

Laboratory plasma dynamos, astrophysical dynamos and magnetic helicity evolution

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ABSTRACT

The term ‘dynamo’ means different things to the laboratory fusion plasma and astrophysical plasma communities. To alleviate the resulting confusion and to facilitate interdisciplinary progress, we pinpoint conceptual differences and similarities between laboratory plasma dynamos and astrophysical dynamos. We can divide dynamos into three types: 1. magnetically dominated helical dynamos which sustain a large-scale magnetic field against resistive decay and drive the magnetic geometry towards the lowest energy state, 2. flow-driven helical dynamos which amplify or sustain large-scale magnetic fields in an otherwise turbulent flow and 3. flow-driven non-helical dynamos which amplify fields on scales at or below the driving turbulence. We discuss how all three types occur in astrophysics whereas plasma confinement device dynamos are of the first type. Type 3 dynamos require no magnetic or kinetic helicity of any kind. Focusing on Types 1 and 2 dynamos, we show how different limits of a unified set of equations for magnetic helicity evolution reveal both types. We explicitly describe a steady-state example of a Type 1 dynamo, and three examples of Type 2 dynamos: (i) closed volume and time dependent; (ii) steady state with open boundaries; (iii) time dependent with open boundaries.

Key words: accretion, accretion discs – magnetic fields – MHD – methods: laboratory – Sun: coronal mass ejections (CMEs) – stars: coronae.

1 INTRODUCTION

1.1 General motivation

Dynamos describe the amplification and/or sustenance of magnetic fields in electrically conducting media, such as plasmas and liquid metals. For 50 yr, the detailed dynamics and mechanisms of dynamos have been subjects of active research in plasma physics, astrophysics, geophysics and non-linear dynamics. Historically, much dynamo research in these fields has progressed independently. Unsurprisingly, when different communities interact, confusion arises when the term ‘dynamo’ is used. This is partly because laboratory plasmas are typically magnetically dominated whereas the interiors of astrophysical rotators are typically flow dominated. Astrophysicists familiar only with dynamos in flow-dominated environments wonder what role a dynamo could possibly play in a magnetically dominated environment as they are used to thinking of a dynamo as a flow-driven amplification of an initially weak magnetic field.

Presently, there is timely motivation to alleviate this confusion and increased opportunity for interdisciplinary research. Flow-driven dynamos and magnetohydrodynamics (MHD) instabilities are being increasingly studied in liquid metals (Peffley et al. 2000; Ji, Goodman & Kageyama 2001; Gailitis et al. 2002, 2003; Noguchi et al. 2002; Sisan et al. 2004) to address some principles of traditional astrophysical and planetary dynamos, and the magnetically dominated dynamos of confinement plasmas are now realized to have direct analogies in astrophysical coronae (e.g. Ji et al. 2004; Blackman 2005).

Generally, it is important to clarify what is meant by a dynamo in each context so that all research communities can appreciate the common principles and differences. A particular unifying question for dynamo theories is to what extent are helical dynamos independent

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of the resistivity (or dissipation). While different answers to this question arise in different contexts, we aim to guide the reader towards understanding these different answers in a unified framework that draws from recent work on magnetic helicity evolution.

1.2 Distinguishing dynamo

Laboratory plasma dynamos arise in magnetically dominated conditions that have been studied in the context of fusion plasma confinement, such as the Reversed Field Pinch (RFP) configuration (Bodin & Newton 1990; Ji & Prager 2002). These dynamos describe how a large-scale magnetic configuration adjusts towards its relaxed state, in the presence of external driving away from the relaxed state by a magnetic field aligned electric field (or equivalently, external injection of one sign of magnetic helicity) (Strauss 1985, 1986; Bhattacharjee & Hameiri 1986; Holmes et al. 1988; Ortolani & Schnack 1993; Bhattacharjee & Yuan 1995; Gruzinov & Diamond 1995; Ji 1999; Bellan 2000; Ji & Prager 2002). The magnetic helicity injection actually drives the system away from the relaxed state, but also generates small amplitude fluctuation via kink mode instabilities from large currents, or tearing modes from large current gradients. The fluctuations produce a correlation between fluctuating velocity and magnetic fields – the turbulent electromotive force (EMF) $\overline{\mathcal{E}} = \langle \mathbf{v} \times \mathbf{b} \rangle$, which allows the system to evolve back towards a relaxed state. The turbulent EMF reduces the field-aligned current to restore stability by driving a spatial flow of magnetic helicity. This enables the magnetic field structure to evolve towards the largest helical scale available (subject to boundary conditions), as this is the lowest energy state (Taylor 1986). Continuous injection of magnetic helicity typically leads to a quasi-steady dynamical equilibrium with (sawtooth type) oscillations as the system is driven away from, and then evolves back towards the relaxed state. If the injection is turned off, the fully relaxed state can be reached, but the field eventually resistively decays. Via the dynamo, the injection therefore also sustains the large-scale helical field against decay.

In short, laboratory plasma dynamos involve: (i) external injection of one sign of magnetic helicity (ii) a change in the magnetic field structure with conversion of magnetic flux from toroidal to poloidal (or vice versa) geometries, (iii) an increase in the scale of the field as the scale of magnetic helicity increases in the relaxation phase and (iv) sustenance of helical field against dissipation. Boundary value studies have focused on all of the above, as well as the specific instabilities that drive the small amplitude fluctuations. Laboratory plasma dynamos involve both favourable and unfavourable features for plasma confinement; on the one hand, they sustain a large-scale field in an ordered configuration. However, to sustain this state, instabilities are required, which produce unwanted dissipation and heat transport.

The magnetically dominated dynamos of laboratory plasmas can be contrasted to the mean field helical dynamo originally applied inside of flow-dominated astrophysical rotators (Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin & Sokoloff 1983). The latter involves an initially weak large-scale field which is subsequently amplified via strong, large amplitude helical velocity fluctuations. (In this context, fluctuations and turbulence are used interchangeably.) That laboratory plasma dynamos involve small amplitude fluctuations around a strong large-scale field implies that high-order correlations of fluctuations can be justifiably ignored. In contrast, the fact that fluctuations typically dominate in a velocity-driven dynamo requires a more sophisticated closure for a rigorous theory because high-order correlations of fluctuations cannot be straightforwardly neglected.

An important feature of velocity-driven helical dynamos is that the large-scale field is amplified and sustained on scales significantly larger than the scale of the driving turbulence. This contrasts the magnetically driven laboratory plasma dynamo, where the fluctuations vary on time-scales short compared to the mean field evolution but can be large scale in space. The key similarity between the laboratory plasma dynamos and the flow-driven helical dynamos is that both thrive from a finite $\overline{\mathcal{E}} \cdot \overline{\mathbf{B}}$. This quantity drives the large-scale helical field relaxation in the laboratory plasma dynamo and amplifies large-scale helical fields in the flow-driven case. For the latter, the source of $\overline{\mathcal{E}}_{\parallel}$ (where \parallel indicates along $\overline{\mathbf{B}}$) typically, though not exclusively, is the kinetic helicity, $\langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle$, a pseudo-scalar correlation arising from the interplay between stratified turbulence and rotation. For a real system with an outward decreasing density, this can be sustained as rising eddies expand and rotate oppositely to the underlying mean rotation to conserve angular momentum. Falling eddies rotate in the same direction as the mean rotation. Both rising and falling eddies thus statistically contribute the same sign of kinetic helicity (or twist) $\langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle$. The Northern and Southern hemispheres have opposite signs of this helicity. Inside astrophysical rotators, the ultimate source of energy for the turbulence is typically convection or differential rotation. The magnetic field is not dominant therein, but the turbulence which facilitates the large-scale field growth is necessarily accompanied by magnetic fluctuations with energy density comparable to that in the velocity fluctuations. Recently, boundary terms, helicity fluxes and anisotropic contributions from mean velocity flows have been considered as additional contributions to the EMF driving the velocity-driven helical dynamo. (e.g. Blackman & Field 2000a; Vishniac & Cho 2001; Rädler, Kleeorin & Rogachevskii 2003; Subramanian & Brandenburg 2004; Brandenburg & Subramanian 2005c).

The growth of small-scale magnetic fluctuations in the velocity-driven dynamo environments of astrophysical rotators also highlights another important concept that distinguishes flow-driven dynamos from the magnetically driven dynamos in laboratory plasmas; laboratory dynamos always involve helicity, whereas both helical and non-helical flow-driven dynamos exist. Non-helical dynamos (Kazansev 1968; Schekochihin et al. 2002, 2004; Haugen, Brandenburg & Dobler 2003; Haugen & Brandenburg 2004; Maron, Cowley & McWilliams 2004) do not involve a mean turbulent EMF, just a turbulent velocity which amplifies magnetic energy via random walk line stretching and shear. Non-helical and helical flow-driven dynamos also differ in that the former amplifies magnetic energy only up to the input driving scale, whereas the latter can amplify fields on even larger scales, as needed to explain observed large-scale dynamo cycle periods in astrophysical objects. An important complication is that small-scale dynamos can operate concurrently with the large-scale helical dynamo, and the effect of the small-scale field growth on the large-scale dynamo has been a subject of considerable research.

While astrophysical dynamos are typically thought of as flow dominated, it is important to note that coronae above astrophysical rotators, such as stars and accretion discs, are likely magnetically dominated (e.g. Galeev, Rosner & Vaiana 1979; Field & Rogers 1993; Haardt & Maraschi 1993; Schrijver & Zwaan 2000). Therefore, astrophysical coronae are in fact sites for magnetically dominated dynamos driven by helical field injection from the astrophysical rotator below (Blackman & Field 2004; Blackman 2005). For all extra-terrestrial astrophysical rotators except galaxies, we observe at most the coronal fields, not the interior field. In the Sun, stars and accretion engines, there is evidence for the presence of large-scale coronal magnetic fields. For example, the coronal holes of the Sun are sites of large-scale ‘open’ field lines along which the solar wind propagates (e.g. Schrijver & Zwaan 2000). The reversal in sign of these fields indicates that they need to be regenerated every 11 yr. Analogously, the jets from accretion engines, such as young stellar objects, active galactic nuclei (AGNs) and even gamma-ray bursts, are likely magnetically mediated either from fling-type models (see e.g. Pudritz 2004) or from magnetic towers (e.g. Lynden-Bell 2003; Uzdensky & MacFadyen 2006). The needed large-scale coronal fields can be produced by the opening up of smaller scale loops from within the rotator below, as seen in the Sun (Wang & Sheeley 2003). These circumstances reveal that magnetically driven helical dynamos, in addition to the flow-driven helical dynamo, also play an important role in astrophysical magnetic field evolution. Indeed, the most direct analogy of laboratory plasma dynamos in astrophysics is the magnetic field relaxation in astrophysical coronae subject to helicity injection at the base. The injection drives the system away from the relaxed state but instabilities arise, allowing the system to relax. Steady injection would lead to a quasi-steady dynamical equilibrium.

Recent progress towards understanding helical dynamos has resulted from a combination of numerical and analytic work that dynamically incorporates magnetic helicity evolution (Pouquet, Frisch & Léorat 1976; Kleorin & Ruzmaikin 1982; Ji 1999; Kleorin & Rogachevskii 1999; Blackman & Field 2000a; Brandenburg 2001; Rogachevskii & Kleorin 2001; Blackman & Brandenburg 2002, 2003; Blackman & Field 2002; Field & Blackman 2002; Kleorin et al. 2002; Maron & Blackman 2002; Vishniac & Cho 2001; Blackman 2003; Subramanian & Brandenburg 2004; Brandenburg & Subramanian 2005b). However, identification of common principles for laboratory plasma dynamos, helical flow-driven dynamos and magnetically dominated coronal relaxation has been obscured by the particular approximations made or by the details of the application. The purpose of this paper is to show how a range of different dynamo theories incorporates magnetic helicity conservation and how they can be seen to represent different cases of a unified framework. We derive the simplest version of the equations representative of a particular type of helical dynamo. We purposely do not present complete analyses of the solutions (see Brandenburg & Subramanian 2005b, for a review); the main goal here is to identify where the equations fit into the big picture.

In Section 2, we derive the general magnetic helicity density evolution equations for a two-scale system including time dependent and boundary terms. In Section 3, we discuss how specific cases of these equations correspond to four specific dynamo circumstances: (i) a steady-state, magnetically driven laboratory plasma dynamo, (ii) a time-dependent closed flow-driven dynamo, (iii) a steady-state open flow-driven dynamo and its implications for subsequent magnetically driven coronal magnetic relaxation (iv) an open time-dependent flow-driven dynamo. In Section 4, we discuss insights about quenching and steady states that emerge from the unified framework. We conclude in Section 5.

2 UNIFYING EQUATIONS FOR HELICAL DYNAMOS

For completeness, we derive the needed magnetic helicity evolution equations and consolidate them into a useful form for all subsequent sections of this paper (see also e.g. Bellan 2000).

We start with the electric field

$$\mathbf{E} = -\nabla\Phi - \partial_t \mathbf{A}, \quad (1)$$

where Φ and \mathbf{A} are the scalar and vector potentials, respectively. Taking the average (spatial, temporal or ensemble) and denoting averaged quantities by an overbar, we obtain

$$\overline{\mathbf{E}} = -\nabla\overline{\Phi} - \partial_t \overline{\mathbf{A}}. \quad (2)$$

Subtracting equation (2) from equation (1) gives the equation for the fluctuating electric field

$$\mathbf{e} = -\nabla\phi - \partial_t \mathbf{a}, \quad (3)$$

where ϕ and \mathbf{a} are the fluctuating scalar and vector potentials, respectively.

Now, using $\mathbf{B} \cdot \partial_t \mathbf{A} = \partial_t(\mathbf{A} \cdot \mathbf{B}) + \mathbf{E} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{E})$, where the latter two terms result from using Maxwell’s equation $\partial_t \mathbf{B} = -\nabla \times \mathbf{E}$ and the vector identity $\mathbf{A} \cdot \nabla \times \mathbf{E} = \mathbf{E} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{E})$, we take the dot product of equation (1) with \mathbf{B} to obtain

$$\partial_t(\mathbf{A} \cdot \mathbf{B}) = -2(\mathbf{E} \cdot \mathbf{B}) - \nabla \cdot (\Phi \mathbf{B} + \mathbf{E} \times \mathbf{A}) = -2(\mathbf{E} \cdot \mathbf{B}) - \nabla \cdot (2\Phi \mathbf{B} + \mathbf{A} \times \partial_t \mathbf{A}). \quad (4)$$

Equation (4) describes the time evolution of magnetic helicity density. The same procedure used to derive equation (4) can be used, after dotting equations (2) and (3) with $\overline{\mathbf{B}}$ and \mathbf{b} , respectively, to obtain equations for the time evolution of the mean magnetic helicity density

$$\partial_t(\overline{\mathbf{A}} \cdot \overline{\mathbf{B}}) = -2\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} - \nabla \cdot (\overline{\Phi} \overline{\mathbf{B}} + \overline{\mathbf{E}} \times \overline{\mathbf{A}}) = -2\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} - \nabla \cdot (2\overline{\Phi} \overline{\mathbf{B}} + \overline{\mathbf{A}} \times \partial_t \overline{\mathbf{A}}), \quad (5)$$

and fluctuating magnetic helicity density

$$\partial_t(\mathbf{a} \cdot \mathbf{b}) = -2\mathbf{e} \cdot \mathbf{b} - \nabla \cdot (\phi \mathbf{b} + \mathbf{e} \times \mathbf{a}) = -2\mathbf{e} \cdot \mathbf{b} - \nabla \cdot (2\phi \mathbf{b} + \mathbf{a} \times \partial_t \mathbf{a}). \quad (6)$$

We can eliminate the electric fields from equations (4)–(6) by using Ohm’s law. Here, we consider the basic Ohm’s law with only a resistive term. For the total electric field, we have

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{J}. \quad (7)$$

Taking the average gives the mean field Ohm’s law

$$\overline{\mathbf{E}} = -\overline{\mathbf{E}} - \overline{\mathbf{V}} \times \overline{\mathbf{B}} + \eta \overline{\mathbf{J}}, \quad (8)$$

where $\overline{\mathbf{E}} \equiv \overline{\mathbf{v} \times \mathbf{b}}$ is the turbulent EMF. Subtracting equation (8) from equation (7) gives Ohm’s law for the fluctuating field

$$\mathbf{e} = \overline{\mathbf{E}} - \mathbf{v} \times \mathbf{b} - \mathbf{v} \times \overline{\mathbf{B}} - \overline{\mathbf{V}} \times \mathbf{b} + \eta \mathbf{j}. \quad (9)$$

Plugging equation (7) into equation (4) gives

$$\partial_t(\mathbf{A} \cdot \mathbf{B}) = -2\eta(\mathbf{J} \cdot \mathbf{B}) - \nabla \cdot (\Phi \mathbf{B} + \mathbf{E} \times \mathbf{A}) = -2\eta(\mathbf{J} \cdot \mathbf{B}) - \nabla \cdot (2\Phi \mathbf{B} + \mathbf{A} \times \partial_t \mathbf{A}). \quad (10)$$

Plugging equation (8) into equation (5) and equation (9) into equation (6) gives, respectively,

$$\partial_t(\overline{\mathbf{A}} \cdot \overline{\mathbf{B}}) = 2\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} - 2\eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} - \nabla \cdot (\Phi \overline{\mathbf{B}} + \overline{\mathbf{E}} \times \overline{\mathbf{A}}) = 2\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} - 2\eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} - \nabla \cdot (2\Phi \overline{\mathbf{B}} + \overline{\mathbf{A}} \times \partial_t \overline{\mathbf{A}}) \quad (11)$$

and fluctuating magnetic helicity

$$\partial_t \overline{\mathbf{a}} \cdot \overline{\mathbf{b}} = -2\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} - 2\eta \overline{\mathbf{j}} \cdot \overline{\mathbf{b}} - \nabla \cdot (\overline{\phi \mathbf{b}} + \overline{\mathbf{e} \times \mathbf{a}}) = -2\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} - 2\eta \overline{\mathbf{j}} \cdot \overline{\mathbf{b}} - \nabla \cdot (2\overline{\phi \mathbf{b}} + \overline{\mathbf{a} \times \partial_t \mathbf{a}}). \quad (12)$$

Dotting equation (9) with \mathbf{b} and averaging reveals the important relation

$$\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} = \overline{\mathbf{e} \cdot \mathbf{b}} - \eta \overline{\mathbf{j}} \cdot \overline{\mathbf{b}}. \quad (13)$$

All helical dynamos require a finite $\overline{\mathbf{E}} \cdot \overline{\mathbf{B}}$. Equations (11)–(13) are the key equations to be used in subsequent sections.

3 EXAMPLES OF DYNAMOS FROM MAGNETIC HELICITY EVOLUTION

Here, we show how different types of dynamos can be understood as limiting cases of the equations in the previous section.

3.1 Steady state, magnetically dominated laboratory plasma dynamo

In this section, we work in the context of a torus. We take mean quantities to be time averages and spatial averages over periodic directions ϕ (locally $\hat{\mathbf{z}}$) and θ , but not over radius r (where $r = 0$ corresponds to an azimuthal ring at the centre of the torus’ cross-section). While realistic RFPs and Tokamaks are time dependent, to illustrate the relevant dynamo simply, we assume mean quantities are steady, and that fluctuations vary on time-scales much less than the averaging time. For RFPs, sawtooth oscillations and crashes occur over millisecond time-scales and fluctuations occur in 100 μs or shorter (e.g. Ortolani & Schnack 1993; Ji & Prager 2002). Quantities averaged over $\gtrsim 10$ ms can be approximated as steady. We write the steady-state limit of equation (12) as

$$\overline{\mathcal{E}}_{\parallel} = \frac{\overline{\mathbf{B}}}{B^2} (\nabla \cdot \overline{\mathbf{h}} - \eta \overline{\mathbf{j}} \cdot \overline{\mathbf{b}}), \quad (14)$$

where $\overline{\mathbf{h}} \equiv -\overline{\phi \mathbf{b}} + \frac{1}{2} \overline{\mathbf{a} \times \partial_t \mathbf{a}}$. Dotting equation (8) with $\overline{\mathbf{B}}$, and using equation (8), gives

$$\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} = \nabla \cdot \overline{\mathbf{h}} - \eta \overline{\mathbf{j}} \cdot \overline{\mathbf{b}} = \eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} - \overline{\mathbf{E}} \cdot \overline{\mathbf{B}}, \quad (15)$$

where, from equation (5), we also have

$$\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} = -\nabla \cdot \left(\Phi \overline{\mathbf{B}} + \frac{1}{2} \overline{\mathbf{A}} \times \partial_t \overline{\mathbf{A}} \right). \quad (16)$$

The dynamo effect in laboratory plasma configurations, such as an RFP, emerges when a large-scale electric field $\overline{\mathbf{E}}$ is externally applied along the initial toroidal magnetic field (this represents helicity injection usually via the divergence term, as emphasized below equation 18). Were there no induced EMF $\overline{\mathcal{E}}$, the measured current term on the right-hand side of equation (15) would have to balance the applied large-scale electric field along the magnetic field $\overline{\mathbf{E}}_{\parallel}$ in a steady state. However, for sufficiently large applied $\overline{\mathbf{E}}_{\parallel}$, RFP experiments reveal (Caramana & Baker 1984; Bodin & Newton 1990; Ji et al. 1994; Ji & Prager 2002) that $\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} = \eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} > 0$ only at a single radius $0 < r = r_c < a$, where a is the minor radius of the torus and r_c is measured from the toroidal axis. For $r < r_c$, $\overline{\mathbf{E}} \cdot \overline{\mathbf{B}} > \eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} > 0$ and for $r > r_c$, $\eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} > 0 > \overline{\mathbf{E}} \cdot \overline{\mathbf{B}}$. Excluding pressure gradient and inertial terms in Ohm’s law, such measurements imply that $\overline{\mathcal{E}}_{\parallel} \neq 0$. Moreover, since $\eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} - \overline{\mathbf{E}} \cdot \overline{\mathbf{B}}$ changes sign from negative to positive moving outwards through r_c (while $\overline{\mathbf{J}} \cdot \overline{\mathbf{B}}$ keeps the same sign), $\overline{\mathcal{E}}_{\parallel}$ must also change sign from negative to positive across $r = r_c$. Because the third term in equation (14) is often negligible, equation (14) shows that the divergence of the small-scale helicity flux $\overline{\mathbf{h}}$ must change sign through r_c . In the RFP, the presence of $\overline{\mathcal{E}}_{\parallel}$ is sustained by fluctuations induced by tearing or kink-mode instabilities when the applied $|\overline{\mathbf{E}}_{\parallel}|$ exceeds a critical value.

Because averaged quantities of this subsection remain functions of r , the steady dynamo just described operates locally in r . Taking the volume integral of equation (15) and using equation (16) gives

$$\int \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} dV = \int \overline{\mathbf{h}} \cdot d\mathbf{S} = \int (\eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} - \overline{\mathbf{E}} \cdot \overline{\mathbf{B}}) dV = \int \eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} dV + \int \left(\Phi \overline{\mathbf{B}} + \frac{1}{2} \overline{\mathbf{A}} \times \partial_t \overline{\mathbf{A}} \right) \cdot d\mathbf{S}, \quad (17)$$

where we have dropped the third term of equation (14) as it is typically negligible, and used Gauss' theorem to convert the divergence integrals to a surface integrals, keeping in mind that for doubly connected topologies we must make sure that $\bar{\mathbf{h}}$ is analytic everywhere. This is ensured because our averaged quantities depend only on radius. Accordingly, using vector identities, the second surface integral in equation (17) is

$$\int \left(\overline{\Phi \mathbf{B}} + \frac{1}{2} \overline{\mathbf{A}} \times \partial_t \overline{\mathbf{A}} \right) \cdot d\mathbf{S} = \int_s \overline{\Phi \mathbf{B}} \cdot d\mathbf{S} - \frac{1}{2} \int_s \overline{\mathbf{E}} \cdot d\mathbf{z} \int_s \overline{\mathbf{A}} \cdot r d\hat{\theta} = \int \overline{\Phi \mathbf{B}} \cdot d\mathbf{S} - \frac{1}{2} V_s \Psi_s, \quad (18)$$

where V_s is the externally applied voltage drop in the toroidal direction on the surface of integration (applied experimentally via gaps) and Φ_s is the toroidal magnetic flux within the surface. On the outer radial surface, there is no normal component of the field and the penultimate term would vanish for that surface. The latter term of equation (18) represents the helicity injection. This helicity injection is unihelical (i.e. one sign), which is important because the magnetically driven dynamo relaxation can be thought of as a process driving the injected magnetic helicity the largest scale available subject to the boundary conditions.

Because of the conducting boundaries, there is no net small-scale helicity flow through the torus. However, it is instructive to separately consider the inner core ($r < r_c$) and shell ($r > r_c$) regions (Ji & Prager 2002). The discussion below equation (16) implies that for $r < r_c$ the left-hand side of equation (17) must be negative. Therefore, the second term of equation (17) must also be negative, which from the definition of $\bar{\mathbf{h}}$, implies an outward flux of positive fluctuating magnetic helicity through r_c . Analogously, positive fluctuating helicity accumulates into the volume defined by $r > r_c$. The helicity flux through r_c therefore provides the local dynamo.

Instead of dotting $\overline{\mathcal{E}}_{\parallel}$ with $\overline{\mathbf{B}}$ (to get equation 15), the dynamo is sometimes expressed by dotting equation (8) with $\overline{\mathbf{J}}_{\parallel}$. Using equation (8) and $\Lambda \equiv \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} / \overline{B}^2$, this simply gives Λ times equation (15), so this offers nothing new beyond equation (17) regarding the local nature of the dynamo defined as a sustained $\overline{\mathcal{E}}_{\parallel}$. However, multiplying equation (14) by Λ and taking the volume integral, we obtain

$$\int \overline{\mathcal{E}}_{\parallel} \cdot \overline{\mathbf{J}} dV = \int \Lambda \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} dV = \int \Lambda (\nabla \cdot \bar{\mathbf{h}} - \eta \overline{\mathbf{J}} \cdot \bar{\mathbf{b}}) dV = \int \Lambda (\eta \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} dV - \overline{\mathcal{E}} \cdot \overline{\mathbf{B}}) dV. \quad (19)$$

Using

$$\int \Lambda \nabla \cdot \bar{\mathbf{h}} dV = \int (\Lambda \bar{\mathbf{h}}) \cdot d\mathbf{S} - \int \bar{\mathbf{h}} \cdot \nabla \Lambda dV, \quad (20)$$

ignoring the third term of equation (19) and eliminating Λ in favour of $\overline{\mathbf{J}}_{\parallel}$, we can rewrite equation (19) as

$$\int \overline{\mathcal{E}}_{\parallel} \cdot \overline{\mathbf{J}} dV = \int (\Lambda \bar{\mathbf{h}}) \cdot d\mathbf{S} - \int \bar{\mathbf{h}} \cdot \nabla \Lambda dV = \int (\eta \overline{\mathbf{J}}_{\parallel}^2 - \overline{\mathcal{E}} \cdot \overline{\mathbf{J}}_{\parallel}) dV, \quad (21)$$

where we have used Gauss' theorem to obtain the surface integral, just as described below equation (17). Here, too the only potentially non-trivial surface terms are radial surface terms. If we integrate over the full volume of the plasma (i.e. for all $r \leq a$), then the surface term in equation (21) vanishes, as $\bar{\mathbf{h}}$ is measured to vanish at $r = a$. However, equation (21) then shows that a finite $\nabla \Lambda$ can produce a {global} dynamo effect defined by $\int_{\text{tot}} \overline{\mathcal{E}}_{\parallel} \cdot \overline{\mathbf{J}} dV \simeq \int_{\text{tot}} \eta \overline{\mathbf{J}}_{\parallel}^2 dV$, where the subscript 'tot' indicates the full volume. The quantity $\bar{\mathbf{h}} \cdot \nabla \Lambda$ gradient need not be finite everywhere for such an effect, only a sufficiently non-zero $\nabla \cdot \bar{\mathbf{h}}$ to ensure finite $\overline{\mathcal{E}}_{\parallel}$ and sufficiently non-zero integral in the third term of equation (21) are required. For the quasi-local case, integration is taken over a subrange of radii and either of the two middle terms of equation (21) could dominate. Thus, $\int \bar{\mathbf{h}} \cdot \nabla \Lambda dV \neq 0$ emerges as a sufficient but not a necessary condition for a quasi-local dynamo. This is consistent with the actual calculation in Bhattacharjee & Hameiri (1986), but disagrees with Bellan (2000) as the latter drops the second term of equation (20).

For the global dynamo as defined below equation (21), consider a case in which the magnetic helicity associated with the mean field has a locally positive sign in the Coulomb gauge. Then Λ is also positive. If $\int_{\text{tot}} \eta \overline{\mathbf{J}}^2 dV$ is dominant on the right-hand side of equation (21), then equation (21) implies that the outward flux of positive small-scale helicity (represented by $\bar{\mathbf{h}} > 0$) is antiparallel to the direction of increasing Λ overall. Therefore, on average, the dynamo acts to homogenize the overall scale of magnetic helicity. This homogenization results because the magnetic helicity is injected with one sign, so the dynamo acts to drive the characteristic scale of magnetic twist to the largest scale available subject to the boundary conditions. Note that for the velocity-driven dynamo discussed later in Section 3.2, kinetic helicity rather than magnetic helicity is injected. There the dynamo acts instead to spectrally segregate magnetic helicity of opposite signs while largely preserving a net zero magnetic helicity. In general, laboratory plasma dynamos involve the injection and evolution of one sign of net magnetic helicity (unihelical) whilst kinetic helicity driven dynamos are bihelical.

The parallel component of the EMF can be written as $\overline{\mathcal{E}}_{\parallel} = \alpha \overline{\mathbf{B}}$. The pseudo-scalar α for the laboratory case is given from equations (13) and (15) by

$$\alpha = \frac{\overline{\mathcal{E}} \cdot \overline{\mathbf{B}}}{\overline{B}^2} \sim \frac{1}{\overline{B}^2} \nabla \cdot \bar{\mathbf{h}} = \frac{\overline{\mathbf{e}} \cdot \bar{\mathbf{b}}}{\overline{B}^2} \simeq \frac{\overline{\mathbf{e}_{\perp}} \cdot \bar{\mathbf{b}}_{\perp}}{\overline{B}^2}, \quad (22)$$

where the last similarity follows because the fluctuations are primarily perpendicular to the strong mean fields. The right-hand side of equation (22) is measured to be larger than any resistive contribution (Ji et al. 1994), and the values are consistent with dynamo models based on the principles described here. We note that the above descriptions can be extended to the case when global quantities are time dependent (Ji et al., in preparation), as required for more precise applications to the RFP.

3.2 Time dependent, closed, flow-driven helical dynamo

In this section, instead of considering the averages to be time averages, we consider global spatial averages over a closed or periodic volume such that the surface terms vanish but the mean quantities remain time dependent. For the laboratory plasma dynamo discussed in Section 3.1, the EMF arises via magnetic helicity injection through a boundary term. Here, kinetic helicity is assumed to be injected into the system and the boundary terms are not invoked. These are the circumstances considered in recent analytic work (Blackman & Field 2002) and numerical simulations (Brandenburg 2001). Another important difference between the laboratory plasma dynamo of the previous section and the flow-driven helical dynamo here is that the former involves weak fluctuations on a strong mean field, whereas the latter involves initially weak fields and strong fluctuations.

Following Blackman & Field (2002), we distinguish the global volume averages from the large-scale $k = 1$ quantities, by using brackets to indicate the former and an overbar to indicate the latter. For pseudo-scalars, we assume the two averages are equal (e.g. $\overline{\mathbf{a} \cdot \mathbf{b}} = \langle \mathbf{a} \cdot \mathbf{b} \rangle$). In this case, equations (11) and (12) become

$$\partial_t \langle \mathbf{a} \cdot \mathbf{b} \rangle = -2\langle \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \rangle - 2\eta \langle \mathbf{j} \cdot \mathbf{b} \rangle \quad (23)$$

and

$$\partial_t \langle \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \rangle = 2\langle \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \rangle - 2\eta \langle \overline{\mathbf{J}} \cdot \overline{\mathbf{B}} \rangle. \quad (24)$$

In the absence of boundary terms, the magnetic helicity is gauge invariant, but choosing the Coulomb gauge we can relate the current helicities to the magnetic helicities via $\langle \mathbf{J} \cdot \mathbf{B} \rangle = \langle k^2 \mathbf{A} \cdot \mathbf{B} \rangle$. To complete the set of equations to be solved, we need an equation for $\overline{\mathcal{E}}$. From its definition,

$$\partial_t \overline{\mathcal{E}} = \overline{\partial_t \mathbf{v} \times \mathbf{b}} + \overline{\mathbf{v} \times \partial_t \mathbf{b}}. \quad (25)$$

We now need equations for the fluctuating velocity and magnetic field $\partial_t \mathbf{b}$ and $\partial_t \mathbf{v}$. For $\nabla \cdot \mathbf{v} = 0$, we have from the induction and momentum density equations, respectively,

$$\partial_t \mathbf{b} = \overline{\mathbf{B}} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \overline{\mathbf{B}} + \nabla \times (\mathbf{v} \times \mathbf{b}) - \nabla \times \overline{\mathbf{v} \times \mathbf{b}} + \lambda \nabla^2 \mathbf{b}, \quad (26)$$

and

$$\partial_t v_q = P_{qi} (\overline{\mathbf{B}} \cdot \nabla b_i + \mathbf{b} \cdot \nabla \overline{B}_i - \mathbf{v} \cdot \nabla v_i + \overline{\mathbf{v} \cdot \nabla v_i} + \mathbf{b} \cdot \nabla b_i - \overline{\mathbf{b} \cdot \nabla b_i}) + \nu \nabla^2 v_q + f_q, \quad (27)$$

where f is a divergence-free forcing function uncorrelated with \mathbf{b} , ν is the viscosity and $P_{qi} \equiv (\delta_{qi} - \nabla^{-2} \nabla_q \nabla_i)$ is the projection operator that arises after taking the divergence of the incompressible momentum density equation to eliminate the fluctuating pressure (magnetic + thermal). Reynolds rules (Rädler 1980) allow the interchange of brackets and time or spatial derivatives, so the 5th term of equation (26) and the 4th and 6th terms in the parentheses of equation (27) do not contribute when put into averages and can be ignored.

The contribution to $\partial_t \overline{\mathcal{E}}$ from the 3rd term in equation (25) can be derived by direct use of equation (26) in configuration space. Following Blackman & Field (2002), we assume isotropy of the resulting velocity and magnetic field correlations for terms linear in $\overline{\mathbf{B}}$ and also retain the triple correlations. The contribution to $\partial_t \overline{\mathcal{E}}$ from the 2nd term in equation (25) also contributes terms linear in $\overline{\mathbf{B}}$, and triple correlations. Here, the terms linear in $\overline{\mathbf{B}}$ are best derived in Fourier space. For this, Gruzinov & Diamond (1995) invoke the Fourier transform of the terms linear in $\overline{\mathbf{B}}$ contributing to $\overline{\partial_t \mathbf{v} \times \mathbf{b}}$, supplemented by a linear expansion of the projection operator in $k_1 \ll 1k_2$, where k_1 is the characteristic wavenumber of the bracketed or mean quantities and k_2 is the characteristic wavenumber of the fluctuating quantities \mathbf{b} and \mathbf{v} . Collecting all surviving terms, we then have for equation (25)

$$\partial_t \overline{\mathcal{E}} = \frac{1}{3} (\overline{\mathbf{b} \cdot \nabla \times \mathbf{b}} - \overline{\mathbf{v} \cdot \nabla \times \mathbf{v}}) \overline{\mathbf{B}} - \frac{1}{3} \overline{v^2} \nabla \times \overline{\mathbf{B}} + \nu \overline{\nabla^2 \mathbf{v} \times \mathbf{b}} + \lambda \overline{\mathbf{v} \times \nabla^2 \mathbf{b}} + \mathbf{T}^M + \mathbf{T}^V, \quad (28)$$

where $\mathbf{T}^M = \overline{\mathbf{v} \times \nabla \times (\mathbf{v} \times \mathbf{b})}$ and $\mathbf{T}_j^V = (\epsilon_{jqn} \langle P_{qi} (\mathbf{b} \cdot \nabla b_i - \mathbf{v} \cdot \nabla v_i) b_n \rangle)$ are the triple correlations. Note that the 3rd, 4th, 6th and 8th terms in equation (28) come from the $\overline{\mathbf{v} \times \partial_t \mathbf{b}}$ term of equation (25) and the 2nd 5th and 7th terms come from the $\overline{\partial_t \mathbf{v} \times \mathbf{b}}$ term of equation (25). For $\overline{\mathcal{E}}_{||}$, we have

$$\partial_t \overline{\mathcal{E}}_{||} = (\overline{\partial_t \mathbf{v} \times \mathbf{b}} + \overline{\mathbf{v} \times \partial_t \mathbf{b}}) \cdot \overline{\mathbf{B}} / |\overline{\mathbf{B}}| + \overline{\mathbf{v} \times \mathbf{b}} \cdot \partial_t (\overline{\mathbf{B}} / |\overline{\mathbf{B}}|). \quad (29)$$

Substituting equation (28) into equation (29) gives

$$\partial_t \overline{\mathcal{E}}_{||} = \tilde{\alpha} \overline{\mathbf{B}^2} / |\overline{\mathbf{B}}| - \tilde{\beta} \overline{\mathbf{B}} \cdot \nabla \times \overline{\mathbf{B}} / |\overline{\mathbf{B}}| - \tilde{\zeta} \overline{\mathcal{E}}_{||}, \quad (30)$$

where $\tilde{\alpha} = (1/3) (\overline{\mathbf{b} \cdot \nabla \times \mathbf{b}} - \overline{\mathbf{v} \cdot \nabla \times \mathbf{v}})$, $\tilde{\beta} = (1/3) \overline{v^2}$ and $\tilde{\zeta}$ accounts for microphysical dissipation terms, the last term of equation (29) and $\mathbf{T}^M + \mathbf{T}^V \neq 0$.

The incorporation of the triple correlations via $\tilde{\zeta}$ was subsequently named the ‘minimal τ ’ closure (Blackman & Field 2003). In reality, $\tilde{\zeta}$ can be a function of wavenumber when spectral models are considered (Kleeorin, Mond & Rogachevskii 1996). However, the simple minimal τ closure is an improvement over the first-order smoothing approximation (Moffatt 1978; Parker 1979; Vishniac & Cho 2001) in which triple correlations are ignored, and simpler than the eddy damped quasi-normal Markovian closure (Pouquet et al. 1976). In general,

triple correlations cannot be ignored because flow-driven dynamos involve fluctuations which are generally not small compared to the mean magnetic field. Moreover, it is these very triple correlation terms which lead to a turbulent cascade, and the presence of a turbulent spectrum. The efficacy of the minimal τ closure used for passive scalar diffusion by Blackman & Field (2003) was verified numerically (Brandenburg, Käpylä & Mohammed 2004), and shows reasonable agreement with numerical tests in the MHD regime (Brandenburg & Subramanian 2005c).

The equations to be solved for the closed, helical flow-driven dynamo case here are equations (23), (24) and (29). The large-scale field which grows is fully helical, and so the magnetic energy and magnetic helicity equations are essentially the same. The resulting dynamo is a modern representation of the α^2 dynamo (e.g. Moffatt 1978) that incorporates the dynamical backreaction of the magnetic field on the kinetic helicity driving the flow and the conservation of magnetic helicity. Solutions of these three equations agree with simulations as described in Blackman & Field (2002). The essence of the dynamo growth is as follows: the initial system is assumed to be driven with a finite $\overline{\mathbf{v} \cdot \nabla \times \mathbf{v}} \simeq \langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle$. This grows a finite $\overline{\mathcal{E}}_{\parallel}$ which then grows large-scale magnetic helicity via equation (23). However, due to magnetic helicity conservation, as seen in equation (24), the small-scale magnetic helicity $\overline{\mathbf{a} \cdot \mathbf{b}}$ must then grow in opposite sign to that of the large scale. This also grows the small-scale current helicity $\overline{\mathbf{j} \cdot \mathbf{b}}$ of the same sign. This in turn quenches $\tilde{\alpha}$. The system reaches a steady state; if the decay of the large-scale helicity were to supersede growth, the small-scale helicity would also deplete, and growth of the large-scale field would again begin if $\overline{\mathbf{v} \cdot \nabla \times \mathbf{v}}$ is steadily driven.

In the traditional α^2 dynamo of the standard texts (Moffatt 1978), the equation for the large-scale field is solved, with an imposed form of the EMF and the linear growth equation results. Blackman & Brandenburg (2003) emphasize that neither magnetic helicity is conserved in the equations nor the diagrams of the dynamo in the standard texts. In the modern version just discussed, the large-scale field evolution equation was replaced by the large-scale magnetic helicity evolution equation and the additional time-dependent equations for small-scale helicity evolution and turbulent EMF evolution are coupled into the theory dynamically.

It should be noted that the two-scale analytic approach has been generalized to a four-scale approach (Blackman 2003) to assess whether the small-scale magnetic helicity tends towards the dissipation scale or towards the forcing scale (with the large-scale magnetic helicity migrating towards the even larger box scale). The analysis shows that the small-scale magnetic helicity first appears towards the resistive scale, but migrates towards the forcing scale before the end of the kinematic regime. Numerical simulations of helical dynamos in a periodic box (Brandenburg 2001; Maron & Blackman 2002) also show that the magnetic helicity in saturation peaks with opposite signs at the forcing scale and box scale, respectively. This is a bihelical dynamical equilibrium state in which the small- and large-scale magnetic helicities both migrate to the largest scales available to them. The driving kinetic helicity ensures that these two scales are distinct and prevents the small-scale magnetic helicity from migrating to the box scale.

The dynamical backreaction approach of this section can be generalized to an α - Ω dynamo. Then the large-scale magnetic helicity evolution equation must be replaced by the vector equation for the large-scale field itself, but the small scale and turbulent EMF equations are also included dynamically (Blackman & Brandenburg 2002).

3.3 Steady state, open flow-driven helical dynamo and implications for coronal magnetic relaxation

Consider now the limit of equations (11) and (12) in which the time evolution and resistive terms are ignored, but the divergence terms are kept. We then have, respectively,

$$0 = 2\langle \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \rangle - \nabla \cdot \langle \overline{\Phi \mathbf{B}} + \overline{\mathbf{E}} \times \overline{\mathbf{A}} \rangle \quad (31)$$

and

$$0 = -2\langle \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \rangle - \nabla \cdot \langle \phi \mathbf{b} + \mathbf{e} \times \mathbf{a} \rangle. \quad (32)$$

Combining these two equations reveals that the fluxes of large- and small-scale helicity through the system boundary are equal and opposite. This has important implications for a helical flow-driven dynamo inside an astrophysical rotator; if helical motions were to sustain kinetic helicity inside of the rotator, large- and small-scale magnetic helicities of opposite sign grow as discussed in the previous section. The existence of a steady state with open boundaries implies that the boundary fluxes of magnetic helicity contribute to the respective loss terms in the large- and small-scale magnetic helicities. The corona would be supplied with bihelical structures (Blackman & Field 2000b; Blackman & Brandenburg 2003; Brandenburg & Blackman 2003a,b). This is consistent with time-averaged steady coronae of the Sun (e.g. Schrijver & Zwaan 2000) and AGN accretion discs (e.g. Field & Rogers 1993; Haardt & Maraschi 1993). The bihelical nature of the field supplied by the dynamo and the sign dependence of the injected helicity on whether surface shear operates on a scale larger or smaller than that of a given loop's footpoint separation are reasons why extracting the dominant sign of the solar coronal magnetic helicity in each hemisphere of the Sun has been somewhat elusive (e.g. Démoulin et al. 2002).

The evolution of magnetic structures injected into a corona is conceptually analogous to the evolution of a magnetically dominated laboratory plasmas to injection of magnetic helicity such as in a Spheromak (e.g. Bellan 2000; Hsu & Bellan 2002). Even though the corona in the astrophysical case receives injected helicity of both signs, the guiding principles understood from laboratory plasmas are applicable. The experiment of Hsu & Bellan (2002) provides a direct analogy to helical loops of flux rising into an astrophysical corona from its rotator below. The loops coalesce at the symmetry axis and form a magnetic tower. For large enough helicity injection, the tower can break off a Spheromak blob from the kink instability. An astrophysical corona can also be modelled as a statistical aggregate of magnetically loops and the corona can be thought of as a single dynamical entity (Blackman & Field 2004). The helicity flux to the corona in

astrophysical rotators acts as a seed for subsequent magnetic relaxation therein. The relaxation opens field lines that the form coronal holes or jets.

3.4 Time dependent, open, flow-driven helical dynamos

In general, both the time dependent and the flux terms in equations (11) and (12) should be included dynamically. We briefly describe two calculations of flow-driven dynamos which incorporate both, using different sets of approximations. In this section, we assume that the overbars indicate spatial averages.

In the context of the Galaxy, Shukurov et al. (2006) have solved the mean field induction equation for $\overline{\mathbf{B}}$ with $\overline{\mathcal{E}}_{\parallel}$ determined from setting $\partial_t \overline{\mathcal{E}}_{\parallel} = 0$ in equation (2). The $\overline{\mathcal{E}}_{\parallel}$ involves the difference between the kinetic and current helicities which can be related to small-scale magnetic helicity in the Coulomb gauge. Shukurov et al. (2006) formally use a gauge invariant helicity density, derived by Subramanian & Brandenburg (2006), to replace the use of the magnetic helicity density but the key role of the boundary terms is conceptually independent of this. Effectively, Shukurov et al. (2006) therefore solve the induction equation for $\overline{\mathbf{B}}$ (which depends on $\overline{\mathcal{E}}_{\parallel}$ and thus $\langle \mathbf{a} \cdot \mathbf{b} \rangle$) and equation (12) for $\langle \mathbf{a} \cdot \mathbf{b} \rangle$. The divergence term in equation (12) can be replaced with one of the form $\propto \nabla \cdot (\langle \mathbf{a} \cdot \mathbf{b} \rangle \overline{\mathbf{V}})$, where $\overline{\mathbf{V}} = (0, 0, \overline{V}_z)$ is the mean velocity advecting the small-scale helicity out of the volume. This mean velocity also appears in the induction equation for $\overline{\mathbf{B}}$, highlighting that the loss terms in the small-scale helicity equation also imply advective loss of mean field. This approach supports the concept (Blackman & Field 2000a) that a flow of small-scale helicity towards the boundary may help to alleviate the backreaction of the small-scale magnetic helicity on the kinetic helicity which drives the dynamo in $\overline{\mathcal{E}}_{\parallel}$. However, if \overline{V}_z is too large, it may carry away too much of the mean field which the dynamo is trying to grow in the first place. In general, more work is needed to calculate \overline{V}_z from first principles and its effect on large- and small-scale fields. Coupling the dynamo in the rotator to the magnetic helicity evolution in the corona above is also of interest.

A more restrictive time-dependent dynamo that includes boundary terms, maintains the time dependence in equation (11), but implicitly assumes that equation (12) reaches a steady state has also been studied (Vishniac & Cho 2001). This approach explicitly incorporates the role of shear into the helicity flux. Although the approach involves assumptions that have now been avoided in more general calculations of helicity fluxes (e.g. Subramanian & Brandenburg 2004) (one being the first-order smoothing approximation which can be avoided by the ‘minimal tau’ closure discussed in Section 3.2), the paper (Vishniac & Cho 2001) identifies how a time-dependent flow-driven dynamo in a Keplerian shear flow might be sustained by a magnetic helicity flux. We choose to outline this approach in more detail here in part because the paper (Vishniac & Cho 2001) is less transparent than the more recent paper of Shukurov et al. (2006), but did first explicitly use the boundary flux to solve a time-dependent dynamo equation. In consolidating the calculation here, we show how it dovetails into the unified framework of Section 2.

If we ignore the resistive terms, the relevant forms of equations (11) and (12) are

$$\partial_t (\overline{\mathbf{A}} \cdot \overline{\mathbf{B}}) = 2\overline{\mathcal{E}} \cdot \overline{\mathbf{B}} - \nabla \cdot (\overline{\Phi \mathbf{B}} + \overline{\mathbf{E}} \times \overline{\mathbf{A}}) \quad (33)$$

and

$$0 = -2\overline{\mathcal{E}} \cdot \overline{\mathbf{B}} - \nabla \cdot (\overline{\phi \mathbf{b}} + \overline{\mathbf{e}} \times \overline{\mathbf{a}}). \quad (34)$$

Using equation (34), $\overline{\mathcal{E}}_{\parallel}$ can be directly written in terms of the small-scale helicity flux as

$$\overline{\mathcal{E}}_{\parallel} = -\frac{\overline{\mathbf{B}}}{B^2} \nabla \cdot (\overline{\phi \mathbf{b}} + \overline{\mathbf{e}} \times \overline{\mathbf{a}}) = -\frac{\overline{\mathbf{B}}}{B^2} \nabla \cdot (\overline{-\nabla \phi + \mathbf{e}} \times \overline{\mathbf{a}}). \quad (35)$$

Vishniac & Cho (2001) then use equation (35) in the equation for the mean magnetic field applied to an accretion disc whose mean quantities are axisymmetric. The mean field equation is

$$\partial_t \overline{\mathbf{B}} = \nabla \times \overline{\mathcal{E}} + \nabla \times (\overline{\mathbf{V}} \times \overline{\mathbf{B}}) + \lambda \nabla^2 \overline{\mathbf{B}}. \quad (36)$$

Solving equation (36) requires use of equation (35). Vishniac & Cho (2001) invoke a correlation time τ_c such that $\mathbf{a} \simeq -(\mathbf{e} + \nabla \phi) \tau_c$ (note: they define $\mathbf{e}_{mf} \equiv -\mathbf{e}$ and work with \mathbf{e}_{mf}). This reduces the last term of equation (35) to

$$2 \frac{\overline{\mathbf{B}}}{B^2} \nabla \cdot \overline{\mathbf{e} \times \nabla \phi} \equiv -\frac{\overline{\mathbf{B}}}{B^2} \nabla \cdot \mathbf{J}_H, \quad (37)$$

which defines the helicity flux \mathbf{J}_H . (Vishniac & Cho 2001 are missing a factor of 2.)

To proceed, ϕ can be inverted in terms of \mathbf{e} by Fourier transform, which gives

$$J_{H,i} = 2 \int \frac{d^3 \mathbf{r}}{4\pi r} \epsilon_{ijk} e_j(\mathbf{x}) \partial_k \partial_l e_l(\mathbf{x} + \mathbf{r}), \quad (38)$$

where the spatial derivatives operate on \mathbf{r} and an assumption of isotropy in wavenumber (not explicitly stated in Vishniac & Cho 2001 was used to obtain this). The notoriously troubling but useful first-order smoothing approximation, where terms non-linear in fluctuating quantities are ignored (see Section 3.2), was then used for \mathbf{e} to obtain $\mathbf{e} \simeq -\mathbf{v} \times \overline{\mathbf{B}} - \overline{\mathbf{V}} \times \mathbf{b}$. Ignoring $\overline{\mathbf{V}}$ here in the equation for \mathbf{e} (not necessarily justified) and assuming $\mathbf{r} \ll \mathbf{x}$, equation (38) becomes, after some algebra,

$$J_{H,i} \sim 2l^2 \tau_c \overline{(\mathbf{x}) \partial_k \partial_l e_l(\mathbf{x})} = 2l^2 \tau_c \epsilon_{lts} (\overline{B_n B_t v_i(\mathbf{x}) \partial_n \partial_l v_s(\mathbf{x})} - \overline{B_i B_t v_k(\mathbf{x}) \partial_k \partial_l v_s(\mathbf{x})}), \quad (39)$$

where l is a spatial correlation scale. Assuming incompressible flow for the fluctuations, and assuming that total spatial derivatives of the velocity correlations are small, the third term on the right-hand side of equation (39) vanishes and integrating the last term by parts gives

$$J_{H,i} \sim 2l^2 \tau_c \overline{\mathbf{B} \cdot \nabla v_i \mathbf{B} \cdot \nabla \times \mathbf{v}}. \quad (40)$$

This current is then used in equation (36) to allow growth of $\overline{\mathbf{B}}$. Subramanian & Brandenburg (2004) and Brandenburg & Subramanian (2005c) show that this is one of a number of current terms that emerge in a more general calculation which avoids the first-order smoothing approximation (see Section 3.2) with additional fluxes arising when $\overline{\mathbf{V}}$ is included in \mathbf{e} . Nevertheless, the importance of equation (40) is that it, in principle, allows dynamo growth of the large-scale field to be driven entirely by the small-scale helicity flux without any kinetic helicity. Determining whether this works in practice needs further work.

The role of the Vishniac–Cho flux has been investigated numerically in a few experiments with somewhat mixed results; Brandenburg & Subramanian (2005a) found that the flux can sustain the field growth in the absence of kinetic helicity but only if the field already exceeds ~ 70 per cent of the equipartition value of the turbulent field. On the other hand, when kinetic helicity is present, the Vishniac–Cho flux has been numerically shown (Brandenburg 2005) to alleviate catastrophic quenching from small-scale helicity build up by allowing ejection of small-scale helicity through the boundary, consistent with conceptual suggestions of the role of the boundary terms (e.g. Blackman & Field 2000a).

The role of the boundary flux may be particularly important when a rapid, unquenched, cycle period is involved, such as in the Sun (Blackman & Brandenburg 2003). However, as also emphasized in Section 3.3, large-scale helicity flux likely accompanies any small-scale helicity flux. Significant loss of the large-scale field would imply removal of the large-scale field that the dynamo is invoked to generate inside the rotator thereby lowering its maximum value inside the rotator. Care in identifying the relative amount of large- and small-scale helicity flux is warranted. The same issue also arises in the Galactic field calculation of Shukurov et al. (2006).

A flux-driven helical dynamo is conceptually distinct from the closed volume kinetic helicity driven dynamo discussed in Section 3.2. The flux-driven dynamo is more analogous to the laboratory plasma dynamo discussed in Section 3.1 in the steady-state approximation, but with a key difference. The source of the helicity flux for the velocity-driven case depends on fluctuations driven by, and coupled to, the velocity shear (e.g. Balbus & Hawley 1998), not fluctuations driven by current-driven magnetically dominated instabilities.

4 INSIGHTS FROM THE UNIFIED FRAMEWORK

To exemplify the benefit of the unified approach to helical dynamos, we consider insights that emerge from synthesizing the cases discussed in the previous sections. In particular, we consider what can be learned about saturation, dissipation versus boundary terms and dynamical equilibria versus relaxed states. As mentioned in the Section 1, non-linear saturation and the extent to which helical dynamos depend on dissipation has been a long standing topic of study, particularly for flow-driven helical dynamos. The magnetically driven example of Section 3.1 can actually provide insight on the flow-driven cases.

First, the fact that the large-scale magnetic field dominates the fluctuating field in Section 3.1 highlights that there is no universal requirement that the mean field energy in the steady state be less than that of the fluctuating magnetic field for a working helical dynamo. This highlights that the Zeldovich relations (Zeldovich 1956; Zeldovich et al. 1983), relating the mean and fluctuating fields when a uniform field is imposed and the small-scale field is amplified non-helicity, were never derived as a constraint on helical dynamos (Blackman & Field 2005). Secondly, the laboratory plasma experiments show that the resistive terms are small in the RFP dynamos. Equation (22) reveals that the consistent explanation for the sustenance of the mean magnetic field via $\overline{\mathcal{E}}_{\parallel}$ in an RFP dynamo is the flux of small-scale magnetic helicity. In a steady state, the field is neither growing nor decaying, but observed RFP sawtooth oscillations occur on a faster-than-resistive time-scale and can be accommodated by equation (22) when augmented by the term $(1/B^2) \partial_t \overline{\mathbf{A} \cdot \mathbf{B}}$ from the left-hand side of equation (11). As discussed in the previous section, flux terms can allow the dynamo to incur cycles or oscillations on time-scales not limited by resistivity. Note the distinction between Sections 3.1 and 3.2. In the latter, we consider a closed volume time dependent, flow-driven helical dynamo. As described therein, the growth starts out fast and formally independent of the magnetic dissipation rate. Then, as the small-scale magnetic helicity builds up, the dynamo slows to become resistively limited and a steady-state balance occurs between the EMF and the resistive terms. However, unlike the case of Section 3.1, the helicity flux terms vanish in Section 3.2. Therefore, the only terms that can balance the EMF in the steady state are the resistive terms for Section 3.2.

The role of the helicity flux for a flow-driven dynamo is further exemplified in the cases of Sections 3.3 and 3.4. When the averages are taken locally, the helicity flux terms for a flow-driven dynamo can dominate resistive terms. In an open astrophysical object like a star or accretion disc, the flux terms send magnetic helicity and magnetic energy to the corona. Indeed, for either the time-dependent case of Section 3.4 or the steady state of Section 3.3, the EMF is balanced by a combination of time evolution of magnetic helicity and boundary terms. In both of these cases, the dissipation terms are negligible.

Finally, a further comment on the relation between dynamical equilibria and fully relaxed states. When kinetic helicity is steadily injected for a velocity-driven dynamo or when magnetic helicity is steadily injected for a magnetically driven dynamo, an equilibrium or a quasi-steady state can be reached. The latter is exemplified by sawtooth oscillations in an RFP dynamo or as the 22-yr solar dynamo cycle. The driving or injection takes the system away from the relaxed state, but the dynamo, fuelled by induced instabilities, continually competes to take the system back towards the relaxed state. An important difference between injecting one sign of magnetic helicity versus one sign of kinetic helicity is that the latter leads to a spectral or spatial segregation of helicities of opposite sign (the bihelical dynamical equilibrium for the

α^2 dynamo discussed in Section 3.2). When magnetic helicity of one sign is injected, that magnetic helicity seeks the largest scale available to it. When the driving is turned off and the resistivity is small, the system can fully relax, but eventually the field decays. The case of a forced versus decaying magnetically dominated turbulent dynamo was studied in Blackman & Field (2004). Numerical simulations of the relaxation of helical MHD turbulence with initially equal kinetic and magnetic energy densities were studied in Christensson, Hindmarsh & Brandenburg (2001). These works show the tendency of a single sign of magnetic helicity to migrate towards the largest scale for magnetic helicity injected with a single sign.

5 SUMMARY AND CONCLUSION

We have used the equations for magnetic helicity evolution as a unifying framework for helical dynamos, and we have discussed both magnetically driven laboratory plasma dynamos and flow-driven astrophysical dynamos within this framework. We summarize the common principles and distinguishing features of these dynamos here.

Laboratory plasma helical dynamos typically involve a magnetically dominated initial state with a dominant mean magnetic toroidal magnetic field. When an external toroidal electric field is applied, a current is driven along the field which injects magnetic helicity of one sign into the system and generates a poloidal field. For sufficiently strong applied electric fields, the system is driven sufficiently far away from its relaxed state that helical tearing or kink-mode instabilities occur. The resulting fluctuations produce a turbulent EMF that drives the system back towards the relaxed state. The relaxed state for such a unihelical dynamo is the state in which the magnetic helicity is at the largest scale possible, subject to boundary conditions. When the unihelical helicity injection is externally sustained for a real system, a dynamical equilibrium with oscillations can incur as the system evolves towards and away from the relaxed state. The time-averaged EMF is maintained by a spatial (radial) flux of small-scale magnetic helicity within the plasma. The injection of helicity is balanced by the dynamo relaxation so the dynamo sustains the field configuration against decay.

Like the laboratory plasma dynamos, the flow-driven helical dynamos require a turbulent EMF to grow or sustain magnetic helicity at large scales. These dynamos are often invoked as an explanation for the large-scale fields of astrophysical rotators. Unlike the laboratory plasma dynamos, for the canonical flow-driven helical dynamo, the initial mean field is weak and the velocity fluctuations are strong. For the simplest time-dependent case in a closed volume, the EMF is proportional to the difference between the current helicity and the kinetic helicity densities. The latter initially dominates and this drives growth of the large-scale magnetic helicity by sending one sign of the magnetic helicity to large scales and the other sign to scales at or below the scale of the dominant velocity fluctuations. Here, unlike the laboratory plasma dynamo, no magnetic helicity is injected and so the dynamo acts to segregate magnetic helicity of opposite signs spatially or spectrally. The build up of the small-scale magnetic helicity also grows the small-scale current helicity which offsets the kinetic helicity contributions to the EMF and quenches the dynamo into a steady state. In the absence of boundary terms, the steady state is one in which growth is balanced by resistive dissipation. When boundary flux terms are allowed, the EMF can be sustained by a magnetic helicity flux, just as in the laboratory case. Such a helicity flux can arise from a Keplerian velocity shear and stratification rather than the current-driven instabilities of laboratory plasma dynamos.

In the astrophysical case, when the resistive terms in the magnetic helicity evolution equation are small, equal in magnitude (but opposite in sign) small- and large-scale fluxes of magnetic helicity are injected to coronae in the steady state. For a time-dependent situation, open boundaries can, in principle, allow the dynamo to overcome any long-term resistive saturation by ejecting the offending small-scale helicity from a volume of interest. However, such ejection will undoubtedly involve ejection of large-scale field as well, which can reduce the steady-state saturation value of the large-scale field inside the rotator volume compared to the asymptotic saturation of the closed case.

In the magnetically dominated corona, energy associated with this helicity can be extracted into high-energy particles. Astrophysical coronae, with their underlying rotators acting as magnetic helicity injectors, are the astrophysical circumstance most directly analogous to the magnetically driven dynamo physics of laboratory plasma devices such as Spheromaks and RFPs. In particular, a single loop with footpoints being sheared or twisted provides a direct analogy to the precursor to Spheromak formation. Two subtle differences that arise when considering the analogy between coronae and laboratory plasmas more carefully are that (i) the corona is composed of many injection sites each analogous to a Spheromak, and (ii) the dynamo in the rotator below injects both signs of magnetic helicity into the corona. Nevertheless, relaxation of magnetic fields in astrophysical coronae can produce the very largest scale fields associated with coronal holes and jets and the principles of laboratory plasma dynamos are applicable.

Our goal has been to provide a base that anchors common principles and differences between laboratory and astrophysical helical dynamos to foster further cross-disciplinary work. We have avoided a detailed exposition about each type of dynamo in specific laboratory and astrophysical systems in order to focus on the basic concepts.

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