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Fast expansion of a plasma beam controlled by short-circuiting effects in a longitudinal magnetic field

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Abstract. The transverse expansion of a plasma beam with finite ion gyroradius injected along a magnetic field has been treated analytically and numerically in low β plasma. It has been demonstrated that the transverse expansion of a beam is much faster than predicted by standard ambipolar diffusion owing to the short-circuiting currents in the system. In the vicinity of an injector, plasma expands with the sound speed while at larger distances the transverse velocity tends to zero and a cylindrical shape of the beam is formed.

The propagation of a fully ionized plasma beam along a magnetic field with an ion gyroradius large compared to the beam transverse width is an important problem related to the space plasma, plasma beam injection into magnetic traps, active experiments in the ionosphere and the magnetosphere, plasma of vacuum arcs and unipolar arcs in tokamaks, etc [1].

In the one-dimensional case of an infinite in z (along $\boldsymbol{B}$) direction cylinder, the transversal expansion can be simply obtained as follows. Owing to their large gyroradius, ions are trapped by the radical electric field. Consequently, an ambipolar (currentless) radial potential corresponds to the Boltzmann distribution of ions $\Phi = T_i \ln n$. The difference between the ion and electron diamagnetic azimuth velocities is given by $u_{e\varphi} - u_{i\varphi} = c[T_e/eB(\partial \ln n/\partial r - e/T_e \partial\Phi/\partial r)]$. The azimuth friction force $\nu_{ei} me(u_{e\varphi} - u_{i\varphi})$ results in the formation of radial fluxes. Using the expression for electric potential one can receive radial ion and electron fluxes:

$$nu_{er} = nu_{ir} = -cm_e \nu_{ei} u_{e\varphi}/eB = -D_\perp \partial n/\partial r$$

where $D_\perp$ is the classical diffusion coefficient:

$$D_\perp = (T_e + T_i)\nu_{ei}/m_e\omega_{ce}^2 \tag{1}$$

where ν_{ei} is electron–ion collision frequency.

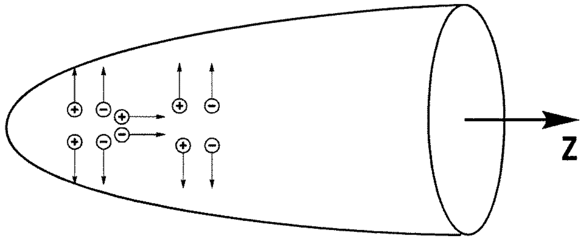
According to conventional wisdom the two-dimensional situation should be similar to the one-dimensional case: the transverse expansion of the plasma beam is to be controlled by ambipolar diffusion. The plasma moves along $\boldsymbol{B}$ with large supersonic velocity and expands in a radial direction with the ambipolar diffusion coefficient $D_\perp$. Moreover, it seems evident that the beam shape is to be a sort of a cone [2] with the width increasing along $\boldsymbol{B}$ as $l_\perp^2 = D_\perp z/u$ where the z-axis is the coordinate along $\boldsymbol{B}$ and u is the beam velocity along $\boldsymbol{B}$. But such a simple picture is totally incorrect. The reason is that ambipolar (currentless) diffusion requires a specific ambipolar potential that should be Boltzmann's for electrons along $\boldsymbol{B}$ and also Boltzmann's for ions across $\boldsymbol{B}$ simultaneously. Such a potential profile can exist only if the plasma density profile can be factorized: $n(r, z) = f_1(z) f_2(r)$ and the potential is the sum $\Phi = \ln f_1(z) + \ln f_2(r)$. In any other case the diffusion is non-ambipolar and is accompanied by the short-circuiting currents in the plasma. We shall consider such a situation of a plasma supersonic beam, which is injected along $\boldsymbol{B}$ from a small source. Electrons close to the beam axis move along $\boldsymbol{B}$ faster than ions diffuse across $\boldsymbol{B}$ and return backwards along $\boldsymbol{B}$ at the beam periphery (see figure 1). Shorting the polarization field accelerates injected ions in the transverse direction in the vicinity of the injection point and decelerates them at larger z. Thus, only the global ambipolarity condition is fulfilled, which means that only when integrated over z are the transverse fluxes of ions and electrons equal to each other. Furthermore, the plasma near the injector expands across a magnetic field with the velocity of the order of sound, which is much faster than the ambipolar diffusion velocity. Far from the injector the beam has a cylindrical shape and its transverse expansion is determined by electron diffusion.

A supersonic quasineutral beam with parallel velocity u large compared with the ion sound speed c_s, injected at $z = 0$ parallel to the uniform magnetic field, is considered. Particle gyroradii ρ_{ci}, ρ_{ce} are related to the transverse beam width $l_\perp$ as

$$\rho_{ce} \ll l_\perp \ll \rho_{ci}. \tag{2}$$

It is assumed that across $\boldsymbol{B}$ ions can move faster than electrons, so that

$$\frac{c_s}{l_\perp} \gg \frac{D_\perp}{l_\perp^2}. \tag{3}$$



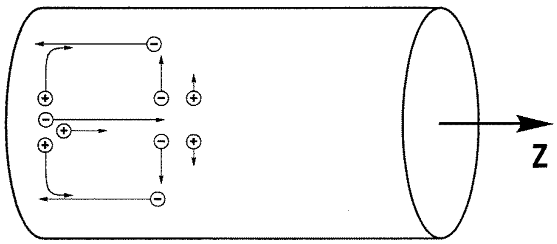


Figure 1. The scheme of non-ambipolar diffusion accompanied by short-circuiting currents. Arrows show the electron and ion motion. The arrow length represents schematically the value of particle fluxes. (a) The hypothetical ambipolar expansion. (b) Real non-ambipolar fluxes.

Then the condition (3) is equivalent to

$$\lambda \gg \rho_{ci}\rho_{ce}/l_\perp \tag{4}$$

where λ is the ion mean free path. The inertial and gaseous pressures are supposed to be small compared to the magnetic pressure and therefore the magnetohydrodynamic (MHD) effects of magnetic field disturbance are neglected.

The basic equations in the cylindrical geometry read:

$$\partial n/\partial t + \mathrm{div}(n\boldsymbol{u}_i) = 0 \tag{5}$$

$$\mathrm{div}\boldsymbol{j} = 0 \tag{6}$$

$$m_i \frac{\mathrm{d}\boldsymbol{u}_i}{\mathrm{d}t} = -e\nabla\Phi - T_i\nabla n/n \tag{7}$$

$$nu_{er} = -D_\perp(\partial n/\partial r - en/T_e\partial\Phi/\partial r)/(1 + T_i/T_e) \tag{8}$$

$$j_z = -\sigma_z(\partial\Phi/\partial z - T_e/e\partial\ln(n)/\partial z) \qquad \sigma_z = 1.96ne^2/m_e\nu_{ei}. \tag{9}$$

The Lorentz and friction forces are neglected in equation (7) according to inequalities (2) and (3). Since the electrons are magnetized, their radial flux is determined by azimuth friction forces. They are proportional to the electron azimuth velocity, which is the sum of the diamagnetic drift and $\boldsymbol{E}\times\boldsymbol{B}$ drift. Equation (8) determines the electron radial flux which follows from the transverse components of the electron momentum balance equations neglecting electron inertia. Indeed, the radial momentum balance for electrons reads

$$en\partial\Phi/\partial r - T_e\partial n/\partial r - enu_{e\varphi}B/c = 0.$$

In this equation the radial friction force is also neglected with respect to the pressure gradient. It follows from the estimate for the friction force $nm_e\nu_{ei}c_s$ in combination with the inequalities (1) and (2). So the azimuth electron velocity is given by

$$u_{e\varphi} = -cT_e/eB[\partial\ln(n)/\partial r - e/T_e\partial\Phi/\partial r].$$

The ion azimuth velocity is negligible since $\rho_{ci} \gg l_\perp$. From the azimuth component of the momentum balance equation for electrons one obtains:

$$u_{er} = cm_e\nu_{ei}u_{e\varphi}/eB.$$

Combining the equations for radial and azimuth electron velocities, one comes to equation (8). The radial ion drift produced by the azimuth friction force is neglected because their radial velocity is much larger—of the order of the sound speed, as follows from equation (7). Temperatures are assumed to be constant. The boundary conditions are given by

$$z = 0 : nu_{ez} = nu_{iz} = \Gamma(r)$$

$$z = z_A : nu_{ez} = nu_{iz}. \tag{10}$$

Here z_A is the right boundary of a beam (in space applications $z_A \to \infty$). These conditions imply that no current is flowing through the boundaries. Since the electron conductivity is very high, for $z_A \ll \sqrt{l_\perp T_e/(c_s m_e \nu_{ei})}$ from equations (5)–(9) it follows:

$$\Phi = T_e/e\ln n + \Psi(r) \tag{11}$$

where $\Psi(r)$ is an arbitrary function which determines the electron radial flux according to equation (6). This function is determined by the condition of global ambipolarity:

$$\int_0^z nu_{er}\,\mathrm{d}z = \int_0^z nu_{ir}\,\mathrm{d}z \tag{12}$$

which follows from equation (6) and the boundary conditions given by equation (10). Equation (12) means that the total transverse current integrated along the magnetic field direction is equal to zero. Substituting equations (8) and (11) into (12) and combining with (7), one obtains

$$\frac{\partial u_{ir}}{\partial t} + \left(u_{ir}\frac{\partial}{\partial r} + u\frac{\partial}{\partial z}\right)u_{ir} = -c_s^2\left(\frac{\partial\ln n}{\partial r} + \int_0^z nu_{ir}\mathrm{d}z \Big/ \int_0^z D_\perp n\,\mathrm{d}z\right) \tag{13}$$

where $c_s = \sqrt{(T_e+T_i)/m_i}$. Since the parallel ionic velocity is supersonic and usually [2] large compared to sound, $u_z \equiv u =$ constant, equation (13) together with the continuity equation (5) completes the system.

It can be easily seen from equation (13) that the solution for the plasma, expanding radially much slower than c_s, can be obtained only if the density is factorized

$$n(r,z) = f_1(z)f_2(r). \tag{14}$$

Equation (14) follows from the fact that if $u_{ir} \ll c_s$ the right-hand side (RHS) of equation (13) is almost zero while

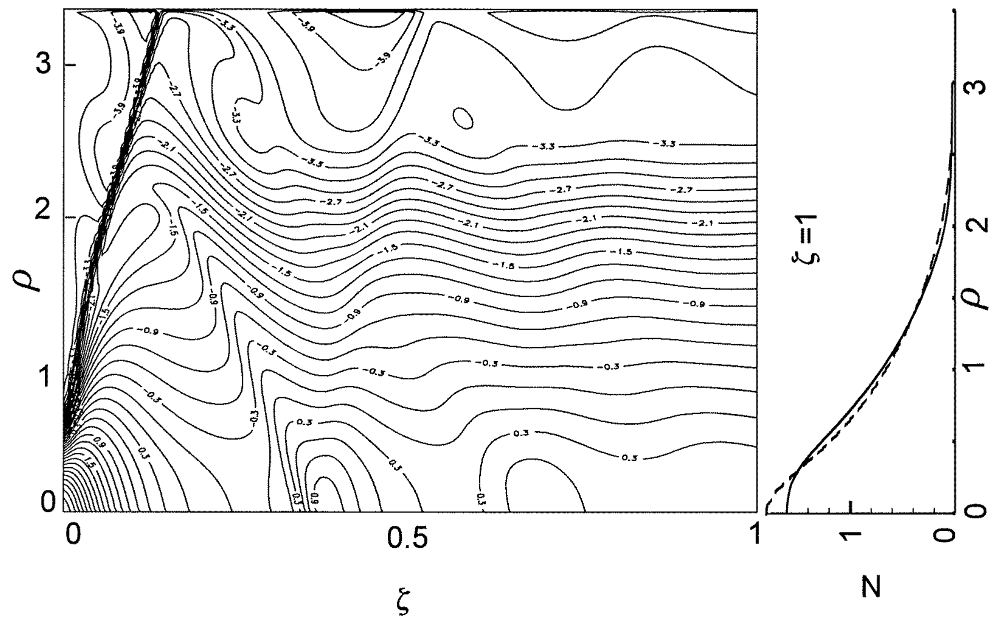


Figure 2. Isodensities $\ln \tilde{n}$ = constant at $\tau = 1.25$ (practically the steady state). $\zeta = z/z_A$, $\rho = r/l_\perp$. In the right-hand side the radial profile at $\zeta = 1$ is compared with the predicted profile; the solution of equation (17) is represented by the broken curve.

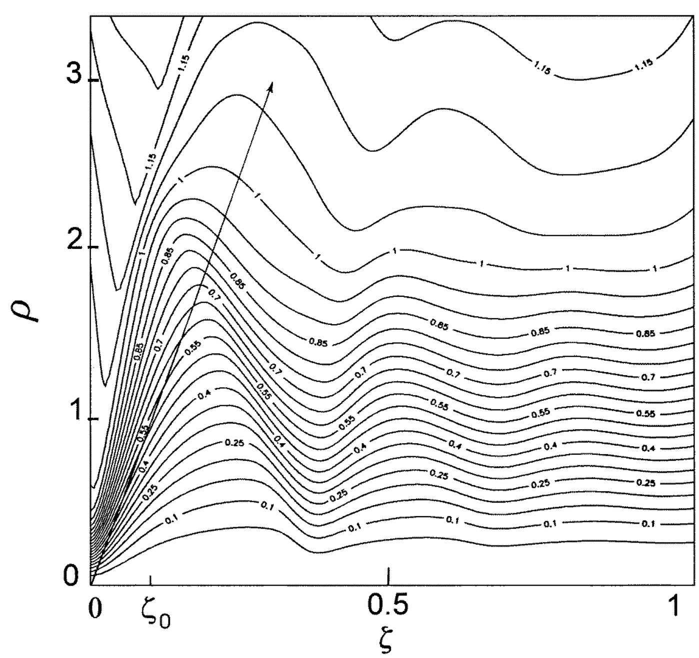


Figure 3. Streamlines G = constant at $\tau = 1.25$, $G = 2\pi \int_0^r n u_{ir} \mathrm{d}r/I$. ζ_0 corresponds to transitional region of length $l_\perp u/c_s z_A$. The arrow shows the characteristic angle of plasma expansion with sound velocity $\Theta = \tan^{-1}(c_s/u)$.

the last term is a function only of r. The potential profile in this case is given by

$$\varphi(r, z) = \frac{T_e}{e} \ln f_1(z) - \frac{T_i}{e} \ln f_2(r) + \text{constant}. \qquad (15)$$

If $f_1(z)$ = constant the solution given by equations (14) and (15) corresponds to the well known case of the infinite in the z direction cylinder. Here the pressure gradient of ions is almost totally balanced by the radial electric field. The resulting small ion velocity is equal to the electron diffusion velocity given by equations (8) and (15).

For a real plasma injected from the source the situation is more complicated. It can be easily seen that near the injector, where the plasma is expanding both radially and in the z direction, the density profile cannot be factorized according to equation (14). Furthermore, it is clear that the plasma near the injector expands radially with u_{ir} of the order of c_s until the profile given by equations (14) and (15) is formed. The length of the transition region z_0 is given by $z \le z_0 \ll z_A$, as it will be shown below. At $z \gg z_0$ the RHS of equation (13) is small and the density is factorized, see equation (14). Since there is no characteristic scale along z, the cylindrical profile with $n = n_c(r)$ is created. Thus, integrating the continuity equation (5) over z and inserting equation (15) into equation (8) for the main contribution to the integral, we have

$$\frac{\partial N}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(rD_\perp \frac{\partial N}{\partial r}\right) + un(r, z_A) - \Gamma(r, 0) = 0 \quad (16)$$

$$N = \int_0^z n\,\mathrm{d}z.$$

For $t \ll z_A/u$ the term with $n(r, z_A)$ is negligible and equation (16) is a diffusion equation with the given source $\Gamma(r)$. Since $D_\perp$ is proportional to n, the solution could be obtained only numerically (if $D = \mathrm{constant}(n)$ in the case of anomalous diffusion, the analytical solution of equation (16) is well known [3]). The width of the beam $l_\perp$ increases with time as $\sqrt{D_\perp t}$. For $t \gg z_A/u$ the steady state density profile is established. Since $n(z)$ is constant, the shape of the main part of the beam is cylindrical with $n_c = N/z_A$, and n_c is given by the equation

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rD_\perp \frac{\mathrm{d}n_c}{\mathrm{d}r}\right) + \frac{un_c}{z_A} = \frac{\Gamma(r)}{z_A}. \quad (17)$$

The characteristic width $l_\perp$ of the cylinder and the typical density n_c^* are easily estimated from equation (17)

$$l_\perp = \sqrt{D_\perp(n_c^*)z_A/u} \qquad n_c^* = I/(2\pi u l_\perp^2) \quad (18)$$

where $I = 2\pi \int_0^\infty \Gamma(r) r\,\mathrm{d}r$ is the total flux. Resolving equation (18), one obtains

$$l_\perp = \left(\frac{\mathrm{d}D_\perp}{\mathrm{d}n}\cdot\frac{Iz_A}{2\pi u^2}\right)^{1/4} \qquad n_c^* = \left[I \Big/ \left(2\pi \frac{\mathrm{d}D_\perp}{\mathrm{d}n} z_A\right)\right]^{1/2}. \quad (19)$$

It is now easy to determine the longitudinal scale of the transitional region, where the plasma is expanding practically freely

$$z_0 = l_\perp \frac{u}{c_s} \ll z_A.$$

The process of the transition from fast expansion to slow diffusion with $n = n_c$, given by equation (13), has an oscillatory character. Indeed, equation (13) has the form of a well known momentum balance equation with the pressure gradient and external force in the RHS. Consequently, sound oscillations of the beam with wavelength of the order of z_0 are expected.

Results of the numerical modelling of equations (5) and (13) are shown in figures 2 and 3. The dimensionless variables are defined as:

$$\zeta = z/z_A \qquad \rho = r/l_\perp \qquad \tilde{n} = n/n_c^* \qquad \tilde{\Phi} = e\Phi/T_e$$

$$\tau = tu/z_A \qquad \tilde{u}_{ir} = u_{ir}/u_{ir0}$$

$$u_{ir0} = z_A/ul_\perp = \left(u^2 \frac{\mathrm{d}D_\perp}{\mathrm{d}n}\cdot\frac{I}{2\pi z_A^3}\right)^{1/4}. \quad (20)$$

In these variables the system of equations (5) and (13) reads:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \rho} n u_{ir} + \frac{\partial}{\partial \zeta} n = 0$$

$$\frac{\partial u_{ir}}{\partial t} + \left(u_{ir}\frac{\partial}{\partial r} + \frac{\partial}{\partial \zeta}\right) u_{ir}$$

$$= -\frac{c_s^2}{u_{ir0}^2}\left(\frac{\partial \ln n}{\partial r} + \int_0^\zeta n u_{ir}\,\mathrm{d}\zeta \Big/ \int_0^\zeta n^2\,\mathrm{d}\zeta\right). \quad (21)$$

The parameters chosen for the numerical simulation are the normalized sound velocity:

$$c_s/u_{ir0} = c_s \Big/ \left(u\frac{2\mathrm{d}D_\perp}{\mathrm{d}n}\cdot\frac{I}{2\pi z_A^3}\right)^{1/4} = 10.1$$

and the normalized source density:

$$\frac{\Gamma(r)}{un_c^*} = 31.4\exp(-\rho/0.252)^2.$$

Correspondingly, $\zeta_0 = z_0/z_A = l_\perp u/z_A c_S = 0.1$; the value of the angle $\Theta = \tan^{-1}(u/c_s) \sim 0.1$, the expected characteristic angle of expansion coincides well with the observed one (see figure 3), at the periphery region the plasma expands with supersonic velocity. As always in expansion in vacuum, the average velocity is about the velocity of sound.

One can follow the process of fast plasma expansion near the injector and formation of a cylinder with $n \sim n_c^*$ ($\tilde{n} \sim 1$). The sound oscillations and the shock formation are observed. The latter is essential for the hydrodynamic expansion given by equation (21).

If the ion gyroradius is smaller than the beam transverse scale, the presented results are also applicable. In this case the ion transverse unipolar flux is controlled by ion viscosity:

$$nu_r = \Delta^*\left[\frac{0.3nT_i\nu_{ii}}{m_i^2\omega_{ci}^4}\left(\frac{\partial nT_i}{n\partial r} + e\frac{\partial \Phi}{\partial r}\right)\right]$$

$$\Delta^* = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{1}{r^2} \quad (22)$$

where ω_{ci} is the ion cyclotron frequency [4]. This flux is larger than the diffusion flux if the ion gyroradius is not too small:

$$\rho_{ci} > l_\perp(m_e/m_i)^{1/4}. \quad (23)$$

If condition (23) is fulfilled, non-ambipolar expansion still takes place.

Conclusions

The evolution of a plasma beam injected along a magnetic field corresponds to the following scenario. If the beam length z_A is not too large, so that its transverse spread l is small with respect to its initial width, then the form of the beam is cylindrical. The cylinder radius is determined by the source. In the opposite case, due to the short-circuiting effect, the beam shape in the source vicinity is a sort of cone. Such a shape corresponds to the sonic radial expansion. At larger distances along $\boldsymbol{B}$ the expansion is suppressed, and the beam shape is also cylindrical. Its radius is determined by the transverse electron diffusion during the ion transit time.

It should be mentioned that the main results of the theory could be extended to the case of a magnetized partially ionized plasma.

The cylindrical shape of the plasma beam has been observed experimentally in vacuum arcs in a magnetic field [5], where the conditions given by equations (1) and (2) were satisfied. Unfortunately the lack of detailed experimental data makes it difficult to perform the quantitative comparison with the presented theory.

Acknowledgments

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