~~A Sketch of~~ the Traditional Multiscale Asymptotic Derivation of Gyrokinetics

4/~~28~~/2014

Greg Hammett

Frieman & Chen (1983), Sugama & Horton (1998), Abel et al (2013)
(Here I follow Dorland's Thesis (1993), app. C. & other refs.)
For simplicity, focus on slab limit, $\underset{\sim}{B} = \hat{z} B_0$, $\underset{\sim}{E} = -\nabla\phi$

Key ordering assumptions (based on experimental evidence on drift wave turbulence & insights into key nonlinear dynamics):

$$\underset{\sim}{F} = F_0(\underset{\sim}{R}, v_\parallel, \mu) + F_1(\underset{\sim}{R}, v_\parallel, \mu, \alpha, t)$$

Equilibrium, evolves very slowly, on transport time scale.

Small amplitude, but nonlinear, fluctuations

$$\frac{\partial F_0}{\partial t} \sim D_{GB} \frac{\partial^2 F}{\partial r^2} \sim \frac{v_t \rho^2}{L} \frac{F_0}{L^2}$$

$$\sim \varepsilon^2 \frac{v_t}{L} F_0$$ neglect on turbulence time scale.

$$\frac{\partial F_1}{\partial t} \sim \omega F_1$$

turbulence time scale

$$\omega \sim \omega_* \sim \underbrace{k_\theta \rho}_{\sim 1} \frac{v_t}{L} \sim \frac{v_t}{L}$$

"Standard" ordering assumptions

(can probably be generalized, the fundamental assumption is that things vary slowly compared to gyrofrequency):

$$\nabla F_0 \sim \frac{F_0}{L}$$

$$\nabla_\perp F_1 \sim k_\perp F_1$$

$$\nabla_\parallel F_1 \sim k_\parallel F_1$$

$$\varepsilon \sim \frac{\omega}{\Omega_c} \sim \frac{\rho}{L} \sim \frac{F_1}{F_0} \sim \frac{e\phi}{T} \sim \frac{V_{E\times B}}{V_t} \sim \frac{k_\parallel V_t}{\Omega_c} \ll 1$$

but $k_\perp \rho \sim \mathcal{O}(1)$

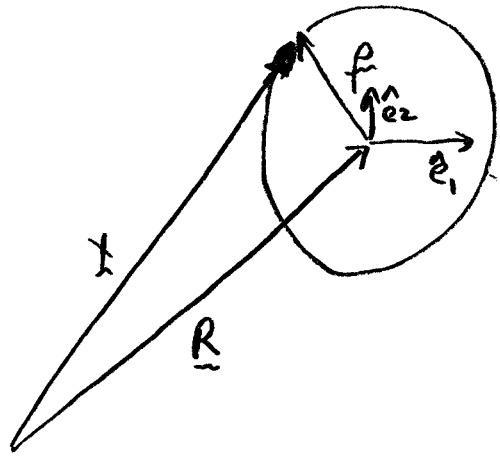
Vlasov Eq.: (order relative to $\frac{V_t}{L} F_0$):

$$\underbrace{\frac{\partial F_0}{\partial t}}_{\sim\varepsilon^2} + \underbrace{\frac{\partial F_1}{\partial t}}_{\sim\varepsilon} + \underbrace{\underset{\sim}{v}\cdot\nabla F_0}_{\sim 1} + \underbrace{\underset{\sim}{v}_\perp\cdot\nabla F_1}_{\sim 1} + \underbrace{\underset{\sim}{v}_\parallel\cdot\nabla F_1}_{\sim\varepsilon}$$

$$+ \underbrace{\frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right)\cdot\frac{\partial F_0}{\partial \underset{\sim}{v}}}_{\sim 1 \quad \sim\varepsilon^{-1}} + \underbrace{\frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right)\cdot\frac{\partial F_1}{\partial \underset{\sim}{v}}}_{\sim\varepsilon \quad \sim 1} = 0$$

Expect solution to be simpler in guiding center coordinates, where fast gyromotion is separated from slow guiding center drifts. Use Catto Guiding Center transformation (i.e., not just cylindrical coordinates in $\underset{\sim}{v}$, but also use a guiding center spatial coordinate):

$$(\underset{\sim}{x}, \underset{\sim}{v}) \Longrightarrow (\underset{\sim}{R}, v_{\parallel}, \mu, \alpha)$$



$$\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{\rho}$$

$$= \underset{\sim}{R} + \frac{\hat{b} \times \underset{\sim}{v}}{\Omega}$$

$$\underset{\sim}{v} = v_{\parallel} \hat{b} + v_{\perp} (\hat{e}_1 \cos\alpha + \hat{e}_2 \sin\alpha)$$

$$\mu = \frac{1}{2} \frac{v_{\perp}^2}{B} = \frac{|\underset{\sim}{v} - \hat{b}\hat{b} \cdot \underset{\sim}{v}|^2}{2B}$$

(An aside: This is okay for std. low flow regime, $V_{E \times B} \sim \varepsilon V_t$, but a better expression for high-flow regimes is: $\mu = \dfrac{|\underset{\sim}{v} - v_{\parallel} \hat{b} - \underset{\sim}{V}_{E \times B}|^2}{2B}$)

Vlasov Eq. of form:

$$\frac{\partial F(\underset{\sim}{z},t)}{\partial t} + \dot{z}_i \frac{\partial F}{\partial z_i} = 0$$

Will do a straight forward coordinate transformation

$$\underset{\sim}{z} = (\underset{\sim}{x}, \underset{\sim}{v}) \Rightarrow \hat{\underset{\sim}{z}} = (\underset{\sim}{R}, v_{\parallel}, \mu, \alpha)$$
$$= \hat{\underset{\sim}{z}}(\underset{\sim}{z})$$

Physicists are often sloppy about coordinates, functions, & coordinate transformations (or use notation with implicit assumptions that depends on context): In above eq., z_i is set of independent variables, & so $\dot{z}_i$ doesn't make sense in a std framework. So instead denote velocities in phase space by u_i

$$\frac{\partial F}{\partial t} + u_i(\underset{\sim}{z}) \frac{\partial F}{\partial z_i} = 0$$

& introduce coordinate transformation:

(assume coord. transf. indep. of t for now.)

$$F(\underset{\sim}{z},t) = \hat{F}(\hat{\underset{\sim}{z}},t)$$
$$= \hat{F}(\hat{\underset{\sim}{z}}(\underset{\sim}{z}),t)$$

(Sometimes write $F(\underset{\sim}{z},t) \Rightarrow F(\hat{\underset{\sim}{z}},t)$, but they are different functions, & so use $\hat{F}(\hat{\underset{\sim}{z}},t)$.)

Straightforward chain rule:

$$\frac{\partial F}{\partial t} + u_i(\underset{\sim}{z})\frac{\partial F}{\partial z_i} = 0$$

partial derivative holding other z_j fixed (for $j \neq i$).

$$\frac{\partial \hat{F}}{\partial t} + u_i \frac{\partial \hat{F}}{\partial \hat{z}_j}\frac{\partial \hat{z}_j}{\partial z_i} = 0$$

partial derivative holding other $\hat{z}_k$ fixed (for $k \neq j$).

$$\frac{\partial \hat{F}}{\partial t} + u_i(\underset{\sim}{z}(\hat{\underset{\sim}{z}}))\frac{\partial \hat{z}_j}{\partial z_i}\frac{\partial \hat{F}}{\partial \hat{z}_j} = 0$$

$$\frac{\partial \hat{F}}{\partial t} + \underset{\sim}{v}\cdot\left(\nabla \underset{\sim}{R}\cdot\nabla_{\underset{\sim}{R}}\hat{F} + \nabla\mu\frac{\partial \hat{F}}{\partial \mu} + \nabla_{\parallel}v_{\parallel}\frac{\partial \hat{F}}{\partial v_{\parallel}} + \nabla\alpha\frac{\partial \hat{F}}{\partial \alpha}\right)$$

$$+\frac{q}{m}\left(\underset{\sim}{E} + \frac{\underset{\sim}{v}\times\underset{\sim}{B}}{c}\right)\cdot\left(\nabla_v \underset{\sim}{R}\cdot\nabla_R\hat{F} + \nabla_v\mu\frac{\partial F}{\partial \mu} + \nabla_v v_{\parallel}\frac{\partial F}{\partial v_{\parallel}} + \nabla_v\alpha\frac{\partial F}{\partial \alpha}\right) = 0$$

Note: everything in this Eq. is a function of guiding center coordinates

$$\underset{\sim}{E}(\underset{\sim}{x},t) = \underset{\sim}{E}(\underset{\sim}{R}+\underset{\sim}{\rho},t)$$

Could have transformed $\underset{\sim}{E}$ like $\underset{\sim}{F}$:

$$\underset{\sim}{E}(\underset{\sim}{z}) = \hat{\underset{\sim}{E}}(\hat{\underset{\sim}{z}}(\underset{\sim}{z}))$$

But in the final Eq. we want to express everything in $\hat{\underset{\sim}{z}}$ coordinates, so:

$$\underset{\sim}{E}(\underbrace{\underset{\sim}{z}(\hat{\underset{\sim}{z}})}_{\text{inverse coordinate transformation}}) = \hat{E}(\hat{\underset{\sim}{z}})$$

$$= \underset{\sim}{E}(\underset{\sim}{x} + \underset{\sim}{f})$$

An aside: physicists often write expressions such as:

$$y = y(x)$$

for coordinate transformations, etc. But this can be confusing to some because y is used here as both a variable (a coordinate) & as a function, & have to know which is meant from the context.

Cary & Littlejohn '83 prefer to always distinguish between variables & functions with different symbols:

$$y = f(x)$$

They devote a full appendix to this & related notational subtleties. "It is important in transformation theory to distinguish between value relationships & functional relationships. These distinctions are elementary, but they cause a great deal of confusion if insufficient attention is given to them."

Expressing new coordinates $(\underset{\sim}{R}, v_{\parallel}, \mu, \alpha)$ in terms of old coord. $(\underset{\sim}{x}, \underset{\sim}{v})$

$$\underset{\sim}{R} = \underset{\sim}{x} - \frac{\hat{b} \times \underset{\sim}{v}}{\Omega}$$

$$v_{\parallel} = \underset{\sim}{v} \cdot \hat{b}$$

$$\mu = \frac{|\underset{\sim}{v} - \hat{b}\hat{b} \cdot \underset{\sim}{v}|^2}{2B}$$

$$\alpha = \operatorname{atan}\left(\frac{\underset{\sim}{v} \cdot \hat{e}_1}{|\underset{\sim}{v} - \hat{b}\hat{b} \cdot \underset{\sim}{v}|} \right)$$

& use identities:

$$\nabla \mu = 0 = \nabla v_{\parallel} = \nabla \alpha$$

$$\underset{\sim}{v} \cdot \nabla \underset{\sim}{R} = \underset{\sim}{v}$$

$$\nabla_v \underset{\sim}{R} = \frac{\underset{\sim}{1} \times \hat{b}}{\Omega}$$

$$\nabla_v \mu = \frac{\underset{\sim}{v}_{\perp}}{B}$$

$$\nabla_v v_{\parallel} = \hat{b}$$

$$\nabla_v \alpha = \frac{\hat{b} \times \underset{\sim}{v}}{v_{\perp}^2}$$

(These are all for the unsheared slab limit $\nabla \underset{\sim}{B} = 0$. Dorland '93 Thesis, App. C has generalized results.)

The proofs are straight forward. The most steps are on the last one:

$$\underset{\sim}{v} = v_{\parallel}\,\hat{b} + v_{\perp}\left(\hat{e}_1 \cos\alpha + \hat{e}_2 \sin\alpha\right)$$

$$\underset{\sim}{v}\cdot\hat{e}_1 = v_{\perp}\cos\alpha$$

apply ∇_v: $\quad \hat{e}_1 = \left(\nabla_v v_{\perp}\right)\cos\alpha - v_{\perp}\sin\alpha\,\nabla_v\alpha$

~~use $\nabla_v v_{\perp} = \nabla_v \sqrt{v^2 - v_{\parallel}^2}$~~

$$\cancel{= \frac{1}{2}\frac{2\underset{\sim}{v} - 2\hat{b}v_{\parallel}}{v_{\perp}} = \frac{\underset{\sim}{v_{\perp}}}{v_{\perp}}}$$

But could also use

$$\underset{\sim}{v}\cdot\hat{e}_2 = v_{\perp}\sin\alpha$$

apply ∇_v: $\quad \hat{e}_2 = \left(\nabla_v v_{\perp}\right)\sin\alpha + v_{\perp}\cos\alpha\,\nabla_v\alpha$

Multiply & add:

$$-\sin\alpha\,\hat{e}_1 + \cos\alpha\,\hat{e}_2 = v_{\perp}\,\nabla_v\alpha$$

$$\frac{1}{v_{\perp}}\,\hat{b}\times\underset{\sim}{v} = v_{\perp}\,\nabla_v\alpha$$

$$\boxed{\nabla_v\alpha = \frac{\hat{b}\times\underset{\sim}{v}}{v_{\perp}^2}}$$

$$\frac{\partial \hat{F}}{\partial t} + \left(\underset{\sim}{v}_{\parallel} + \frac{c}{B}\underset{\sim}{E}\times\hat{b}\right)\cdot\frac{\partial \hat{F}}{\partial \underset{\sim}{R}} - \Omega\frac{\partial \hat{F}}{\partial \alpha}$$

$$\sim\varepsilon \qquad \sim 1 \qquad \sim\varepsilon \qquad \sim\varepsilon^{-1}$$

$$+ \frac{q_\alpha}{m}\underset{\sim}{E}\cdot\left(\frac{\underset{\sim}{v}_\perp}{B}\frac{\partial \hat{F}}{\partial \mu} + \hat{b}\frac{\partial \hat{F}}{\partial v_\parallel} + \frac{\hat{b}\times\underset{\sim}{v}_\perp}{v_\perp^2}\frac{\partial \hat{F}}{\partial \alpha}\right) = 0$$

$$\sim 1 \qquad \sim\varepsilon \qquad \sim 1$$

Order of terms relative to $\frac{v_t}{L}\hat{F}_0$. Example:

$$\frac{T_5}{\frac{v_t}{L}F_0} \sim \frac{\frac{q_\alpha}{m}\frac{E}{B}\cdot\cancel{v_t}\frac{\cancel{F_0}}{(v_t^2/B)}}{\frac{\cancel{v_t}}{L}\cancel{F_0}} \sim \frac{q_\alpha B}{mc}\frac{v_E}{v_t}\frac{1}{v_t}L$$

$$\sim \frac{v_E}{v_t}\frac{L}{\rho}$$

$$\sim \varepsilon\frac{1}{\varepsilon} \sim 1$$

$\boxed{\text{Drop hats: } \hat{F} \to F}$

Lowest order $\mathcal{O}(\varepsilon^{-1})$: $\dfrac{\partial F_0}{\partial \alpha} = 0$

Next order, $\mathcal{O}(\varepsilon^{0})$:

$$\underset{\sim}{v}_{\parallel} \cdot \frac{\partial F_0}{\partial \underset{\sim}{R}} + \frac{q}{mB} \underset{\sim}{E} \cdot \underset{\sim}{v}_{\perp} \frac{\partial F_0}{\partial \mu} - \Omega \frac{\partial F_1}{\partial \alpha} = 0$$

Recall $\underset{\sim}{E} = \underset{\sim}{E}(\underset{\sim}{R} + \underset{\sim}{\rho})$

$$= -\nabla_R \phi(\underset{\sim}{R} + \underset{\sim}{\rho})$$

so $\underset{\sim}{v}_{\perp} \cdot \underset{\sim}{E} = -\underset{\sim}{v}_{\perp} \cdot \nabla_R \phi(\underset{\sim}{R} + \underset{\sim}{\rho})$

But $\underset{\sim}{v}_{\perp}$ & $\underset{\sim}{\rho}$ are related:

$$\underset{\sim}{\rho} = \frac{\hat{b} \times \underset{\sim}{v}}{\Omega} = \frac{v_{\perp}}{\Omega} (\hat{e}_2 \cos\alpha - \hat{e}_1 \sin\alpha)$$

$$\frac{\partial \underset{\sim}{\rho}}{\partial \alpha} = \frac{-\underset{\sim}{v}_{\perp}}{\Omega}$$

So $\underset{\sim}{v}_{\perp} \cdot \underset{\sim}{E} = \Omega \dfrac{\partial \phi(\underset{\sim}{R} + \underset{\sim}{\rho}(\alpha))}{\partial \alpha}$

So $\mathcal{O}(\varepsilon^0)$ Eq. is:

$$\underset{\sim}{v}_{\parallel} \cdot \frac{\partial F_0}{\partial \underset{\sim}{R}} + \frac{q}{mB} \Omega \frac{\partial \phi(\underset{\sim}{R}+\underset{\sim}{\rho})}{\partial \alpha} \frac{\partial F_0}{\partial \mu} - \Omega \frac{\partial F_1}{\partial \alpha} = 0$$

avg. over α & just get constraint that the equilibrium is constant along $\underset{\sim}{B}$: $v_{\parallel} \hat{b} \cdot \frac{\partial F_0}{\partial \underset{\sim}{R}} = 0$

Soln. to full $\mathcal{O}(\varepsilon^0)$ Eq is

$$F_1 = \overline{F}_1 + \tilde{F}_1(\alpha)$$

$$= \overline{F}_1 + \underbrace{\frac{q}{mB}\left(\phi(\underset{\sim}{R}+\underset{\sim}{\rho}) - \langle\phi\rangle\right)\frac{\partial F_0}{\partial \mu}}_{\text{gyro-phase dependent part, gives variation of } F \text{ around gyro orbit due to Boltzmann response to } \phi.}$$

gyro-phase independent part

$F_0 + \overline{F}_1$ = guiding center density at $\underset{\sim}{R}$.

$$\langle\phi\rangle = \frac{1}{2\pi}\int_0^{2\pi} d\alpha \, \phi(\underset{\sim}{R} + \underset{\sim}{\rho}(\alpha))$$

so that $\langle \tilde{F} \rangle = 0$

& $\langle F_1 \rangle = \overline{F}_1$

For a Maxwellian

$$F_0 \propto e^{-m\mu B/T}$$

$$\frac{\partial F_0}{\partial \mu} = -\frac{mB}{T} F_0$$

so $$F_1 = \bar{F}_1 - \underbrace{\frac{q}{T}\left(\phi(\underline{R}+\underline{\rho}) - \langle\phi\rangle\right) F_0}_{\text{Boltzmann response of } F \text{ to } \phi \text{ variations around gyro-orbit. (Because } \omega \ll \Omega)}$$

[Hamiltonian approaches define higher-order corrections to coordinate transformations so $\frac{\partial F}{\partial \alpha} = 0$ to all orders in ε. Related to existence of an adiabatic invariant.]

[Some approaches, like Abel 2013 & Frieman-Chen '83 write $F = F_0\left(1 - \frac{q\phi}{T}\right) + h$ ← "non-adiabatic" or "non-Boltzmann" part.

& so never worry about α-dependent part of h. Hides polarization density.]

$$\langle \phi \rangle = \langle \phi \rangle (\underset{\sim}{R}, \mu)$$

$$= \frac{1}{2\pi} \int d\alpha \, \phi(\underset{\sim}{R} + \underset{\sim}{\rho}(\alpha))$$

$$= \frac{1}{2\pi} \int d\alpha \sum_{\underset{\sim}{k}} \phi_{\underset{\sim}{k}} \, e^{i \underset{\sim}{k} \cdot (\underset{\sim}{R} + \underset{\sim}{\rho}(\alpha))}$$

Align coordinate system for gyro-average with $\underset{\sim}{k}$:

$$= \sum_{\underset{\sim}{k}} \phi_{\underset{\sim}{k}} \, e^{i \underset{\sim}{k} \cdot \underset{\sim}{R}} \, \frac{1}{2\pi} \int d\alpha \, e^{i k \rho \cos\alpha}$$

$$= \sum_{\underset{\sim}{k}} J_0(k_\perp \rho) \, \phi_{\underset{\sim}{k}} \, e^{i \underset{\sim}{k} \cdot \underset{\sim}{R}}$$

Sometimes write this in operator notation (like in quantum mechanics): $\langle \phi \rangle = J_0 \phi$

$\mathcal{O}(\varepsilon^1)$ Eq:

$$\frac{\partial F_1}{\partial t} + \underset{\sim}{v}_{\parallel} \cdot \frac{\partial F_1}{\partial \underset{\sim}{R}} + \underset{\sim}{v}_E \cdot \nabla (F_0 + F_1) - \Omega \frac{\partial F_2}{\partial \alpha}$$

$$+ \frac{q}{m} \underset{\sim}{E} \cdot \left(\frac{\underset{\sim}{v}_\perp}{B} \frac{\partial F_1}{\partial \mu} + \hat{b} \frac{\partial F_0}{\partial v_\parallel} + \frac{\hat{b} \times \underset{\sim}{v}}{v_\perp^2} \frac{\partial F_1}{\partial \alpha} \right) = 0$$

Use $F_1 = \overline{F}_1 + \tilde{F}_1$ + gyroaverage:

$$\boxed{\begin{aligned} &\frac{\partial \overline{F}_1}{\partial t} + \underset{\sim}{v}_\parallel \cdot \frac{\partial \overline{F}_1}{\partial \underset{\sim}{R}} + \langle \underset{\sim}{v}_E \rangle \cdot \nabla (F_0 + \overline{F}_1) + \langle \underset{\sim}{v}_E \cdot \nabla \tilde{F} \rangle \\ &+ \frac{q}{m} \langle E_\parallel \rangle \frac{\partial F_0}{\partial v_\parallel} + \frac{q}{mB} \langle \underset{\sim}{E} \cdot \underset{\sim}{v}_\perp \rangle \frac{\partial \overline{F}_1}{\partial \mu} \\ &+ \frac{q}{mB} \left\langle \underset{\sim}{E} \cdot \underset{\sim}{v}_\perp \frac{\partial \tilde{F}_1}{\partial \mu} \right\rangle + \frac{q}{m v_\perp^2} \left\langle \underset{\sim}{E} \cdot \hat{b} \times \underset{\sim}{v} \frac{\partial \tilde{F}_1}{\partial \alpha} \right\rangle = 0 \end{aligned}}$$

$$T_5 = \langle \underset{\sim}{v}_E \cdot \nabla \tilde{F} \rangle \propto \left\langle \underset{\sim}{v}_E \cdot \nabla \left((\phi - \langle \phi \rangle) \frac{\partial F_0}{\partial \mu} \right) \right\rangle$$

use $\nabla (\phi F_0) = (\nabla \phi) F_0 + \phi (\nabla F_0)$

$\sim k_\perp \phi F_0 \qquad \sim \frac{\phi F_0}{L}$ (ignore)

So

$$T_5 \propto \langle \underbrace{\underset{\sim}{v}_E \cdot \nabla(\phi}_{=0 \text{ because } \underset{\sim}{v}_E \propto \hat{b}\times\nabla\phi} - \langle\phi\rangle) \frac{\partial F_0}{\partial \mu}$$

$$\propto - \underbrace{\langle \underset{\sim}{v}_E\rangle \cdot \nabla\langle\phi\rangle}_{=0}$$

$$T_{-2} = \frac{q}{mB}\left\langle \underset{\sim}{E}\cdot\underset{\sim}{v}_\perp \frac{\partial F_1}{\partial \mu}\right\rangle = \frac{q\Omega}{mB}\left\langle \frac{\partial\phi}{\partial\alpha}\frac{\partial \tilde{F}_1}{\partial\mu}\right\rangle$$

2 parts.

1 part $\propto \dfrac{\partial^2 F_0}{\partial \mu^2}$, but that leads to

$$\propto \left\langle \frac{\partial\phi}{\partial\alpha}\left(\phi - \langle\phi\rangle\right)\right\rangle \frac{\partial^2 F_0}{\partial\mu^2} \propto \left\langle \frac{\partial}{\partial\alpha}\left(\tfrac{1}{2}\phi^2\right)\right\rangle = 0$$

2nd part $\propto \dfrac{\partial}{\partial\mu}(\phi - \langle\phi\rangle)$

gyro average of this term $\to 0$

$$T_{-2} = \frac{q}{mB}\left\langle \underset{\sim}{E}\cdot\underset{\sim}{v}_\perp \frac{q}{mB}\frac{\partial(\phi - \langle\phi\rangle)}{\partial\mu}\right\rangle \frac{\partial F_0}{\partial\mu}$$

use $\dfrac{\partial\phi}{\partial\mu} = (\nabla_R\phi)\cdot\dfrac{\partial\underset{\sim}{\rho}}{\partial\mu}$

since $\rho \propto \sqrt{\mu}$

$$\frac{\partial\underset{\sim}{\rho}}{\partial\mu} = \frac{\underset{\sim}{\rho}}{2\mu}$$

$$= (\nabla_{\underset{\sim}{R}}\phi)\cdot\frac{\hat{b}\times\underset{\sim}{v}_\perp}{2\mu\Omega}$$

$$\frac{\partial \phi}{\partial \mu} = - \underset{\sim}{E} \cdot \frac{\hat{b} \times \underset{\sim}{v}_\perp}{2 \mu \Omega}$$

So $$T_{-2} = - \frac{q}{mB} \left\langle \underset{\sim}{E} \cdot \underset{\sim}{v}_\perp \frac{q}{mB} \frac{\underset{\sim}{E} \cdot \hat{b} \times \underset{\sim}{v}}{2 \mu \Omega} \right\rangle \frac{\partial F_0}{\partial \mu}$$

$$= - \frac{q}{m B v_\perp^2} \left\langle \underset{\sim}{E} \cdot \underset{\sim}{v}_\perp \frac{q}{m \Omega} \underset{\sim}{E} \cdot \hat{b} \times \underset{\sim}{v} \right\rangle \frac{\partial F_0}{\partial \mu}$$

Compare with T_{-1} (& use p.8):

$$T_{-1} = \frac{q}{m v_\perp^2} \left\langle \underset{\sim}{E} \cdot \hat{b} \times \underset{\sim}{v} \frac{q}{m B \Omega} \underset{\sim}{E} \cdot \underset{\sim}{v}_\perp \right\rangle \frac{\partial F_0}{\partial \mu}$$

so $T_{-2} + T_{-1} = 0$

Gyro-averaging the first order Eq. from p.7. gives us:

$$\frac{\partial \bar{F}_1}{\partial t} + v_\parallel \hat{b} \cdot \frac{\partial \bar{F}_1}{\partial \underset{\sim}{R}} + \langle \underset{\sim}{v}_E \rangle \cdot \nabla (F_0 + \bar{F}_1)$$

$$+ \frac{q}{m} \langle E_\parallel \rangle \frac{\partial F_0}{\partial v_\parallel} = 0$$

adding in small terms from higher order, can write this in the "Full-F" form: $\bar{F} = F_0 + \bar{F}_1$

$$F(\underset{\sim}{R}, v_\parallel, \mu, \alpha, t) = \bar{F}(\underset{\sim}{R}, v_\parallel, \mu, t) + \tilde{F}(\underset{\sim}{R}, v_\parallel, \mu, \alpha, t)$$

$$\boxed{\frac{\partial \bar{F}}{\partial t} + \left(v_\parallel \hat{b} + \langle \underset{\sim}{v}_E \rangle\right) \cdot \frac{\partial \bar{F}}{\partial \underset{\sim}{R}} + \frac{q}{m} \langle E_\parallel \rangle \frac{\partial \bar{F}}{\partial v_\parallel} = 0}$$

[Fine for the turbulence time scale, but to get exact energy conservation, which only matters on long transport time scales, have to include small corrections from $H_2 \sim \varepsilon^2 T$ that can be found from a Hamiltonian approach, or by going to the $O(\varepsilon^2)$ Eq. in this asymptotic, multiscale approach to determine the transport Eq. for $\frac{\partial F_0}{\partial t} = \cdots$]

While this determines $\bar{F}_1$ in <u>guiding-center coordinates</u>, Maxwell's Eqs. (& quasineutrality) are satisfied in <u>particle coordinates</u>

("pull-back transformation" in the language of differential geometry):

$\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{\rho}$ (see next page).

(& time)

$$n_e(\underset{\sim}{x},t) = n_i(\underset{\sim}{x},t) = \int d^3v \; F_{p.c}(\underset{\sim}{x},\underset{\sim}{v}) \quad \leftarrow \text{in particle coordinates}$$

$$= \int d^3v \; F_{g.c}(\underset{\sim}{x}-\underset{\sim}{\rho},\underset{\sim}{v}) \quad \leftarrow \text{in guiding-center coordinates}^{\star}$$

$$= \int d^3v \; \bar{F}(\underset{\sim}{x}-\underset{\sim}{\rho}, v_{\parallel}, \mu) + \int d^3v \; \tilde{F}(\underset{\sim}{x}-\underset{\sim}{\rho}, v_{\parallel}, \mu, \alpha)$$

$$= \underbrace{\int d^3v \; \bar{F}(\underset{\sim}{x}-\underset{\sim}{\rho}, v_{\parallel}, \mu)}_{n_{i,g.c}} \underbrace{- \frac{q}{T} \int d^3v \; F_0 \left(\phi(\underset{\sim}{x}) - \langle\phi\rangle(\underset{\sim}{x}-\underset{\sim}{\rho})\right)}_{n_{i,pol}}$$

$$= \int d^3v \; J_0 \bar{F}(\underset{\sim}{x}, v_{\parallel}, \mu)$$

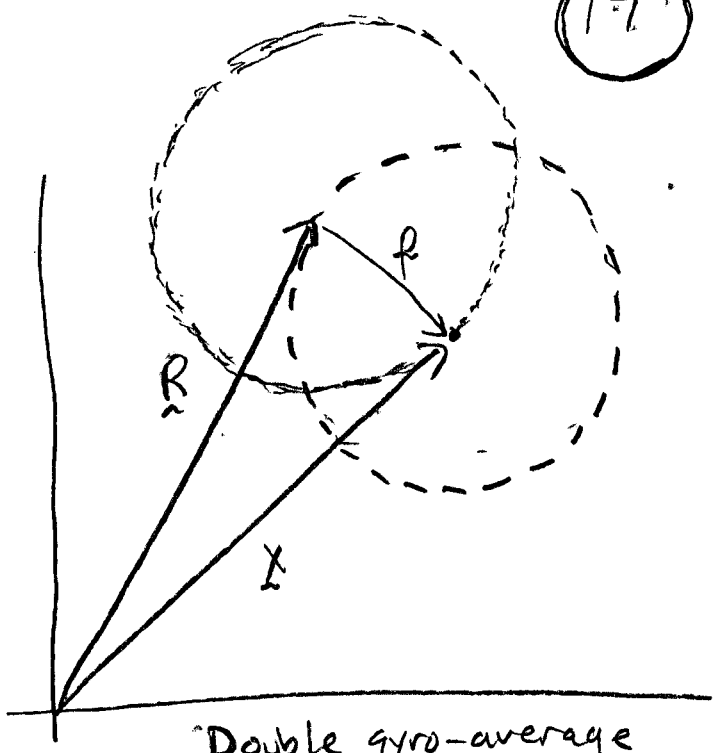
$\star$ [i.e., $F_{pc}(\underset{\sim}{x},\underset{\sim}{v}) = F(\underset{\sim}{z}) = \hat{F}(\hat{\underset{\sim}{z}}(\underset{\sim}{z})) = F_{gc}(\underset{\sim}{R}, v_{\parallel}, \mu, \alpha)$]

Integrate over all gyroangles α

$$\underset{\sim}{\rho} = \frac{\hat{b} \times \underset{\sim}{v}(\alpha)}{\Omega}$$

$$\underset{\sim}{R} = \underset{\sim}{x} - \rho(\alpha)$$

that contribute to particles at $\underset{\sim}{x}$.



Double gyro-average

$$n_{ipoe} = -\frac{q}{T} n_0 \left(\phi(\underset{\sim}{x}) - \underbrace{\int d^3v}_{= \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} dv_\perp v_\perp \int_0^{2\pi} d\alpha} \frac{F_0}{n_0} \sum_{\underset{\sim}{k}} J_0\left(\frac{k_\perp v_\perp}{\Omega}\right) \phi_{\underset{\sim}{k}} e^{i\underset{\sim}{k}\cdot(\underset{\sim}{x}-\rho)} \right)$$

$$= -\frac{q}{T} n_0 \left(\phi(\underset{\sim}{x}) - \int d^3v \frac{F_0}{n_0} \sum_{\underset{\sim}{k}} J_0^2\left(\frac{k_\perp v_\perp}{\Omega}\right) \phi_{\underset{\sim}{k}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \right)$$

For a Maxwellian F_0:

$$= -\frac{q}{T} n_0 \left(\phi(\underset{\sim}{x}) - \sum_{\underset{\sim}{k}} \Gamma_0\left(\frac{k_\perp^2 v_t^2}{\Omega^2}\right) \phi_{\underset{\sim}{k}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} \right)$$

where $\Gamma_0(b) = e^{-b} I_0(b) = e^{-b} J_0(ib)$

"modified Bessel function".

Sometimes write in an operator notation (like in quantum mechanics):

$$n_{i,pol} = -n_0 \left(1 - \Gamma_0\left(k_\perp^2 \rho_t^2\right)\right) \frac{q\phi}{T}$$

$$\Gamma_0 \approx 1 - b + \cdots$$

agrees with fluid derivation through $k_\perp^2 \rho_t^2$ for $k_\perp^2 \rho_t^2 \ll 1$. Generalizes to arbitrary $k_\perp \rho_t$.

Padé approximation much safer!

$$\Gamma_0(b) \approx \frac{1}{1+b}$$

$$1 - \Gamma_0(b) \approx \frac{b}{1+b}$$

(Though a more accurate asymptotic form for $b \gg 1$ is $\Gamma_0 \sim \frac{1}{\sqrt{2\pi b}}$).

GK Poisson / Quasineutrality Eq.

$$\boxed{\begin{aligned} n_e &= n_i \\ &= n_{i,gc} + n_{i,pol} \\ &= \int d^3v\, J_0 \bar{F} - n_0 \frac{q_{Ti}}{T_i}\left(1 - \Gamma_0(k_\perp^2 \rho_i^2)\right)\phi \end{aligned}}$$

For adiabatic electrons & $k_\perp^2 \rho_i^2 \ll 1$

get
$$n_{ipol} \approx -n_0 \frac{e}{T_i} k_\perp^2 \rho_i^2 \phi$$

$$= -n_0 k_\perp^2 \rho_s^2 \frac{e\phi}{T_e}$$

Same as in fluid derivation of polarization density, used in Hasegawa-Mima!

$$n_{e0}\left(1 + k_\perp^2 \rho_s^2\right)\frac{e\phi}{T_{e0}} = \int d^3v\, J_0 \bar{F} - n_{e0}$$

Summary of Gyrokinetic Derivation:

Vlasov Eq.

Catto transformation to guiding-center coordinates.

expansion in $\varepsilon \sim \frac{\omega}{\Omega} \sim \frac{\rho}{L} \sim k_\parallel \rho \sim \frac{e\phi}{T} \sim \frac{F_1}{F_0}$

Scales relative to $\frac{c_s}{L} F_0$:

$\mathcal{O}(\varepsilon^{-1}) \Longrightarrow \frac{\partial F_0}{\partial \alpha} = 0$

(Ω time scale for F_0)

$\mathcal{O}(\varepsilon^{0}) \Longrightarrow$ Polarization response $\tilde{F}_1 \propto \frac{e(\phi - \langle\phi\rangle)}{T} F_0$

(Ω time scale for F_1)

$\mathcal{O}(\varepsilon^{1}) \Longrightarrow$ GK Eq. for $\frac{\partial \bar{F}_1}{\partial t}$

(turbulence time scale).

$\mathcal{O}(\varepsilon^{2}) \Longrightarrow$ Transport time scale Eq. $\frac{\partial F_0}{\partial t} = \ldots$