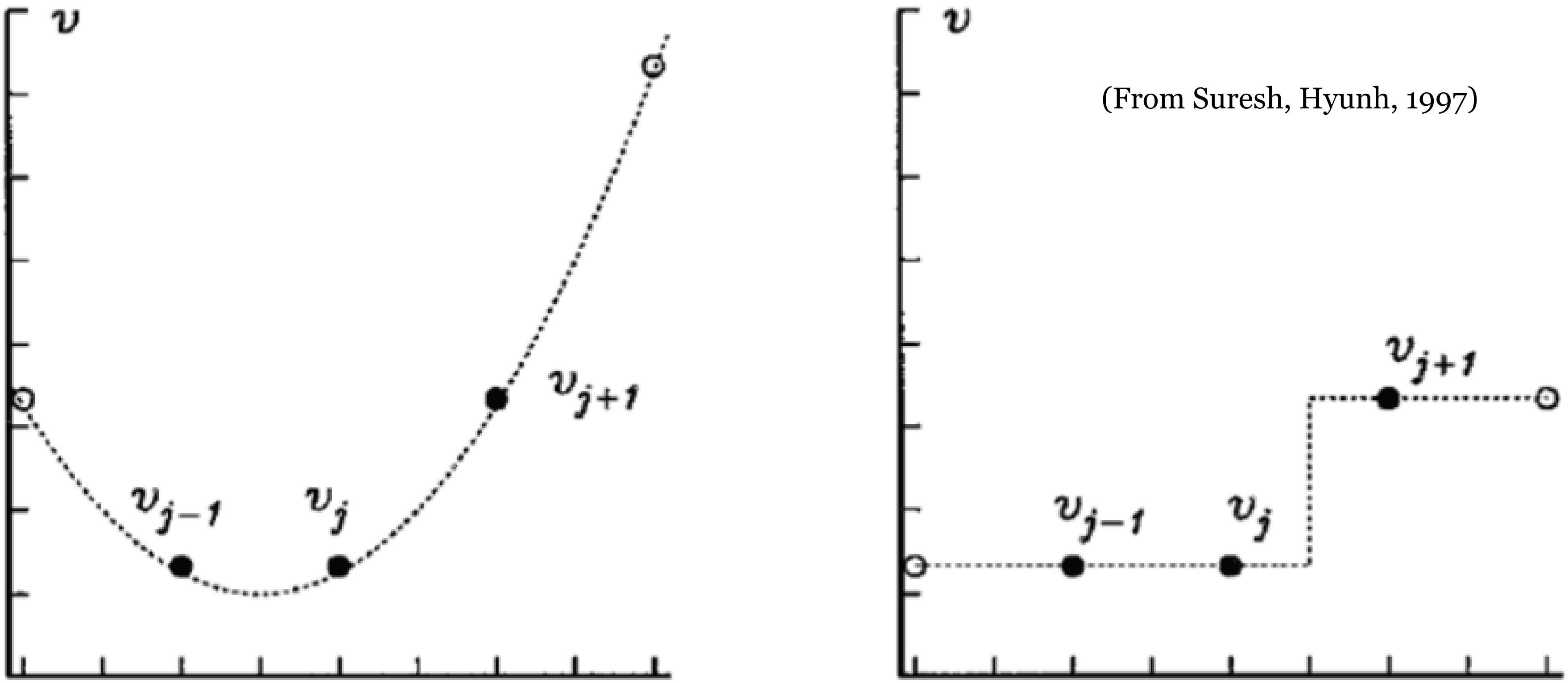


New XPPM



- Recent advances in non-clipping limiters for finite volume schemes e.g. XPPM, Collela & Sekora 2008
- Detect discontinuities (allowed for hyperbolic problems), revert to low order in non smooth regions, introduce minimum diffusion to preserve monotonicity

Non-clipping limiters



- In finite volume methods, information from three adjacent cells is not sufficient to distinguish smooth extremum from a discontinuity.

Future work

- A number of recent papers work on extending moment style limiters to unstructured meshes and high dimensions without compromising computation efficiency
  - Investigate these approaches for their usefulness on edge plasma related test problems

- Can the recent developments in non-clipping finite volume limiters be converted into techniques for high-order, efficient DG limiters?

Acknowledgments

This work was supported by DOE CSGF under grant number DE-FG02-97ER235308 and the Princeton Plasma Physics Laboratory.

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