Discussion of Gyrokinetic Momentum Ordering Issues

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PPPL, Sept. 9, 2014

- Simple illustrations of gyrokinetic momentum ordering issue, 2 ways:
  - Directly from GK Poisson equation
  - Scott's GK momentum conservation expression
- Why isn't the momentum flux from H_1 much larger than from H_3?

  A: Several effects make momentum flux from H_1 much smaller than it would seem at first.
- Related Issues

Summary of Gyrokinetic Momentum Issues

In a series of papers, Parra & Catto pointed out challenges of toroidal momentum transport in a standard regime (gyroBohm ordering, axisymmetric, up-down symmetric, slow toroidal flow of order the diamagnetic velocity $v_* \sim \varepsilon\, v_t$, where $\varepsilon = \rho/L = \rho_*$). In particular, they showed that the standard Lagrangian gyrokinetic approach would require the third order Hamiltonian, $H_3 \sim \varepsilon^3\, T$, to calculate momentum transport accurately in this slow flow ordering. They advocate supplementing with a separate equation for directly solving for toroidal momentum evolution, then need only H_2. (See Krommes & Hammett, PPPL report 4945, 2013. http://bp.pppl.gov/pub_report//2014/PPPL-4945-abs.html)

Our report gives some straightforward ordering arguments (originally due to P&C) demonstrating their point. One should understand the implications in a balanced way. Slow flows in this regime are so slow that usually they would not significantly affect the turbulence, though they might still be important for MHD stability. Flows are usually more important in regimes that break some of these assumptions (like non-gyroBohm scaling near the edge or near transport barriers), but then still need a second order Hamiltonian. P&C deserve credit for pointing out these subtle issues and helping people realize the importance of even H_2 for a complete treatment in other regimes. (Many codes at present neglect H_2.)

Polarization Density Plays Key Role in Gyrokinetic Poisson / Quasineutrality Eq.

At long wavelengths (neglecting higher-order FLR corrections to polarization density):

$$-\nabla_\perp \cdot \left(\frac{\sum_s 4\pi n_s m_s c^2}{B^2} \nabla_\perp \phi \right) = \sum_s 4\pi e_s \int m_s B_{||}^* dv_{||} d\mu d\theta \, \bar{f}(\vec{x} - \vec{\rho}(\mu,\theta), \mu, v_{||})$$

-(Polarization charge density) = guiding center charge density (including gyroaveraging)

Looks like a Poisson equation, but actually is a statement of quasineutrality:

$$0 = \sigma = \sigma_{\rm gc} + \sigma_{\rm pol}$$

Because the polarization density depends on the potential, this is how the potential gets determined. The polarization density can be shown to be related to the higher-order polarization drift:

$$\frac{\partial \sigma_{\rm pol}}{\partial t} = -\nabla \cdot \vec{j}_{\rm pol} \propto \nabla \cdot \frac{\partial \vec{E}}{\partial t}$$

Illustration of Momentum Ordering Issue Directly from GK Poisson Eq.

9/9/2014

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Various versions of GK Poisson Eq.,

at long wavelength all reduce to:

$$-\nabla_\perp \cdot \left(n_i q_i \rho_s^2 \frac{e}{T_e} \nabla_\perp \phi \right) = \sum_s \bar{n}_s q_s \int d\underline{v}\, F_s$$

– ion polarization charge density = guiding-center charge density

[Solve this & the GK Eq. for F_s for a transport time scale, to determine the resulting $\langle E_r \rangle$ (& $\langle u_\varphi \rangle$)

Take ∂_t + use $\partial_t F_s = -\nabla \cdot (\underline{V}_{ds} F_s) + \frac{\partial}{\partial v_\parallel}(\ldots)$

$$-\partial_t \nabla_\perp \cdot \left(n_i q_i \rho_s^2 \frac{e}{T_e} \nabla_\perp \phi \right) = -\nabla \cdot \left(\sum_s \bar{n}_s q_s \int d\underline{v}\, F_s \underline{V}_{ds} \right)$$

Integrate from magnetic axis out to some flux surface:

$$-\partial_t \left\langle n_i q_i \rho_s^2 \frac{e}{T_e} \frac{\partial \phi}{\partial r} \right\rangle = - \left\langle \sum_s \bar{n}_s q_s \int d\underline{v}\, F_s \underline{V}_{ds} \cdot \hat{r} \right\rangle$$

flux-surface avg.

Integrate in time for a transport timescale τ_E to find resulting $\langle E_r \rangle \propto \langle \frac{\partial \phi}{\partial r} \rangle$

$$n_{i0} q_i \rho_s^2 \frac{e}{T_e} \frac{\partial \langle \phi \rangle}{\partial r} \sim \tau_E \left\langle \sum_s n_s q_s \overline{\underline{V}_{ds}} \cdot \hat{r} \right\rangle$$

Toroidal rotation in slow-flow ordering ($u_\varphi \sim \varepsilon c_s$)

corresponds to $u_\varphi \sim \varepsilon c_s \sim \frac{cE}{B} \sim \frac{c}{B} \frac{\partial \phi}{\partial r}$

(not making any large aspect ratio expansions here, for generality, $L_p \sim L_n \sim a \sim R \sim L$).

$$LHS \sim n_i q_i \rho_s^2 \underbrace{\frac{e}{T_e} \frac{B}{c}}_{\frac{1}{\rho_s c_s}} \varepsilon c_s \qquad \frac{cT}{eB} \sim \frac{mc}{eB} \frac{T}{m} \sim \frac{1}{\Omega_c} c_s^2 \sim c_s \rho_s$$

$$\boxed{LHS \sim n_i e \rho_s \varepsilon}$$

For RHS, use $\tau_E \sim \frac{L^2}{D_{GB}} \sim \frac{L^2}{\frac{c_s}{L} \rho_s^2} \sim \frac{L}{c_s} \frac{1}{\varepsilon^2}$

$$RHS \sim \frac{L}{c_s} \frac{1}{\varepsilon^2} n_i e V_{dr}$$

So $V_{dr} \sim \frac{\varepsilon \cancel{n_i} \cancel{e} \rho_s}{L \cancel{n_i} \cancel{e}} c_s \varepsilon^2 \sim \varepsilon^4 c_s$

(This is essentially Parra Ph.D. 2009, Eq. 1.3)

Radial drifts have the form: (3)

$$\bar{V}_{dr} \sim \frac{c}{B} \hat{b} \times \nabla \left(\frac{H}{e} \right)$$

$$\sim \frac{cT}{eB} \, k_\perp \frac{H}{T}$$

$$\sim c_s \underbrace{(k_\perp \rho_s)}_{\sim \mathcal{O}(1)} \frac{H}{T}$$

Comparing w/ last page $\Rightarrow$ accuracy needed is $H \sim \varepsilon^4 T$.

However, the radial flux from the 4^{th} order drift vanishes (or nearly so) after flux-surface averaging:

$$\propto \left\langle n \frac{\partial H_4}{\partial y} \right\rangle \approx \left\langle n_0 \frac{\partial H_4}{\partial y} \right\rangle = n_0 \left\langle \frac{\partial H_4}{\partial y} \right\rangle = 0$$

binormal direction

But 3^{rd} order drift still needed:

$$\propto \left\langle \tilde{n}_1 \frac{\partial H_3}{\partial y} \right\rangle \neq 0 \quad \text{in general.}$$

[This is from Sec. 5.1.2 of Krommes & Hammett '13.]

Illustrating momentum ordering issues by starting from Scott's gyrokinetic momentum conservation law

This is a brief version of Sec. 5.3-5.4 of Krommes & Hammett 2013, slightly rewritten to be closer to Scott's notation. Start from the gyrokinetic momentum conservation law given in Eq.(80) of Scott10 (B. Scott and J. Smirnov, Phys. Plasmas **17**, 112302 (2010)):

$$\frac{\partial}{\partial t}\left(\left\langle\int_{\mathcal{P}} f p_z b_\varphi\right\rangle - \frac{1}{c}\left\langle\vec{P}\cdot\nabla A_\varphi\right\rangle\right) = -\frac{\partial}{\partial V}\left\langle\int_{\mathcal{P}} f p_z b_\varphi \dot{V}\right\rangle - \left\langle\int_{\mathcal{P}} f\frac{\partial H}{\partial\varphi}\right\rangle$$

(This momentum conservation law has also been derived in a somewhat different way in a recent paper by A. Brizard.) In this equation, angle brackets $\langle\ldots\rangle$ represents flux-surface averaging, and $\int_{\mathcal{P}}$ represents integration (without indicating the differentials) over the gyrokinetic velocity space coordinates. (This is somewhat different notation than used in the Scott10 paper, which combines both of these operations into the angle brackets.) The first term on the LHS involves the contribution to the toroidal canonical angular momentum density from the parallel momentum (note that in Scott's notation, $p_z \approx mv_\parallel$ and b_φ is the covariant toroidal component of the magnetic field unit vector, $b_\varphi \equiv \hat{b}\cdot\partial\vec{x}/\partial\varphi \approx R$). The second term on the LHS, involving the polarization vector $\vec{P}$, includes contributions to the toroidal angular momentum from the $\vec{E}\times\vec{B}$ drift and from diamagnetic effects. (Note that because of the way the polarization vector is defined in Lagrangian field-theory approaches to gyrokinetics, it does not necessarily vanish when the electric field is zero, as it also contains other contributions from FLR/diamagnetic effects.) The first term on the RHS represents the effects of the radial flux of the parallel momentum.

Although the last term on the RHS does not look like it is in conservative form, by using some non-trivial identities, Scott demonstrated that it in fact can be written in conservative form as the divergence of a flux. The general form is somewhat complicated, but an example of the form is given in Scott2010's Eq. 83:

$$\left\langle\int_{\mathcal{P}} f\frac{\partial H}{\partial\varphi}\right\rangle = \frac{\partial}{\partial V}\left\langle\int_{\mathcal{P}} f\frac{\partial\phi}{\partial\varphi}\nabla V\cdot\frac{\partial H}{\partial\nabla\phi}\right\rangle + \ldots$$

where ϕ is the electrostatic potential and the volume V is used as a radial flux-surface coordinate.

Now consider the order of magnitude of the first and last term in this gyrokinetic momentum conservation law. For the gyro-Bohm transport time scale in the low flow ordering, the first term is of order

$$LHS \sim \frac{\partial}{\partial t} nmu_\varphi R \sim \frac{c_s}{L}\epsilon^2 nm\epsilon c_s L \sim nT\epsilon^3$$

In the last term, we will use $\partial\phi/\partial\varphi \sim k_\perp L\phi$ and $\nabla V \sim V/L$ (again ignoring ordering-unity factors including geometrical factors of q or B_φ/B_θ), and find

$$RHS \sim \frac{1}{V} nk_\perp L\phi\frac{V}{L}\frac{H}{k_\perp\phi} \sim nH$$

Balancing these two terms, LHS $\sim$ RHS, we find that we need to calculate the Hamiltonian H to the accuracy of $H_3 \sim \epsilon^3 T$, the same result found from the previous quasineutrality order-of-magnitude argument.

Why isn't momentum flux from H_1 much larger than from H_3?

(App. K. of Krommes & Hammett '13)

$$j_{gc,r} = \langle \sum_s \bar{n}_s q_s \int d\underline{v}\, F_s \underline{V}_{ds} \cdot \hat{r} \rangle$$

$\sim n_0 q \, \varepsilon c_s$ is the apparent order of this term, but it needs to be of order

$\sim n_0 q \, \varepsilon^4 c_s$ in order to explain the GyroBohm slow flow regime.

Consider usual slab limit for simplicity (like Hasegawa-Mima, x=radial, z = parallel to $\underline{B}$, & y = binormal direction, has toroidal component)

One order in ε from the fact that F_{0s} doesn't contribute to the current (like before):

$$\langle n V_{ds,r} \rangle \propto \langle n \frac{\partial H}{\partial y} \rangle$$

$$\propto \langle (n_0 + \tilde{n}) \frac{\partial H}{\partial y} \rangle$$

$$\propto \langle \tilde{n} \frac{\partial H}{\partial y} \rangle$$

& $\tilde{n} \sim \varepsilon n_0$

Focus on lowest order $E \times B$ drift:

$$j_{gc,r} = -\frac{c}{B} \Big\langle \underbrace{\sum_s n_s q_s}_{\text{substitute using GK Poisson Eq:}} \frac{\partial \phi}{\partial y} \Big\rangle$$

$$= \frac{c}{B} \Big\langle \nabla_\perp \cdot \Big(n_{i0} q_i \rho_s^2 \frac{e}{T_e} \nabla_\perp \phi \Big) \frac{\partial \phi}{\partial y} \Big\rangle$$

(use n_{i0} here for simplicity, as done in many codes...)

$\nabla_\perp \cdot$ gives two types of terms.

$$\frac{\partial}{\partial y} \text{ term} = \frac{c}{B} \Big\langle \frac{\partial}{\partial y} \Big(n_{i0} q_i \rho_s^2 \frac{e}{T_e} \frac{\partial \phi}{\partial y} \Big) \frac{\partial \phi}{\partial y} \Big\rangle$$

$$\propto \int dy \, \frac{\partial^2 \phi}{\partial y^2} \frac{\partial \phi}{\partial y} = \int dy \, \frac{\partial}{\partial y} \Big(\frac{1}{2} \Big(\frac{\partial \phi}{\partial y} \Big)^2 \Big)$$

$$= 0$$

$\sim k_\perp \sim \frac{1}{\rho_s}$

The other term is:

$$j_{gc,r} = \frac{c}{B} \Big\langle \frac{\partial}{\partial x} \Big(n_{i0} q_i \rho_s^2 \frac{e}{T_e} \frac{\partial \phi}{\partial x} \Big) \frac{\partial \phi}{\partial y} \Big\rangle$$

$\sim \frac{1}{L}$ brings in a factor of ϵ.

$$= \frac{c}{B} \frac{\partial}{\partial x} \Big\langle n_{i0} q_i \rho_s^2 \frac{e}{T_e} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \Big\rangle$$

$$- \frac{c}{B} \Big\langle n_{i0} q_i \rho_s^2 \frac{e}{T_e} \frac{\partial \phi}{\partial x} \frac{\partial}{\partial y} \Big(\frac{\partial \phi}{\partial x} \Big) \Big\rangle$$

Last term

$$\propto \left\langle \frac{\partial \phi}{\partial x} \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) \right\rangle$$

$$\propto \left\langle \frac{\partial}{\partial y} \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 \right\rangle$$

$$= 0$$

So we are left with:

$$j_{gc,r} = \frac{c}{B} \frac{\partial}{\partial x}\left(n_{i0} q_i \rho_s^2 \frac{e}{T_e} \underbrace{\left\langle \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \right\rangle}_{\propto \langle \tilde{u}_y \tilde{u}_x \rangle} \right)$$

$$\Uparrow$$

$$\frac{\partial}{\partial t} \frac{\partial \langle \phi \rangle}{\partial r} \propto j_{gc,r} \quad (p.\ 1)$$

$$\Downarrow$$

$$\frac{\partial}{\partial t} \langle u_y \rangle \propto \frac{\partial}{\partial x} \langle u_x u_y \rangle$$

(Subtlety: turbulent viscosity from $\perp$ Reynold's stress by itself usually negative. But in real tokamak, toroidal momentum transport also involves $\langle \tilde{u}_{\parallel} \tilde{u}_r \rangle$ & interaction w/ ITG/TEM & poloidal flow damping.

$\propto \langle \tilde{u}_y \tilde{u}_x \rangle$

= Reynold's stress

appears to be $\mathcal{O}(\varepsilon^2 c_s^2)$

But $\tilde{u}_x$ & $\tilde{u}_y$ are weakly correlated:

$$\langle \tilde{u}_x \tilde{u}_y \rangle \sim D_{GB} \frac{\partial u_{y0}}{\partial x}$$

$$\sim \frac{cT}{eB} \frac{\rho}{L} \frac{\partial u_{y0}}{\partial x}$$

$$\sim c_s \rho_s \frac{\rho}{L} \frac{u_{y0}}{L}$$

$$\langle \tilde{u}_x \tilde{u}_y \rangle \sim c_s\, \varepsilon^2 U_{y0}$$

$$\sim c_s^2\, \varepsilon^3$$

Thus:

$$j_{gc,r} \sim \frac{1}{L} \frac{B}{c} n_{i0} q_i \rho_s^2 \frac{e}{T_e} \langle \tilde{u}_x \tilde{u}_y \rangle$$

$$\sim \frac{1}{L} \frac{1}{\cancel{c_s \rho_s}} n_{i0}\, e\, \rho_s^{\cancel{2}}\, \varepsilon^3 c_s^{\cancel{2}}$$

$$\sim \varepsilon^3 n_{i0}\, e \frac{\rho_s}{L} c_s$$

$$\sim \varepsilon^4 n_{i0}\, e\, c_s \quad \checkmark$$

<u>Conclusion</u>: momentum flux from H_1 is much smaller than expected, comparable to momentum flux from H_3.

Polarization Density Plays Key Role in Gyrokinetic Poisson / Quasineutrality Eq.

At long wavelengths (neglecting higher-order FLR corrections to polarization density):

$$-\nabla_\perp \cdot \left(\frac{\sum_s 4\pi n_s m_s c^2}{B^2} \nabla_\perp \phi \right) = \sum_s 4\pi e_s \int m_s B^*_{||} dv_{||} d\mu d\theta \, \bar{f}(\vec{x} - \vec{\rho}(\mu,\theta), \mu, v_{||})$$

In order for a non-local/global gyrokinetic code to have a conserved energy-like quantity using just the lowest order drifts (ExB, grad(B) and curvature) from the first order Hamiltonian $H_1 \sim (\rho/L)\ T$, the density on the LHS must be replaced by a time-independent n_{s0}. Okay for short time scales.

In order to allow a time varying n_s, and conserve the energy properly, one must include drifts from the second order Hamiltonian $H_2 \sim (\rho/L)^2\ T$. Natural consequence of Lagrangian field theory approach.

(Local gyrokinetics also satisfies energy conservation, H_2 effects incorporated in the higher-order transport equations.)

Two main types of gyrokinetics

- Original local "δf" iterative/asymptotic gyrokinetics, directly expands Vlasov Eq. and $F = F_0 + \varepsilon F_1$ (Frieman and Chen). Rigorous for small $\rho_* = \rho/L$ gyroBohm limit, important limit to study. Eddy size $L_{eddy} \sim \rho << L$. Simulate small-scale turbulence in a local region where radial variation of parameters ($\omega_*(r), \nu(r)$, etc.) can be neglected. (I.e., both n_0 and dn_0/dr are treated as constant, as in Hasegawa-Mima eq.) The most complete derivations, including both gyrokinetic turbulence equation & next order transport equations:
  - Ian Abel, Rep. Prog. Phys. 76 (2013) 116201 (69 pp)
  - Sugama and Horton, Phys. Plasmas 5 (1998) 2560 (14 pp)

- Global "full F" Lagrangian/Hamiltonian gyrokinetics. Does not break up $F = F_0 + \delta f$. Does not assume eddy sizes $L_{eddy} << L$, and so includes effect of radial variation of parameters and possible non-gyroBohm regimes. (Probably important near plasma edge and near transport barriers.) Maybe consistent only in some case:
  - $\rho \sim L_{eddy} << L$, (gyroBohm regime) or
  - $\rho << L_{eddy} \sim L$, (i.e., $k_\perp \rho << 1$, Bohm regime) but not
  - $\rho \sim L_{eddy} \sim L$ (but perhaps generalizations exist for SOL, ...)
  - First derivation in Lagrangian field theory approach that gave particle+field energy conservation consistently is Sugama (2000), followed quickly by Brizard (2000) and others.

Outline of Iterative local gyrokinetics

- Original "2-scale" local "delta f" gyrokinetics with direct iterative/asymptotic expansion of Vlasov eq. and $F = F_0 + \varepsilon F_1$ (or δf)

- Involves 4 orders of expansion to go through transport time scale (Sugama 98, Barnes 08, Plunk 09, Abel 13):

- ε^{-1}: F_0 independent of gyro-angle:

$$\frac{q}{m}\frac{\vec{v}\times\vec{B}}{c}\cdot\nabla_v F_0 = \Omega\frac{\partial F_0}{\partial\theta} = 0$$

- ε^{0}: parallel force balance and polarization from gyro-phase dependence:

$$\tilde{F}_1 = -\frac{q(\phi - \langle\phi\rangle)}{T}F_0$$

- ε^{1}: standard GK equation on ω_* turbulence time scale (free energy/entropy balance)

$$\frac{\partial F_1}{\partial t} = \ldots$$

- ε^{2}: transport equations for slow variation of F_0 on transport time scale. (energy conservation)

$$\frac{\partial n_0}{\partial t} = \ldots$$