Destabilizing effect of shear flow without inflection point in ideal MHD plasma

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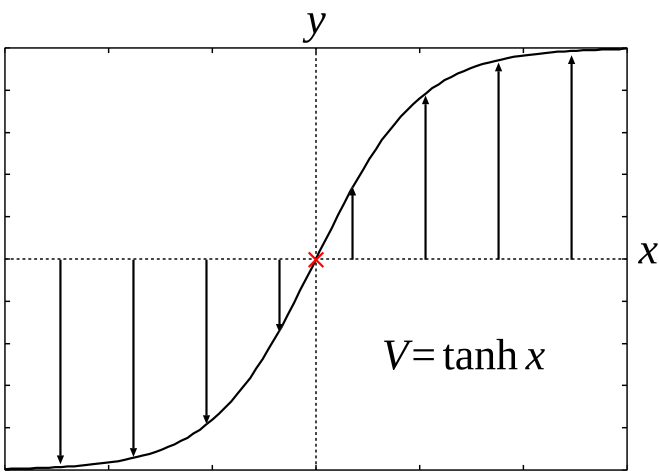
What is inflection point?

Kelvin-Helmholtz instability: In the 2d inviscid incompressible classical fluid mechanics, Rayleigh had shown in 1880 that inflection point is necessary for KH instability.

Rayleigh equation

$$(\partial_t + V_y \partial_y)\Delta\phi = V_y''\partial_y\phi \tag{1}$$



where ϕ is stream function, V_y is ambient flow in y-direction, and prime denotes x-derivative.

Inflection point theorem

$$V_y'' \neq 0 \quad (\forall x) \quad \Rightarrow \quad \text{stable} \tag{2}$$

J. W. S. Rayleigh, Proc. London Math. Soc. **9**, 57 (1880).

Is this Magneto-Rotational Instability? — No!

If motions in galaxy are Keplerian,

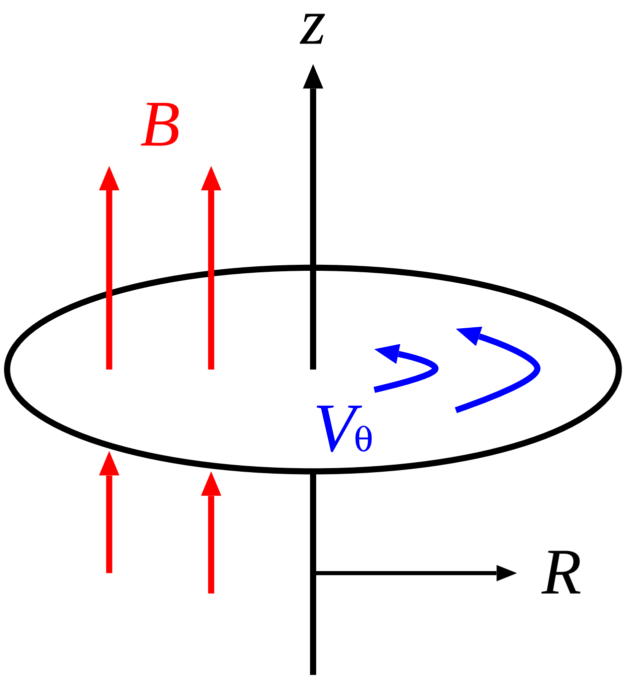
$$V_\theta \sim R^{-3/2} \tag{3}$$

it is fluid-dynamically stable:

$$\frac{\mathrm{d}(R^2\Omega)^2}{\mathrm{d}R} > 0. \tag{4}$$

But once if it is threaded by a weak axial magnetic field, it becomes unstable:

$$\frac{\mathrm{d}\Omega^2}{\mathrm{d}R} < 0. \tag{5}$$



Caused by the combined effect of flow curvature and magnetic field.

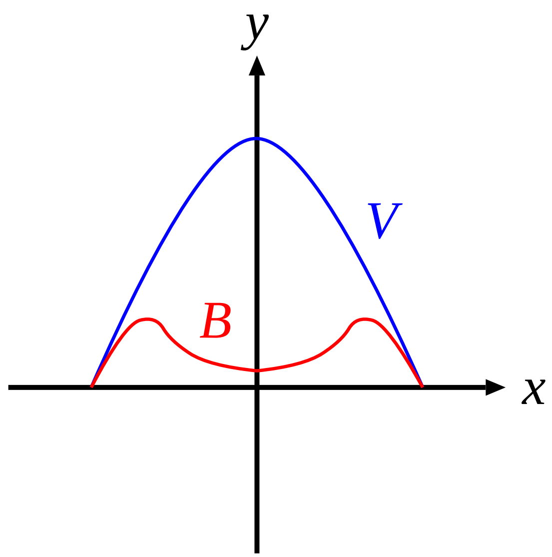
E. P. Velikhov, Soviet Phys. JETP **36**, 995 (1959)

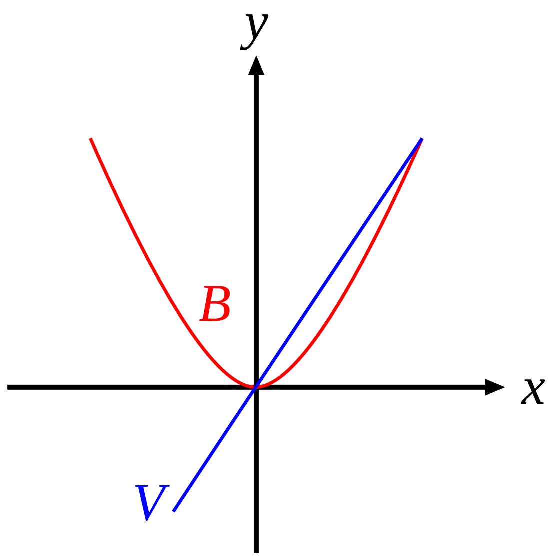
S. Chandrasekhar, Proc. Natl. Acad. Sci. USA **46**, 253 (1960)

S. A. Balbus & J. F. Hawley, Astrophys. J. **376**, 214 (1991)

earlier works

Stern [a] had first pointed out that separately stable flow and field can be combined to give instability. Kent [b] and Chen & Morrison [c] had shown several possible profiles by means of symmetry argument.





The properties of this instability is not known yet including growth rates, eigen-function structure and its nonlinear development.

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[a] M. E. Stern, Phys. Fluids **6**, 636 (1963).

[b] A. Kent, Phys. Fluids **9**, 1286 (1966); J. Plasma Phys. **2**, 543 (1968).

[c] X. L. Chen & P. J. Morrison, Phys. Fluids B **3**, 863 (1991).

Why is this important?

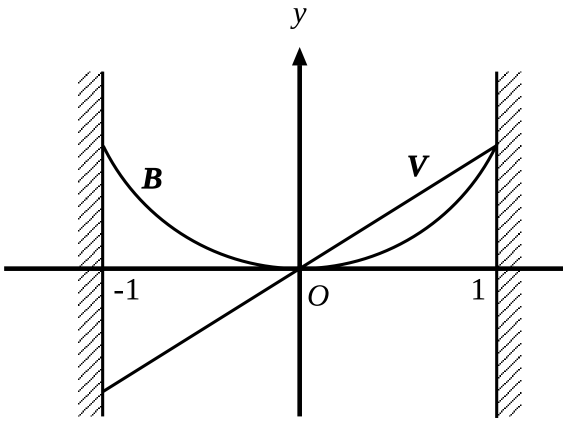
Various applications

1. parasitic instability on MRI [a]
   stop mechanism of nonlinear MRI mode
2. solar tachocline [b]
   possible source of anisotropic turbulence
3. tokamak q-minimum surface
   shear flow around the internal transport barrier

[a] J. Goodman & G. Xu, Astrophys. J. **432**, 213 (1994).
[b] P. A. Gilman & P. A. Fox, Astrophys. J. **484**, 439 (1997).

model

Suppose an incompressible ideal slab plasma in the finite domain.



$\boxed{\text{Equilibrium}}$ (prime denotes x-derivative)

1. inhomogeneous magnetic field:
$$\boldsymbol{B} = (0, B_y(x), B_z(x))$$
2. linear shear flow: $\boldsymbol{V} = (0, x, 0)$

$\boxed{\text{MHD equations}}$ ($\rho = \text{const}$)

$$(\partial_t + \boldsymbol{v} \cdot \nabla)\boldsymbol{v} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p$$
$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{0},$$
$$\nabla \cdot \boldsymbol{v} = 0, \qquad \nabla \cdot \boldsymbol{B} = 0.$$

$\boxed{\text{Normalization}}$: $L = L_x/2$, $V_{\mathrm{A}} = B_y^*/\sqrt{\mu_0 \rho}$; B_y^*: characteristic field.

ODE and condition for (exponential) stability

The ODE is ($\varphi = v_x/\mathit{\Omega}$, $\mathit{\Omega} = \omega - \boldsymbol{k}\cdot\boldsymbol{V}$, $F = \boldsymbol{k}\cdot B$)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\mathit{\Omega}^2 - F^2\right)\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right] - (k_y^2 + k_z^2)\left(\mathit{\Omega}^2 - F^2\right)\varphi = 0, \tag{6}$$

By multiplying $\bar{\varphi}$ and integrating,

$$A\omega^2 - 2B\omega + C = 0, \tag{7}$$

where $[\nabla\varphi = (\mathrm{d}\varphi/\mathrm{d}x, \mathrm{i}k_y\varphi, \mathrm{i}k_z\varphi)]$

$$A = \int |\nabla\varphi|^2\,\mathrm{d}x, \quad B = \int (\boldsymbol{k}\cdot\boldsymbol{V})|\nabla\varphi|^2\,\mathrm{d}x, \quad C = \int [(\boldsymbol{k}\cdot\boldsymbol{V})^2 - F^2]|\nabla\varphi|^2\,\mathrm{d}x. \tag{8}$$

Sufficient condition for stability: $C \leq 0$

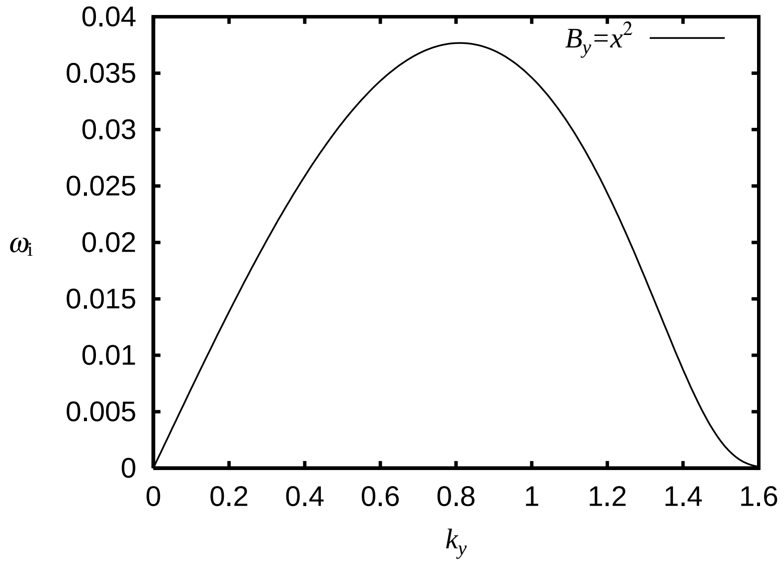
$$\boldsymbol{k}\cdot\boldsymbol{V} \leq F\ (= \boldsymbol{k}\cdot\boldsymbol{B}) \quad (\forall x) \quad \text{in any inertial frame.} \tag{9}$$

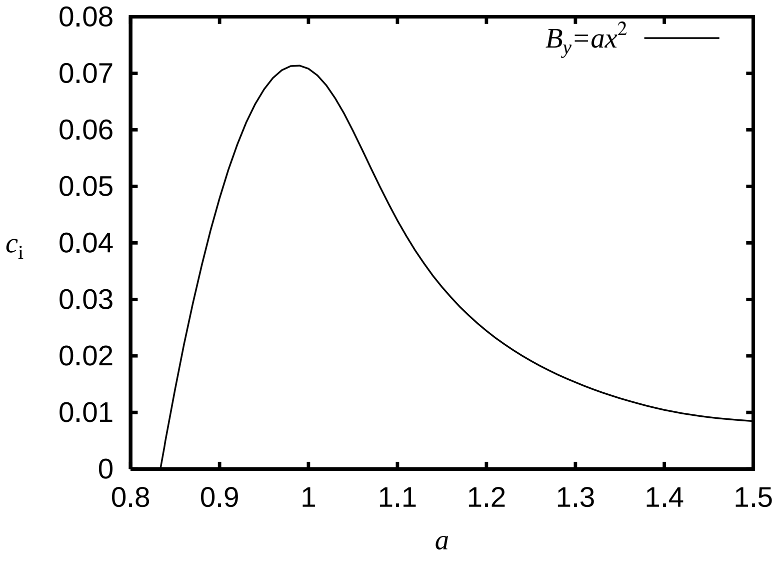
E. Frieman & M. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960);
A. Miura & P. L. Pritchett, J. Geophys. Res. **87**, 7431 (1982).

linear flow and parabolic field

Assume [a]

$$\boldsymbol{V} = (0, x, 0), \quad \boldsymbol{B} = (0, ax^2, 0). \tag{10}$$





The critical value of $a \sim 0.834$ agrees with analytic result obtained by Chen & Morrison.

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[a] X. L. Chen & P. J. Morrison, Phys. Fluids B **3**, 863 (1991).

tokamak $q_{\rm min}$ surface

Let's assume

$$q = x^2 + q_{\rm min} \quad \left(= \frac{rB_T}{RB_P} \right), \quad (11)$$

with

$$r_0/R = 5, \quad q_{\rm min} = 1.7, \quad (12)$$

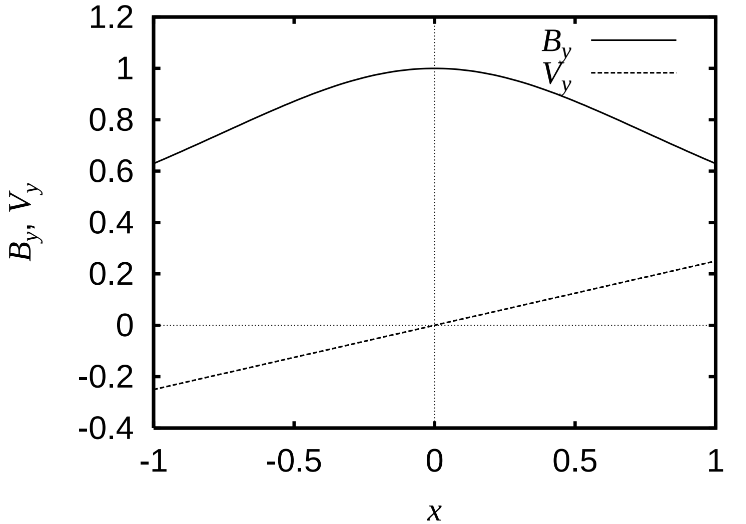
$$(m,n) = (2,1), \quad k_y = 1 \quad (13)$$

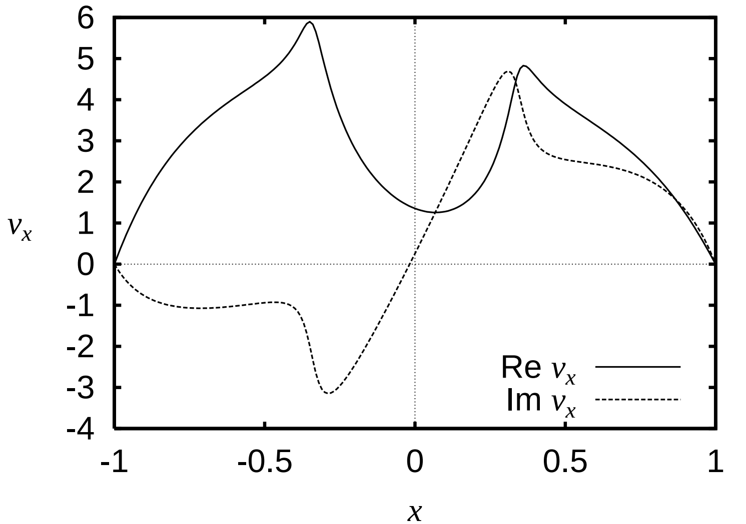
then

$$V_y = 0.25x \quad (14)$$

can cause instability:

$$\omega_{\rm i} = 0.021. \quad (15)$$





summary

We have numerically obtained the growth rate and eigenfunction of Magneto-Flow Instability in the incompressible ideal plasmas.

- shear flow can destabilize a certain field profile without inflection point
- the variation of the poloidal Alfvén velocity must be comparable to that of shear flow
- eigenfunction has peaks around which $V_y \sim |F/k_y|$ suggesting resonant property of this instability

We have developped a two-dimensional nonlinear pseudospectral code with boundary conditions

- observed linear growth
- instability nonlinearly quenched due to reform of backgraound profiles
- nonlinearly turbulent but weak

2d simulation

Two-dimensional pseudospectral code is developed and used which solves:

$$\partial_t \Delta\phi + \{\phi, \Delta\phi\} - \{\psi, \Delta\psi\} = \nu\Delta^2\phi, \tag{16}$$

$$\partial_t \psi + \{\phi, \psi\} = \eta\Delta\psi, \tag{17}$$

where $\{P, Q\} = (\partial_y P)(\partial_x Q) - (\partial_x P)(\partial_y Q)$ is the standard Poisson bracket.

$\boxed{\text{Boundary conditions}}$

$$\left.\begin{array}{ll} \phi = 0 & \text{(no penetration)} \\ v_y = \text{const} & \text{(no slip)} \\ \psi = \text{const} & \text{(ideal conductor)} \end{array}\right\} \text{ at } x = \pm 1, \tag{18}$$

all variables periodic in y where length is L_y.

2d simulation (our scheme)

- basis functions [a]

  Fourier in y + Chebyshev in x $[T_n(\cos\theta) = \cos n\theta]$: FFT possible

  2/3 rule

- time integration

  adaptive AB3 (convective) + Crank-Nicholson (diffusion)

- integration preconditioner [b]

  convert upper triangular differentiation matrix into tridiagonal in terms of 3-term integral formula

- boundary conditions [c]

  convert v_y condition to numerical ones on $\Delta\phi$: spectrally accurate

- projection

  project initial condition such that it meets boundary condition

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[a] J. P. Boyd, *Chebyshev and Fourier Spectral Method* (Dover, 2000).

[b] E. A. Coutsias *et al.*, Houston J. Math. (1996).

[c] E. A. Coutsias & J. P. Lynov, Physica D **51**, 482 (1991).

nonlinear simulation 1

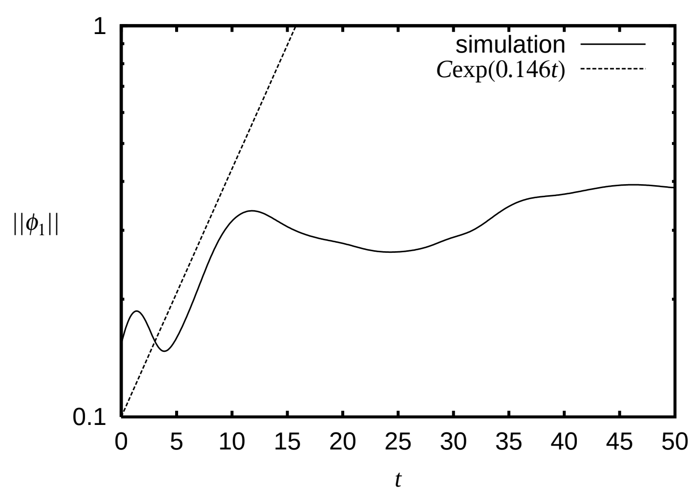
We made a 512×512 grids simulation ($\nu = \eta = 10^{-3}$) with the initial perturbation:

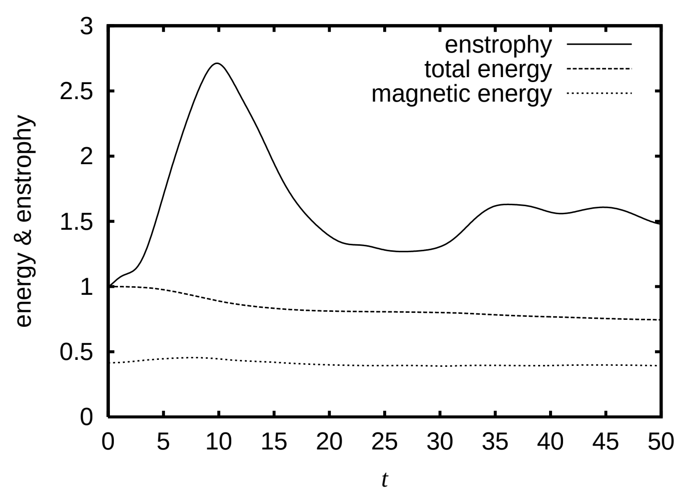
$$\phi_1 = 0.1 \sin\left(\frac{2\pi(x+y)}{L_y}\right)\cos^2\left(\frac{\pi x}{2}\right), \tag{19}$$

where backgraound field is given by

$$B_y = \frac{1}{2}x^2 - \frac{1}{\cosh^2(2x)} + \frac{1}{2}, \tag{20}$$

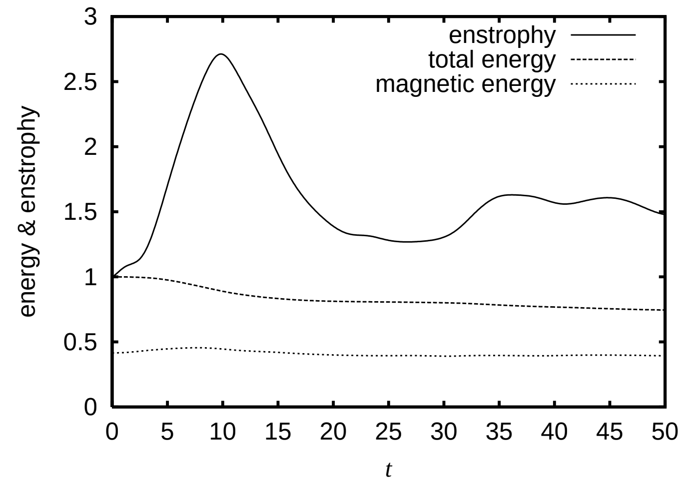
so that resonant surface ($F = 0$) enters in the domain.

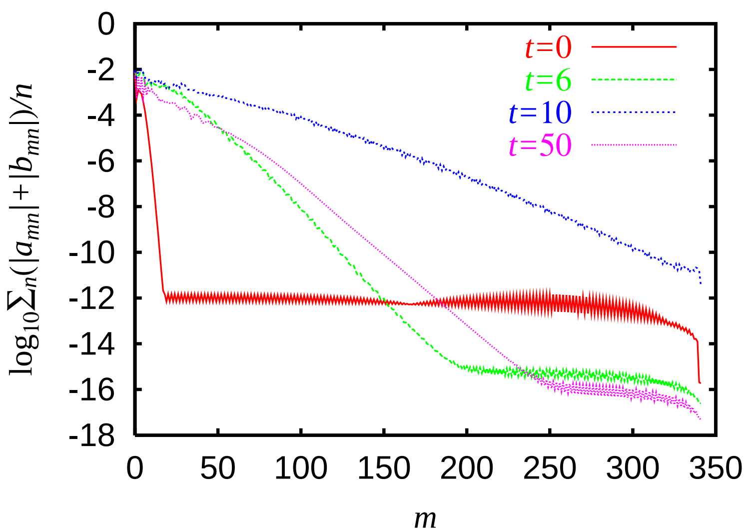


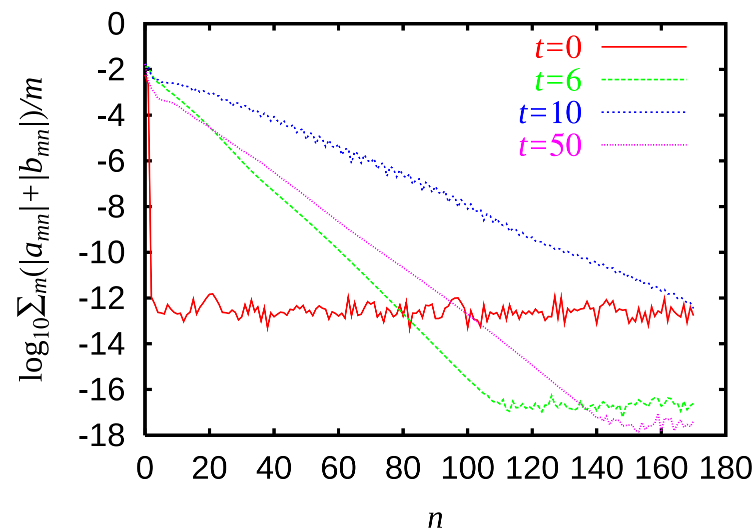


nonlinear simulation 2

$$\Delta\phi = \sum_{m,n} \left[a_{mn}\cos(k_n y) + b_{mn}\sin(k_n y)\right] T_m(x)$$

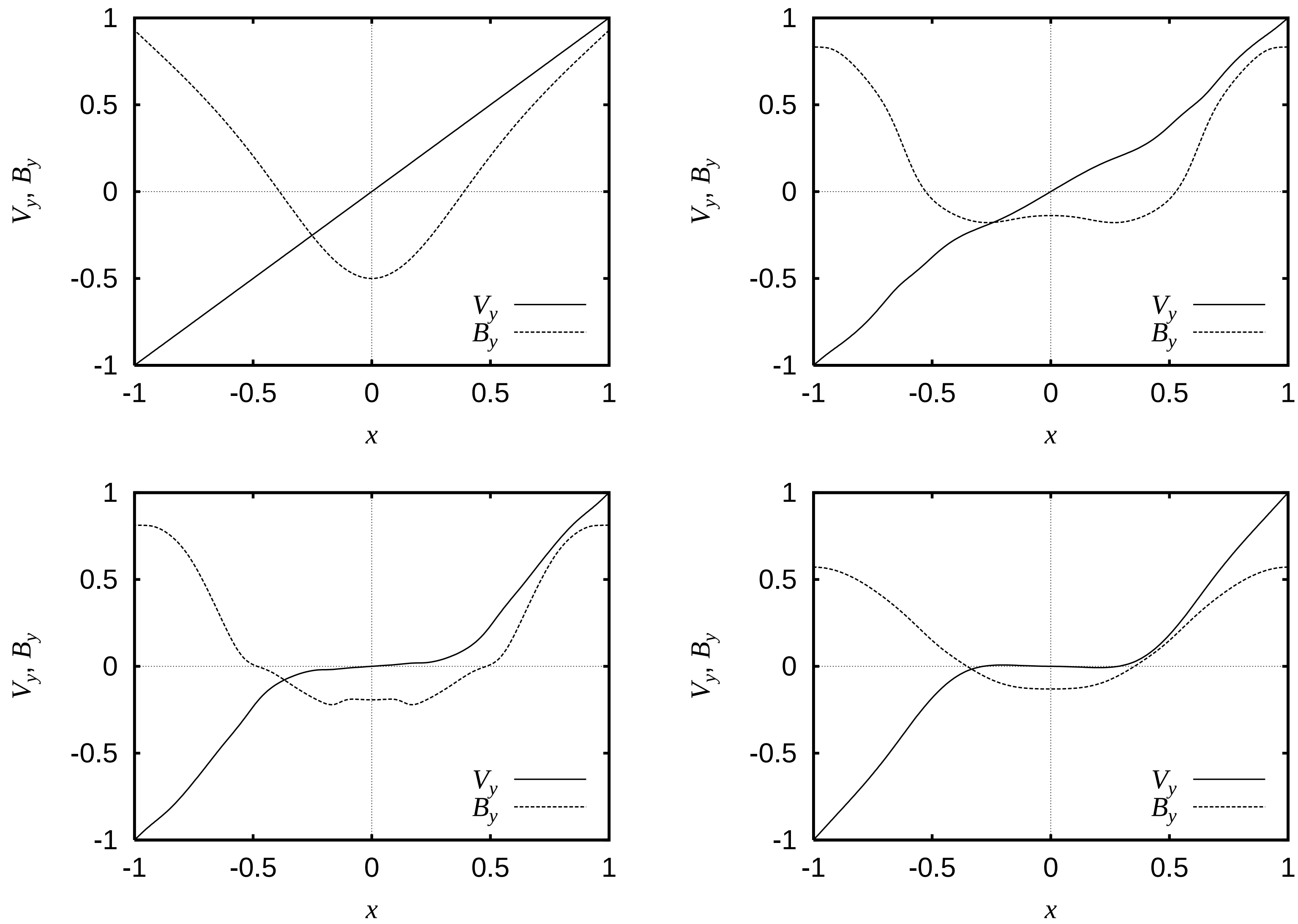






nonlinear simulation 3

Time evolution of the $k_y = 0$ component of V_y and B_y at $t = 0$, 6, 10, and 50.



Magneto-Rotational Instability 1

Stability analysis for axisymmetric perturbations in

- Chandrasekhar (Proc. Natl. Acad. Sci., 1960)

$$\boldsymbol{B} = (0, 0, B_z), \quad \boldsymbol{V} = (0, V_\theta(r), 0) \tag{21}$$

- Balbus & Hawley (Astrophys. J., 1991)

$$\boldsymbol{B} = (0, B_\theta(r, z), B_z(r)), \quad \boldsymbol{V} = (0, V_\theta(r), 0) \tag{22}$$

Both are for $\boldsymbol{k} \cdot \boldsymbol{V} = 0$.
The non-Hermitian (non-self-adjoint) contribution of convective derivative is eliminated.

$$\partial_t \psi + \boldsymbol{V} \cdot \nabla \psi = \cdots \tag{23}$$

Magneto-Rotational Instability 2

Rayleigh instability and MRI are local instabilities.

- Rayleigh's cylindrical system (Rayleigh, 1917) [$\Phi(r) = (r^2 V_\theta^2)'/r^3$]

$$\omega^2 \left\{ \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{1}{r} \frac{\mathrm{d}(ru)}{\mathrm{d}r} \right] - k^2 u \right\} + k^2 \Phi(r) u = 0. \tag{24}$$

By putting $\mathrm{d}/\mathrm{d}r = 0$,

$$\omega^2 - \Phi(r) = 0. \tag{25}$$

- MRI (Chandrasekhar, 1960) [$\Omega = V_\theta / r, \omega_\mathrm{A} = \boldsymbol{k} \cdot \boldsymbol{B} / \sqrt{\mu_0 \rho}$]

$$(\omega^2 - \omega_\mathrm{A}^2) \left\{ \frac{\mathrm{d}}{\mathrm{d}r} \left[\frac{1}{r} \frac{\mathrm{d}(ru)}{\mathrm{d}r} \right] - k^2 u \right\} + k^2 \left(\Phi(r) + \frac{4\Omega^2 \omega_\mathrm{A}^2}{\omega^2 - \omega_\mathrm{A}^2} \right) u = 0. \tag{26}$$

By putting $\mathrm{d}/\mathrm{d}r = 0$ (Balbus & Hawley, 1998),

$$\omega^4 - [2\omega_\mathrm{A}^2 + \Phi(r)]\omega^2 + \omega_\mathrm{A}^2 [\omega_\mathrm{A}^2 + \Phi(r) - 4\Omega^2] = 0. \tag{27}$$

ω^2 is real in both cases.

Magneto-Rotational Instability 3

Parasitic instability on MRI

MRI is a nonlinear solution to MHD and they analyzed the stability of this nonlinear solution, where $\boldsymbol{k} \cdot \boldsymbol{V} \neq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left([(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^2 - \omega_{\mathrm{A}}^2]\frac{\mathrm{d}\xi}{\mathrm{d}x}\right) - k^2[(\omega - \boldsymbol{k}\cdot\boldsymbol{V})^2 - \omega_{\mathrm{A}}^2]\xi = 0, \tag{28}$$

but MRI mode itself ($\boldsymbol{V}$, and ω_{A}) is sinusoidal in z: includes inflection point.

They also mentions about the possibility of second instability (non-KH), but this also seems to originate from the periodicity of nonlinear solution (Floquet's theory).

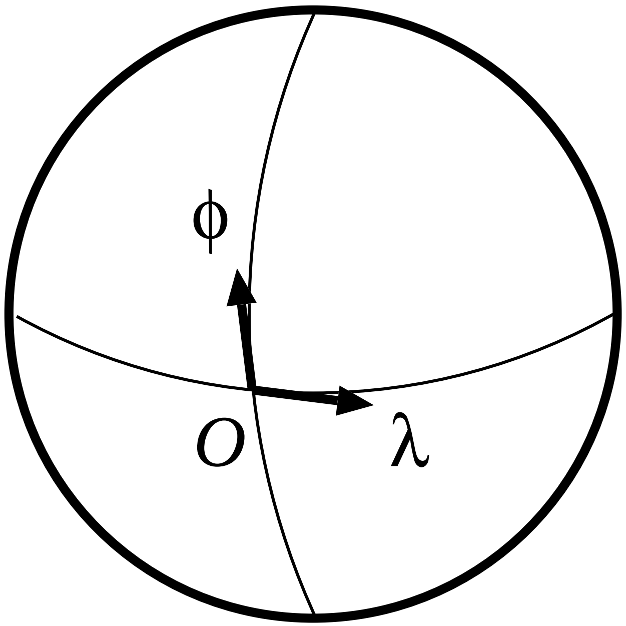
J. Goodman & G. Xu, Astrophys. J. **432**, 213 (1994).

solar tachocline

In a spherical surface coordinate (λ, ϕ),

$$\boldsymbol{V} = (\omega_0(\phi)\cos\phi,\ 0), \tag{29}$$

$$\boldsymbol{B} = (\alpha_0(\phi)\cos\phi,\ 0). \tag{30}$$



By taking $\omega_0 = r - s\sin^2\phi$ and $\alpha_0 = a\sin\phi$, where

$$r = 1, \quad 0.01 \leq s \leq 0.45, \quad 0 \leq a \leq 2 \tag{31}$$

they found instability of the form $\exp[im(\lambda - ct)]$.

- field and velocity are separately stable
- $\boldsymbol{k} \cdot \boldsymbol{V} \neq 0$

P. A. Gilman & P. A. Fox, Astrophys. J. **484**, 439 (1997).

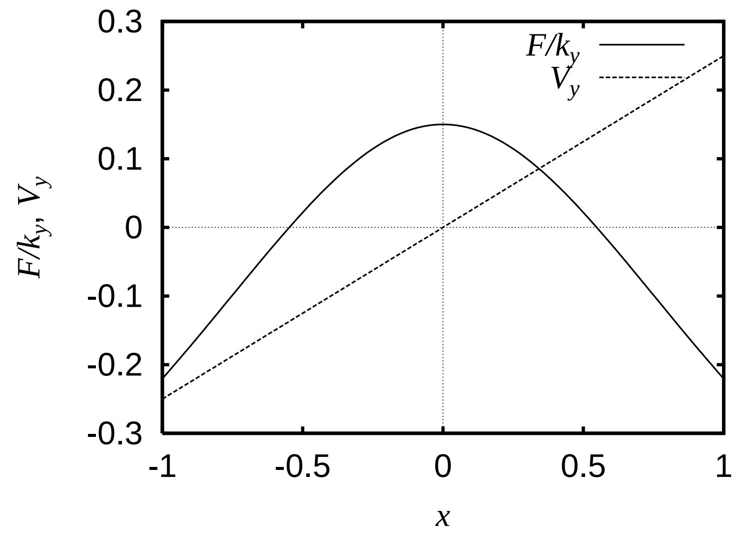
tokamak $q_{\min}$ surface 2

Sufficient condition for stability

$$\boldsymbol{k} \cdot \boldsymbol{V} \leq F \qquad (32)$$

where $F = \boldsymbol{k} \cdot \boldsymbol{B}$.

- B_y can be biased by strong B_z
- V_y may be weaker than both of B_y and B_z



Thus, in the case of slab model with constant B_z, $q_{\min}$ close to rational number, and as far as flow shear is localized, the stability condition may be violated when

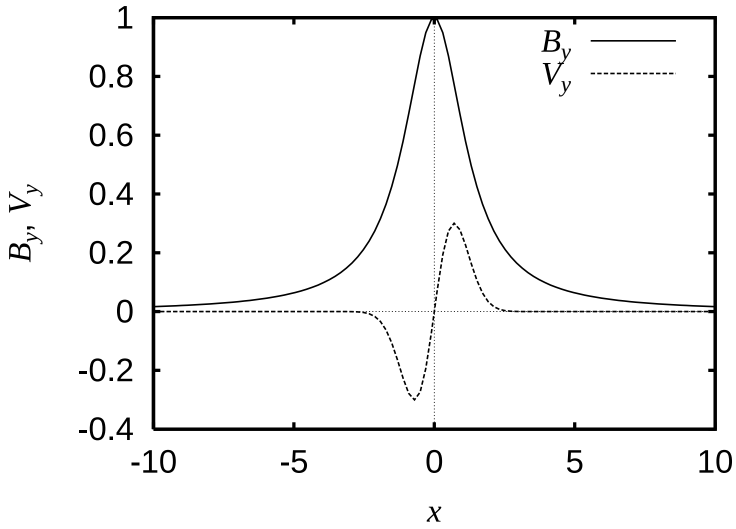
the difference of flow is comparable to the difference of poloidal Alfvén velocity

tokamak $q_{\min}$ surface 3

We extend our domain in order to simulate tokamak in more broad area:

$$B_y = \frac{q_{\min}}{x^2 + q_{\min}}, \tag{33}$$

$$V_y = -x \exp(-x^2). \tag{34}$$



Although $B_y > V_y$ is satisfied in all domain, we obtained an instability:

$$\omega_{\mathrm{i}} = 9.79 \times 10^{-3}. \tag{35}$$

Growth rate is rather small reflecting the small velocity which governs the instability time scale.