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Fluid Theory of Magnetized Plasma Dynamics at Low Collisionality

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Abstract

Finite Larmor radius (FLR) fluid equations for magnetized plasmas evolving on either sonic or diamagnetic drift time scales are derived consistent with a broad low-collisionality hypothesis. The fundamental expansion parameter is the ratio δ between the ion Larmor radius and the shortest macroscopic length scale (including fluctuation wavelengths in the absence of small scale turbulence). The low-collisionality regime of interest is specified by assuming that the other two basic small parameters, namely the ratio between the electron and ion masses and the ratio between the ion collision and cyclotron frequencies are comparable to or smaller than δ^2. First significant order FLR equations for the stress tensors and the heat fluxes are given, including a detailed discussion of the collisional terms that need be retained under the assumed orderings and of the closure terms that need be determined kinetically. This analysis is valid for any magnetic geometry and for fully electromagnetic non-linear dynamics with arbitrarily large fluctuation amplitudes. It is also valid for strong anisotropies and does not require the distribution functions to be close to Maxwellians. With a subsidiary small-parallel-gradient ordering for large-aspect-ratio toroidal plasmas in a strong but weakly inhomogeneous magnetic field, a new system of reduced two-fluid equations is derived, rigorously taking into account all the diamagnetic effects associated with arbitrary density and anisotropic temperature gradients.

I. Introduction.

The fluid description of magnetized plasmas constitutes a very useful framework to analyze their macroscopic behavior. Even though a consistent fluid closure can only be justified rigorously at high collisionality[1−2], the fluid moments of the kinetic equations provide an exact, lower-dimensionality constraint on the complete kinetic description and a good approximation for the dynamics perpendicular to the magnetic field by themselves, regardless of collisionality. For the collisionless or weakly collisional regimes of main interest in space and in magnetic fusion experiments, the hybrid approach based on exploiting the fluid moment information, complemented by a kinetic approximation of the unavailable closure terms, is currently a major area of active research[3−7].

In a previous work[8], a general formalism for magnetized plasmas encompassing a maximum fluid moment information, was developed for the strictly collisionless case. The purpose of the present article is to extend that work to the more realistic case of low but finite collisionality. Considering dynamical evolution away from equilibrium, it will be assumed that the two terms that contribute to the convective time derivatives are comparable ($\partial/\partial t \sim \mathbf{u}_\alpha \cdot \nabla$ where $\mathbf{u}_\alpha$ are the different species macroscopic flow velocities). Then, provided that small scale turbulence effects can be neglected, the strictly collisionless fluid moment analysis can be based on the single expansion parameter $\delta \sim \rho_\iota/L$, the ratio between the ion Larmor radius and the shortest macroscopic length scale including large scale fluctuation wavelengths. The treatment of the finite collisionality terms involves two more independent small parameters, namely the ratio m_e/m_ι between the electron and ion masses and the ratio $\nu_\iota/\Omega_{c\iota}$ between the ion collision and cyclotron frequencies, whose ordering relative to δ is a matter of choice. The present low-collisionality analysis will adopt as basic working hypotheses $(m_e/m_\iota)^{1/2} \lesssim \delta \ll 1$ and $\nu_\iota/\Omega_{c\iota} \lesssim \delta\ (m_e/m_\iota)^{1/2}$. With deuterium ions of density and temperature n and T_ι respectively in a magnetic field B, this means $1.4 \times 10^{-4} T_\iota(eV)^{1/2} B(T)^{-1} \ll L(m) \lesssim 8.7 \times 10^{-3} T_\iota(eV)^{1/2} B(T)^{-1}$ and $2.7 \times 10^{-16} n(m^{-3}) T_\iota(eV)^{-3/2} B(T)^{-1} \lesssim 1$. Such conditions are well satisfied for a wide class of macroscopic modes over most of the plasma parameter range relevant to tokamak fusion experiments, possibly failing only at the very plasma edge where the complex governing physics is beyond the scope of the simple fluid theory and precludes its applicability anyway. It can therefore be argued that these orderings provide a meaningful foundation for a broad low-collisionality fluid theory.

Besides specifying the collisionality regime, the fluid analysis requires specification of the time scales of interest. In terms of dimensionless ratios, this amounts to specifying the orderings of the time derivatives relative to the ion cyclotron frequency and of the macroscopic flow velocities relative to the ion thermal speed, which are linked once $\partial/\partial t \sim \mathbf{u}_\alpha \cdot \nabla$ is assumed. Here we shall be concerned with two frequently considered such orderings. The first one is the "fast dynamics" (or "sonic") ordering characterized by $u_\alpha \sim v_{th\iota}$ and $\partial/\partial t \sim \delta\Omega_{c\iota}$. The second one is the "slow dynamics" (or "drift") ordering where the flow velocities and time derivatives are comparable to the diamagnetic drift velocities and frequencies, $u_\alpha \sim \delta v_{th\iota}$ and $\partial/\partial t \sim \delta^2\Omega_{c\iota}$. As discussed in Ref.8, this slow dynamics ordering has a consistency problem whose resolution requires the adoption of further assumptions. One way of obviating this difficulty is to assume separate length scales parallel and perpendicular to the magnetic field, with a subsidiary small parameter $\epsilon \sim L_\perp/L_\parallel \sim k_\parallel/k_\perp \ll 1$. In this case, as a consequence of the fact that parallel gradients are ordered small, the pressures need to be known only in their zero-Larmor-radius limit, avoiding the problematic evaluation of their $O(\delta^2)$ FLR corrections. This subsidiary small-parallel-gradient ordering for plasmas in a strong but weakly inhomogeneous magnetic field, which leads to the so called "reduced" systems[9–17] where the fast magnetosonic wave is eliminated, will be adopted when considering the slow dynamics. A major improvement over previous reduced fluid models in the slow dynamics ordering[13–17] will be the rigorous treatment of arbitrary density and temperature gradients, especially their associated diamagnetic effects, as well as the allowance for strong temperature anisotropies and arbitrarily large density, temperature and electric potential fluctuation amplitudes.

II. General fluid formalism.

This section presents the general macroscopic equations for the fluid moment variables. It follows the approach of Ref.8, with the addition of the collisional terms and some changes and streamlining in the notation. All the results derived in this section are exact without approximations and valid for each plasma species independently, so the species index is dropped here for convenience. The macroscopic system follows from the velocity moments of the underlying kinetic equation,

$$\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial t} \;+\; v_j \frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial x_j} \;+\; \frac{e}{m}\Big(E_j + \epsilon_{jkl} v_k B_l\Big)\frac{\partial f(\mathbf{v},\mathbf{x},t)}{\partial v_j} \;=\; C(\mathbf{v},\mathbf{x},t), \tag{1}$$

where $f(\mathbf{v},\mathbf{x},t)$ is the distribution function, $C(\mathbf{v},\mathbf{x},t)$ is the collision operator, $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ are the electric and magnetic fields, and m and e are the species mass and electric charge. Conservation of particles in the collisions yields $\int d^3\mathbf{v}\; C(\mathbf{v},\mathbf{x},t) = 0$. The following fluid moments of the distribution function and the collision operator will be considered:

$$n(\mathbf{x},t) \;=\; \int d^3\mathbf{v}\; f(\mathbf{v},\mathbf{x},t), \tag{2}$$

$$n(\mathbf{x},t)\; u_j(\mathbf{x},t) \;=\; \int d^3\mathbf{v}\; v_j\; f(\mathbf{v},\mathbf{x},t), \tag{3}$$

$$\mathsf{P}_{jk}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)\; f(\mathbf{v},\mathbf{x},t), \tag{4}$$

$$\mathsf{Q}_{jkl}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)(v_l-u_l)\; f(\mathbf{v},\mathbf{x},t), \tag{5}$$

$$\mathsf{R}_{jklm}(\mathbf{x},t) \;=\; m^2\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)(v_l-u_l)(v_m-u_m)\; f(\mathbf{v},\mathbf{x},t), \tag{6}$$

$$F_j^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)\; C(\mathbf{v},\mathbf{x},t), \tag{7}$$

$$\mathsf{G}_{jk}^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)\; C(\mathbf{v},\mathbf{x},t), \tag{8}$$

$$\mathsf{H}_{jkl}^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\; (v_j-u_j)(v_k-u_k)(v_l-u_l)\; C(\mathbf{v},\mathbf{x},t). \tag{9}$$

Integrating the appropriately weighed kinetic equation over velocity space, one obtains the following system of macroscopic equations:

$$\frac{\partial n}{\partial t} \;+\; \frac{\partial (n u_j)}{\partial x_j} \;=\; 0, \tag{10}$$

$$mn\left(\frac{\partial u_j}{\partial t} + u_k\frac{\partial u_j}{\partial x_k}\right) \;+\; \frac{\partial \mathsf{P}_{jk}}{\partial x_k} \;-\; en\Big(E_j + \epsilon_{jkl}u_k B_l\Big) \;-\; F_j^{coll} \;=\; 0, \tag{11}$$

$$\frac{\partial \mathsf{P}_{jk}}{\partial t} \;+\; \frac{\partial}{\partial x_l}\left(\mathsf{P}_{jk}u_l + \mathsf{Q}_{jkl}\right) \;+\; \frac{\partial u_{[j}}{\partial x_l}\mathsf{P}_{lk]} \;-\; \frac{e}{m}\epsilon_{[jlm}B_m\mathsf{P}_{lk]} \;-\; \mathsf{G}_{jk}^{coll} \;=\; 0, \tag{12}$$

$$\begin{aligned}\frac{\partial \mathsf{Q}_{jkl}}{\partial t} \;&+\; \frac{\partial}{\partial x_m}\left(\mathsf{Q}_{jkl}u_m + \frac{1}{m}\mathsf{R}_{jklm}\right) \;+\; \frac{\partial u_{[j}}{\partial x_m}\mathsf{Q}_{mkl]} \;-\; \frac{e}{m}\epsilon_{[jmn}B_n\mathsf{Q}_{mkl]} \;-\\ &-\; \frac{1}{mn}\frac{\partial \mathsf{P}_{[jm}}{\partial x_m}\mathsf{P}_{kl]} \;+\; \frac{1}{mn}F_{[j}^{coll}\mathsf{P}_{kl]} \;-\; \mathsf{H}_{jkl}^{coll} \;=\; 0,\end{aligned} \tag{13}$$

where the square brackets around indices represent the minimal sum over permutations of uncontracted indices needed to yield completely symmetric tensors.

The stress and stress-flux tensors can be uniquely split into "Chew- Goldberger-Low" (CGL) and "perpendicular" (noted with a "hat") parts:

$$\mathsf{P}_{jk} \;=\; p_\perp \delta_{jk} \;+\; (p_\| - p_\perp) b_j b_k \;+\; \hat{\mathsf{P}}_{jk} \;=\; \mathsf{P}^{CGL}_{jk} \;+\; \hat{\mathsf{P}}_{jk} \;, \tag{14}$$

$$\mathsf{Q}_{jkl} \;=\; q_{T\|} \delta_{[jk} b_{l]} \;+\; (2q_{B\|} - 3q_{T\|}) b_j b_k b_l \;+\; \hat{\mathsf{Q}}_{jkl} \;=\; \mathsf{Q}^{CGL}_{jkl} \;+\; \hat{\mathsf{Q}}_{jkl} \;, \tag{15}$$

where b_j is the magnetic unit vector, $\hat{\mathsf{P}}_{jj} = \hat{\mathsf{P}}_{jk} b_j b_k = 0$ and $\hat{\mathsf{Q}}_{jkk} b_j = \hat{\mathsf{Q}}_{jkl} b_j b_k b_l = 0$. In the CGL tensors, $p_\perp$ and $p_\|$ are the perpendicular and parallel pressures with the mean scalar pressure defined as $p = (2p_\perp + p_\|)/3$, $q_{T\|}$ is the parallel flux of perpendicular heat and $q_{B\|}$ is the parallel flux of parallel heat. The total heat flux vector is

$$q_j = \mathsf{Q}_{jkk}/2 \;=\; (q_{T\|} + q_{B\|}) b_j + \hat{\mathsf{Q}}_{jkk}/2 \;=\; q_\| b_j + q_{\perp j} \tag{16}$$

and the total flux of parallel heat is

$$q_{Bj} = \mathsf{Q}_{jkl} b_k b_l/2 \;=\; q_{B\|} b_j + \hat{\mathsf{Q}}_{jkl} b_k b_l/2 \;=\; q_{B\|} b_j + q_{B\perp j} \;. \tag{17}$$

For the fourth-rank moment, it is useful to define

$$\mathsf{R}_{jklm} \;=\; \frac{1}{n}\, \mathsf{P}_{[jk}\, \mathsf{P}_{lm]} \;+\; \tilde{\mathsf{R}}_{jklm} \;. \tag{18}$$

The contribution of the factorized first term allows the proper account of temperature gradient effects, as can be seen by bringing this representation to Eq.(13) which becomes

$$\frac{\partial \mathsf{Q}_{jkl}}{\partial t} \;+\; \frac{\partial}{\partial x_m}\left(\mathsf{Q}_{jkl} u_m + \frac{1}{m} \tilde{\mathsf{R}}_{jklm} \right) \;+\; \frac{\partial u_{[j}}{\partial x_m} \mathsf{Q}_{mkl]} \;-\; \frac{e}{m} \epsilon_{[jmn} B_n \mathsf{Q}_{mkl]} \;+$$

$$+\; \frac{1}{m} \mathsf{P}_{[jm} \frac{\partial}{\partial x_m}\left(\frac{1}{n} \mathsf{P}_{kl]} \right) \;+\; \frac{1}{mn} F^{coll}_{[j} \mathsf{P}_{kl]} \;-\; \mathsf{H}^{coll}_{jkl} \;=\; 0. \tag{19}$$

The irreducible term $\tilde{\mathsf{R}}_{jklm}$ would vanish with a Maxwellian distribution function and accounts for purely kinetic effects such as wave-particle resonances and collisionless dissipation.

A formal solution for the stress and stress-flux tensors can be constructed as follows. Bringing the representations (14) and (15) to the evolution equations (12) and (19), these can be written as

$$\epsilon_{[jlm}b_m\hat{\mathsf{P}}_{lk]} \;=\; \mathsf{K}_{jk} \tag{20}$$

and

$$\epsilon_{[jmn}b_n\hat{\mathsf{Q}}_{mkl]} \;=\; \mathsf{L}_{jkl} \;, \tag{21}$$

where

$$\mathsf{K}_{jk} \;=\; \frac{m}{eB}\left[\frac{\partial \mathsf{P}_{jk}}{\partial t} \;+\; \frac{\partial}{\partial x_l}\left(\mathsf{P}_{jk}u_l + \mathsf{Q}_{jkl}\right) \;+\; \frac{\partial u_{[j}}{\partial x_l}\mathsf{P}_{lk]} \;-\; \mathsf{G}^{coll}_{jk}\right] \tag{22}$$

and

$$\mathsf{L}_{jkl} \;=\; \frac{m}{eB}\left[\frac{\partial \mathsf{Q}_{jkl}}{\partial t} \;+\; \frac{\partial}{\partial x_m}\left(\mathsf{Q}_{jkl}u_m + \frac{1}{m}\tilde{\mathsf{R}}_{jklm}\right) \;+\; \frac{\partial u_{[j}}{\partial x_m}\mathsf{Q}_{mkl]} \;+\right.$$

$$\left. +\; \frac{1}{m}\mathsf{P}_{[jm}\frac{\partial}{\partial x_m}\left(\frac{1}{n}\mathsf{P}_{kl]}\right) \;+\; \frac{1}{mn}F^{coll}_{[j}\mathsf{P}_{kl]} \;-\; \mathsf{H}^{coll}_{jkl}\right]. \tag{23}$$

These equations are subject to the solubility constraints $\mathsf{K}_{jj} = \mathsf{K}_{jk}b_jb_k = 0$ and $\mathsf{L}_{jkk}b_j = \mathsf{L}_{jkl}b_jb_kb_l = 0$, which correspond to the dynamic evolution equations for the CGL variables:

$$\frac{3}{2}\left[\frac{\partial p}{\partial t} + \frac{\partial (pu_j)}{\partial x_j}\right] \;+\; \mathsf{P}_{jk}\frac{\partial u_j}{\partial x_k} \;+\; \frac{\partial q_j}{\partial x_j} \;-\; g^{coll} \;=\; 0 \;, \tag{24}$$

$$\frac{1}{2}\left[\frac{\partial p_\parallel}{\partial t} + \frac{\partial (p_\parallel u_j)}{\partial x_j}\right] \;-\; \mathsf{P}_{jk}b_j\left[\frac{\partial b_k}{\partial t} + u_l\frac{\partial b_k}{\partial x_l} - b_l\frac{\partial u_l}{\partial x_k}\right] \;+\; \frac{\partial q_{Bj}}{\partial x_j} \;-\; \mathsf{Q}_{jkl}b_j\frac{\partial b_k}{\partial x_l} \;-\; g^{coll}_B \;=\; 0 \;, \tag{25}$$

$$\frac{\partial q_\parallel}{\partial t} + \frac{\partial (q_\parallel u_j)}{\partial x_j} \;-\; q_j\left[\frac{\partial b_j}{\partial t} + u_k\frac{\partial b_j}{\partial x_k} - b_k\frac{\partial u_k}{\partial x_j}\right] \;+\; \mathsf{Q}_{jkl}b_j\frac{\partial u_k}{\partial x_l} \;+\; \frac{1}{m}\mathsf{P}_{jk}b_l\frac{\partial}{\partial x_j}\left(\frac{3p}{2n}\delta_{kl} + \frac{1}{n}\mathsf{P}_{kl}\right) \;+$$

$$+\; \frac{1}{2m}b_j\frac{\partial \tilde{\mathsf{R}}_{jkll}}{\partial x_k} \;+\; \frac{1}{mn}b_jF^{coll}_k\left(\frac{3p}{2}\delta_{jk} + \mathsf{P}_{jk}\right) \;-\; h^{coll} \;=\; 0 \tag{26}$$

and

$$\frac{\partial q_{B\|}}{\partial t} + \frac{\partial (q_{B\|} u_j)}{\partial x_j} \;-\; 3 q_{Bj} \left[\frac{\partial b_j}{\partial t} + u_k \frac{\partial b_j}{\partial x_k} - b_k \frac{\partial u_k}{\partial x_j} \right] \;+\; \frac{3}{2m} \mathsf{P}_{jk} b_k \frac{\partial}{\partial x_j} \left(\frac{p_\|}{n} \right) -$$

$$- \frac{3}{mn} \mathsf{P}_{jl} \mathsf{P}_{km} b_j b_k \frac{\partial b_l}{\partial x_m} \;+\; \frac{1}{2m} b_j b_k b_l \frac{\partial \tilde{\mathsf{R}}_{jklm}}{\partial x_m} \;+\; \frac{3 p_\|}{2mn} b_j F_j^{coll} \;-\; h_B^{coll} \;=\; 0 \; . \tag{27}$$

Here we have defined the collisional exchange rates $g^{coll} = \mathsf{G}_{jj}^{coll}/2$, $g_B^{coll} = \mathsf{G}_{jk}^{coll} b_j b_k /2$, $h^{coll} = \mathsf{H}_{jkk}^{coll} b_j/2$ and $h_B^{coll} = \mathsf{H}_{jkl}^{coll} b_j b_k b_l /2$. Then, Eqs.(20) and (21) can be inverted to yield:

$$\hat{\mathsf{P}}_{jk} = \frac{1}{4} \epsilon_{[jlm} b_l \mathsf{K}_{mn} \Big(\delta_{nk]} + 3 b_n b_{k]} \Big) \; , \tag{28}$$

$$\hat{\mathsf{Q}}_{jkl} = \frac{1}{3} \epsilon_{[jmn} b_m \mathsf{L}_{nkl]} - \frac{1}{12} \epsilon_{[jmn} b_k b_m b_p \mathsf{L}_{npl]} + \frac{2}{9} \epsilon_{[jmn} \epsilon_{kpq} \epsilon_{lrs]} b_m b_p b_r \mathsf{L}_{nqs} + \frac{5}{6} \epsilon_{[jmn} b_k b_{l]} b_m b_p b_q \mathsf{L}_{npq} \; . \tag{29}$$

Since K_{jk} and L_{jkl} (22,23) are proportional to the inverse of the gyrofrequency, $\Omega_c = eB/m$, these equations are amenable to a perturbative expansion in the case of strong magnetization, thus yielding explicit algebraic representations for $\hat{\mathsf{P}}_{jk}$ and $\hat{\mathsf{Q}}_{jkl}$, and explicit evolution equations for P_{jk}^{CGL} and Q_{jkl}^{CGL}. The tensor $\tilde{\mathsf{R}}_{jklm}$ and the collisional terms are the closure variables that must be provided by kinetic theory.

III The Fokker-Plank collision operator and its fluid moments.

The irreversible part of the plasma dynamics will be modeled with a Fokker-Plank operator for binary Coulomb collisions in the kinetic equation (1). The Fokker-Plank collision operator will be kept in its complete, quadratic form so that the analysis remains valid for far-from-Maxwellian distribution functions. Introducing the species indices (α, β) and adopting the Landau form[18] in the rationalized electromagnetic system of units being used throughout this work,

$$C_\alpha(\mathbf{v}, \mathbf{x}, t) \;=\; - \sum_\beta \; \frac{c^4 e_\alpha^2 e_\beta^2 \ln \Lambda_{\alpha\beta}}{8 \pi m_\alpha} \; \Gamma_{\alpha\beta}(\mathbf{v}, \mathbf{x}, t) \; , \tag{30}$$

where $\ln \Lambda_{\alpha\beta} = \ln \Lambda_{\beta\alpha}$ are the Coulomb logarithms,

$$\Gamma_{\alpha\beta}(\mathbf{v}, \mathbf{x}, t) \;=\; \frac{\partial}{\partial v_j} \int d^3\mathbf{w} \; \mathsf{U}_{jk}(\mathbf{v}, \mathbf{w}) \left[\frac{f_\alpha(\mathbf{v}, \mathbf{x}, t)}{m_\beta} \; \frac{\partial f_\beta(\mathbf{w}, \mathbf{x}, t)}{\partial w_k} \;-\; \frac{f_\beta(\mathbf{w}, \mathbf{x}, t)}{m_\alpha} \; \frac{\partial f_\alpha(\mathbf{v}, \mathbf{x}, t)}{\partial v_k} \right] \tag{31}$$

and

$$\mathsf{U}_{jk}(\mathbf{v},\mathbf{w}) \;=\; \frac{|\mathbf{v}-\mathbf{w}|^2\ \delta_{jk}\ -\ (v_j-w_j)(v_k-w_k)}{|\mathbf{v}-\mathbf{w}|^3}\ . \tag{32}$$

We shall assume a single ion species of unit charge, $\alpha,\beta\in(\iota,e)$, $e_\iota=-e_e=e$ and take now $\alpha\neq\beta$. Then, considering the fluid moments of the collision operator (7-9) and dropping the $(\mathbf{x},t)$ arguments, we obtain after integrations by parts:

$$F^{coll}_{\alpha,j} \;=\; -F^{coll}_{\beta,j} \;=\; -\frac{c^4e^4}{4\pi}\ln\Lambda_{\alpha\beta}\left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\int\!\!\int d^3\mathbf{v}\ d^3\mathbf{w}\ f_\alpha(\mathbf{v})\ f_\beta(\mathbf{w})\ \frac{v_j-w_j}{|\mathbf{v}-\mathbf{w}|^3}\ , \tag{33}$$

$$\begin{aligned}\mathsf{G}^{coll}_{\alpha,jk} \;=\; \frac{c^4e^4}{4\pi}\Bigg[&\frac{\ln\Lambda_{\alpha\alpha}}{m_\alpha}\int\!\!\int d^3\mathbf{v}\ d^3\mathbf{w}\ f_\alpha(\mathbf{v})\ f_\alpha(\mathbf{w})\ \frac{|\mathbf{v}-\mathbf{w}|^2\ \delta_{jk}-3(v_j-w_j)(v_k-w_k)}{|\mathbf{v}-\mathbf{w}|^3}\ +\\ &+\frac{\ln\Lambda_{\alpha\beta}}{m_\alpha}\int\!\!\int d^3\mathbf{v}\ d^3\mathbf{w}\ f_\alpha(\mathbf{v})\ f_\beta(\mathbf{w})\ \frac{|\mathbf{v}-\mathbf{w}|^2\ \delta_{jk}-(v_j-w_j)(v_k-w_k)}{|\mathbf{v}-\mathbf{w}|^3}\ -\\ &-\ \ln\Lambda_{\alpha\beta}\left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\int\!\!\int d^3\mathbf{v}\ d^3\mathbf{w}\ f_\alpha(\mathbf{v})\ f_\beta(\mathbf{w})\ \frac{(v_{[j}-w_{[j})(v_{k]}-u_{\alpha,k]})}{|\mathbf{v}-\mathbf{w}|^3}\Bigg]\end{aligned} \tag{34}$$

and

$$\begin{aligned}\mathsf{H}^{coll}_{\alpha,jkl} = \frac{c^4e^4}{4\pi}\Bigg\{&\frac{\ln\Lambda_{\alpha\alpha}}{m_\alpha}\int\!\!\int d^3\mathbf{v}\ d^3\mathbf{w}\ f_\alpha(\mathbf{v})\ f_\alpha(\mathbf{w})\ \frac{(v_{[j}-u_{\alpha,[j})\Big[|\mathbf{v}-\mathbf{w}|^2\ \delta_{kl]}-3(v_k-w_k)(v_{l]}-w_{l]})\Big]}{|\mathbf{v}-\mathbf{w}|^3}\ +\\ &+\frac{\ln\Lambda_{\alpha\beta}}{m_\alpha}\int\!\!\int d^3\mathbf{v}\ d^3\mathbf{w}\ f_\alpha(\mathbf{v})\ f_\beta(\mathbf{w})\ \frac{(v_{[j}-u_{\alpha,[j})\Big[|\mathbf{v}-\mathbf{w}|^2\ \delta_{kl]}-(v_k-w_k)(v_{l]}-w_{l]})\Big]}{|\mathbf{v}-\mathbf{w}|^3}\ -\\ &-\ \ln\Lambda_{\alpha\beta}\left(\frac{1}{m_\alpha}+\frac{1}{m_\beta}\right)\int\!\!\int d^3\mathbf{v}\ d^3\mathbf{w}\ f_\alpha(\mathbf{v})\ f_\beta(\mathbf{w})\ \frac{(v_{[j}-w_{[j})\Big[(v_k-u_{\alpha,k})(v_{l]}-u_{\alpha,l]})\Big]}{|\mathbf{v}-\mathbf{w}|^3}\ \Bigg\}\ .\end{aligned} \tag{35}$$

IV Asymptotic expansions.

The general fluid moment equations given in the previous two Sections yield a workable fluid description of the macroscopic plasma dynamics after an asymptotic expansion for strong magnetization. The fundamental expansion parameter is the ratio $\delta \sim \rho_\iota / L$ between the ion Larmor radius and the shortest characteristic length other than the gyroradii, typically a fluctuation perpendicular wavelength or a perpendicular gradient scale length. The ensuing macroscopic fluid theory will therefore apply to phenomena where physical effects associated with length scales comparable to the gyroradii (such as small scale turbulence) can be neglected and the ratio δ can indeed be taken as much smaller than unity. In addition, the relative orderings of the ratio m_e/m_ι between the electron and ion masses and the ratio $\nu_\iota/\Omega_{c\iota}$ between the ion collision and cyclotron frequencies must be specified. As discussed in the Introduction, the present low-collisionality macroscopic analysis will adopt as basic working hypotheses $(m_e/m_\iota)^{1/2} \lesssim \delta \ll 1$ and $\nu_\iota/\Omega_{c\iota} \lesssim \delta\ (m_e/m_\iota)^{1/2}$. It will also be assumed that the plasma is quasineutral with a single ion species of unit charge, $n_\iota = n_e = n$, that the ion and electron pressures are comparable, $p_\iota \sim p_e$, and that the pressure anisotropies are arbitrary, $(p_{\alpha\|} - p_{\alpha\perp}) \sim p_\alpha$. The ordering of the partial time derivatives will be linked to the macroscopic flow velocities of the ions and electrons by $\partial/\partial t \sim u_\alpha / L$ and, in order to cover both the fast (sonic) and slow (diamagnetic drift) motions, it will be assumed $\delta v_{th\iota} \lesssim u_\iota \sim u_e \lesssim v_{th\iota}$. Finally, the requirement that the electromagnetic force $\mathbf{j} \times \mathbf{B}$ be balanced by either pressure gradients or inertial forces with sonic or diamagnetic flows yields $\mathbf{j}/(en) = \mathbf{u}_\iota - \mathbf{u}_e \sim \delta v_{th\iota}$.

The ion and electron thermal velocities are defined here as $v_{th\alpha} = [p_\alpha/(m_\alpha n)]^{1/2}$. For the collision frequencies, the following definitions are adopted:

$$\nu_\iota = \frac{c^4 e^4 n \ln \Lambda_{\iota\iota}}{4\pi m_\iota^2 v_{th\iota}^3} , \tag{36}$$

$$\nu_e = \frac{c^4 e^4 n \ln \Lambda_{e\iota}}{4\pi m_e^2 v_{the}^3} , \tag{37}$$

$$\nu_{ee} = \frac{c^4 e^4 n \ln \Lambda_{ee}}{4\pi m_e^2 v_{the}^3} . \tag{38}$$

Within the temperature ranges of interest for magnetic fusion, it can be taken $\ln \Lambda_{ee} = \ln \Lambda_{e\iota}$, hence $\nu_{ee} = \nu_e$. These natural definitions of the collision frequencies differ by numerical factors from the inverse collision times τ_α^{-1} defined in Ref.2 and widely used in the literature: $\nu_\iota = 3\pi^{1/2}\tau_\iota^{-1}$ and $\nu_e = 3(\pi/2)^{1/2}\tau_e^{-1}$.

In order to carry out the asymptotic expansion of the collisional moments, it is useful to introduce the dimensionless velocity space coordinates $\boldsymbol{\xi}$ defined by

$$\mathbf{v} = \mathbf{u}_\alpha + v_{th\alpha}\ \boldsymbol{\xi} \tag{39}$$

and the dimensionless distribution functions $\hat{f}_\alpha(\boldsymbol{\xi})$ defined by

$$f_\alpha(\mathbf{v}) \ = \ f_\alpha(\mathbf{u}_\alpha + v_{th\alpha}\ \boldsymbol{\xi}) \ = \ \frac{n}{v_{th\alpha}^3}\hat{f}_\alpha(\boldsymbol{\xi}) \ , \tag{40}$$

such that

$$\int d^3\boldsymbol{\xi}\ \hat{f}_\alpha(\boldsymbol{\xi}) \ = \ 1 \ , \tag{41}$$

$$\int d^3\boldsymbol{\xi}\ \xi_j\ \hat{f}_\alpha(\boldsymbol{\xi}) \ = \ 0 \ , \tag{42}$$

$$\int d^3\boldsymbol{\xi}\ \xi_j\xi_k\ \hat{f}_\alpha(\boldsymbol{\xi}) \ = \ \frac{1}{p_\alpha}\ \mathsf{P}_{\alpha,jk} \tag{43}$$

and

$$\int d^3\boldsymbol{\xi}\ \xi_j\xi_k\xi_l\ \hat{f}_\alpha(\boldsymbol{\xi}) \ = \ \frac{1}{p_\alpha v_{th\alpha}}\ \mathsf{Q}_{\alpha,jkl} \ . \tag{44}$$

In terms of the above variables, the collisional friction force (33) becomes

$$F_{\iota,j}^{coll} \ = \ -F_{e,j}^{coll} \ = \ \frac{\nu_e p_e}{v_{the}}\left(1+\frac{m_e}{m_\iota}\right)\int\int d^3\boldsymbol{\xi}\ d^3\boldsymbol{\zeta}\ \hat{f}_\iota(\boldsymbol{\xi})\ \hat{f}_e\left(\boldsymbol{\zeta}+\frac{v_{th\iota}}{v_{the}}\boldsymbol{\xi}+\frac{1}{env_{the}}\mathbf{j}\right)\ \frac{\zeta_j}{\zeta^3} \ , \tag{45}$$

which is an expression suitable for the asymptotic expansion under our assumed orderings. By virtue of these, $v_{th\iota}/v_{the} \sim (m_e/m_\iota)^{1/2} \lesssim \delta$ and $j/(env_{the}) \sim \delta(m_e/m_\iota)^{1/2} \lesssim \delta^2$ so, for $\xi = O(1)$, we can Taylor expand:

$$\hat{f}_e\left(\boldsymbol{\zeta}+\frac{v_{th\iota}}{v_{the}}\boldsymbol{\xi}+\frac{1}{env_{the}}\mathbf{j}\right) \ = \ \hat{f}_e(\boldsymbol{\zeta}) + \left(\frac{v_{th\iota}}{v_{the}}\xi_k+\frac{1}{env_{the}}j_k\right)\frac{\partial\hat{f}_e(\boldsymbol{\zeta})}{\partial\zeta_k} + \frac{v_{th\iota}^2}{2v_{the}^2}\xi_k\xi_l\frac{\partial^2\hat{f}_e(\boldsymbol{\zeta})}{\partial\zeta_k\partial\zeta_l} + O(\delta^3) \ . \tag{46}$$

Provided the distribution functions decay sufficiently fast at high energies, the integrals over the $\boldsymbol{\xi}$ variable in (45) can now be carried out using (41-43). Using also the relationship $\zeta_j/\zeta^3 = -\partial(1/\zeta)/\partial\zeta_j$ to integrate by parts with respect to the $\boldsymbol{\zeta}$ variable, we obtain:

$$F^{coll}_{\iota,j} \;=\; -F^{coll}_{e,j} \;=\; \frac{\nu_e p_e}{v_{the}}\left(1+\frac{m_e}{m_\iota}\right)\left[\int d^3\boldsymbol{\zeta}\;\frac{\zeta_j}{\zeta^3}\hat{f}_e(\boldsymbol{\zeta}) \;+\right.$$

$$\left. +\;\frac{1}{env_{the}} j_k \int \frac{d^3\boldsymbol{\zeta}}{\zeta}\;\frac{\partial^2 \hat{f}_e(\boldsymbol{\zeta})}{\partial\zeta_j\partial\zeta_k} \;+\; \frac{m_e}{2m_\iota p_e}\mathsf{P}_{\iota,kl}\int\frac{d^3\boldsymbol{\zeta}}{\zeta}\;\frac{\partial^3\hat{f}_e(\boldsymbol{\zeta})}{\partial\zeta_j\partial\zeta_k\partial\zeta_l} \;+\; O(\delta^3)\right] . \tag{47}$$

For the magnetized plasmas under consideration, the distribution functions can be expanded as

$$\hat{f}_\alpha(\boldsymbol{\xi}) \;=\; \hat{f}^{(0)}_\alpha(\xi,\xi_\parallel) \;+\; \hat{f}^{(1)}_\alpha(\boldsymbol{\xi}) \;+\; O(\delta^2_\alpha) \;, \tag{48}$$

where $\xi_\parallel = \xi_j b_j$ is the dimensionless parallel velocity coordinate, $\hat{f}^{(0)}_\alpha(\xi,\xi_\parallel) = O(1)$ is independent of the gyrophase, $\hat{f}^{(1)}_\alpha(\boldsymbol{\xi}) = O(\delta_\alpha)$ and $\delta_\alpha \sim \rho_\alpha/L \sim \delta\;(m_\alpha/m_\iota)^{1/2}$. In addition, as a consequence of Eqs.(20,22) and our low collisionality orderings, the "perpendicular" part of the ion stress tensor is $\hat{\mathsf{P}}_{\iota,kl} \lesssim \delta p_\iota$. Therefore, within the retained accuracy of $O(\delta^2\nu_e p_e/v_{the})$, we can write:

$$F^{coll}_{\iota,j} \;=\; -F^{coll}_{e,j} \;=\; \frac{\nu_e p_e}{v_{the}}\left[\left(1+\frac{m_e}{m_\iota}\right)b_j\int d^3\boldsymbol{\zeta}\;\frac{\zeta_\parallel}{\zeta^3}\hat{f}^{(0)}_e(\zeta,\zeta_\parallel) \;+\int d^3\boldsymbol{\zeta}\;\frac{\zeta_j}{\zeta^3}\hat{f}^{(1)}_e(\boldsymbol{\zeta}) \;+\right.$$

$$\left. +\;\frac{1}{env_{the}} j_k \int \frac{d^3\boldsymbol{\zeta}}{\zeta}\;\frac{\partial^2 \hat{f}^{(0)}_e(\zeta,\zeta_\parallel)}{\partial\zeta_j\partial\zeta_k} \;+\; \frac{m_e}{2m_\iota p_e}\mathsf{P}^{CGL}_{\iota,kl}\int\frac{d^3\boldsymbol{\zeta}}{\zeta}\;\frac{\partial^3\hat{f}^{(0)}_e(\zeta,\zeta_\parallel)}{\partial\zeta_j\partial\zeta_k\partial\zeta_l} \;+\; O(\delta^3)\right] . \tag{49}$$

If the zeroth-order electron distribution function were isotropic (not necessarily Maxwellian), i.e. $\hat{f}^{(0)}_e = \hat{f}^{(0)}_e(\xi)$, the first and last terms on the right hand side of Eq.(49) would vanish and the leading order collisional friction force would reduce to

$$F^{coll}_{\iota,j} \;=\; -F^{coll}_{e,j} \;=\; \frac{\nu_e p_e}{v_{the}}\left[\int d^3\boldsymbol{\zeta}\;\frac{\zeta_j}{\zeta^3}\hat{f}^{(1)}_e(\boldsymbol{\zeta}) \;-\; \frac{4\pi\hat{f}^{(0)}_e(0)}{3env_{the}}\;j_j\right] \;=\; O\left(\delta_e\frac{\nu_e p_e}{v_{the}}\right) , \tag{50}$$

which contains the results obtained in high-collisionality theories[2,19,20]. However, in the low-collisionality regime of interest here, nothing in principle prevents the distribution function from having a zeroth-order anisotropic part, odd along the direction of the magnetic field, that would contribute to both the first and last terms of (49). In this case, the leading order collisional friction force stems just from the first term and is

$$F^{coll}_{\iota,j} \;=\; -F^{coll}_{e,j} \;=\; \frac{\nu_e p_e}{v_{the}}\; b_j \int d^3\boldsymbol{\zeta}\; \frac{\zeta_\parallel}{\zeta^3} \hat{f}^{(0)}_e(\zeta,\zeta_\parallel) \;=\; O\left(\frac{\nu_e p_e}{v_{the}}\right)\,, \tag{51}$$

the remaining terms giving corrections of order $\delta_e \nu_e p_e / v_{the} \lesssim \delta^2 \nu_e p_e / v_{the}$ or higher.

The higher-rank collisional moments can be expanded in a similar manner. For the second-rank moments, keeping $O(\nu_e p_e)$ and $O(\nu_\iota p_\iota)$, but neglecting $O(\delta_e \nu_e p_e) \sim O(\delta \nu_\iota p_\iota)$ we get:

$$\begin{aligned} \mathsf{G}^{coll}_{e,jk} \;=\; & \frac{1}{2}\, \nu_e p_e\; (3 b_j b_k - \delta_{jk}) \int d^3\boldsymbol{\xi}\; \frac{\xi^2 - 3\xi_\parallel^2}{\xi^3} \Bigg[\hat{f}^{(0)}_e(\xi,\ \xi_\parallel) \,+ \\ & + \int d^3\boldsymbol{\zeta}\;\; \hat{f}^{(0)}_e\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\parallel + \zeta_\parallel\Big)\;\; \hat{f}^{(0)}_e(\zeta,\ \zeta_\parallel) \Bigg] \;+\; O(\delta_e \nu_e p_e) \end{aligned} \tag{52}$$

and

$$\mathsf{G}^{coll}_{\iota,jk} = \frac{1}{2} \nu_\iota p_\iota (3 b_j b_k - \delta_{jk}) \int d^3\boldsymbol{\xi}\; \frac{\xi^2 - 3\xi_\parallel^2}{\xi^3} \int d^3\boldsymbol{\zeta}\;\; \hat{f}^{(0)}_\iota\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\parallel + \zeta_\parallel\Big) \hat{f}^{(0)}_\iota(\zeta,\ \zeta_\parallel) + O(\delta \nu_\iota p_\iota). \tag{53}$$

For the electrons, keeping $O(\nu_e p_e v_{the})$, but neglecting $O(\delta_e \nu_e p_e v_{the})$, the term needed to evaluate the collisional contribution to the third-rank stress-flux tensor is (23):

$$\begin{aligned} & \mathsf{H}^{coll}_{e,jkl} \;-\; \frac{1}{m_e n} F^{coll}_{e,[j}\;\; \mathsf{P}_{e,kl]} \;= \\ & = \; \nu_e p_e v_{the} \Bigg\{ b_{[j}\delta_{kl]} \Bigg[\frac{1}{2} \int\!\!\int d^3\boldsymbol{\xi}\; d^3\boldsymbol{\zeta}\;\; \frac{9\xi_\parallel^2 \zeta_\parallel - \xi^2 \zeta_\parallel - 6\xi_\parallel \boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\;\; \hat{f}^{(0)}_e\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\parallel + \zeta_\parallel\Big)\; \hat{f}^{(0)}_e(\zeta,\ \zeta_\parallel) \,+ \\ & \quad + \int d^3\boldsymbol{\xi}\; \frac{\xi_\parallel\;(3\xi_\parallel^2 - 2\xi^2)}{\xi^3}\;\; \hat{f}^{(0)}_e(\xi,\ \xi_\parallel) \;+ \frac{p_{e\perp}}{p_e} \int d^3\boldsymbol{\xi}\; \frac{\xi_\parallel}{\xi^3}\;\; \hat{f}^{(0)}_e(\xi,\ \xi_\parallel) \Bigg] \,+ \\ & \quad + b_j b_k b_l \Bigg[\frac{9}{2} \int\!\!\int d^3\boldsymbol{\xi}\; d^3\boldsymbol{\zeta}\;\; \frac{\xi^2 \zeta_\parallel - 5\xi_\parallel^2 \zeta_\parallel + 2\xi_\parallel \boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\;\; \hat{f}^{(0)}_e\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\parallel + \zeta_\parallel\Big)\; \hat{f}^{(0)}_e(\zeta,\ \zeta_\parallel) \,+ \\ & + 3 \int d^3\boldsymbol{\xi}\;\; \frac{\xi_\parallel (3\xi^2 - 5\xi_\parallel^2)}{\xi^3}\;\; \hat{f}^{(0)}_e(\xi,\ \xi_\parallel) \;+\; \frac{3(p_{e\parallel} - p_{e\perp})}{p_e} \int d^3\boldsymbol{\xi}\; \frac{\xi_\parallel}{\xi^3}\;\; \hat{f}^{(0)}_e(\xi,\ \xi_\parallel) \Bigg] \Bigg\} \;+\; O(\delta_e \nu_e p_e v_{the}). \end{aligned} \tag{54}$$

For the ions, keeping $O(\nu_\iota p_\iota v_{th\iota})$, but neglecting $O(\delta\nu_\iota p_\iota v_{th\iota})$:

$$\mathsf{H}^{coll}_{\iota,jkl} \;-\; \frac{1}{m_\iota n} F^{coll}_{\iota,[j}\ \mathsf{P}_{\iota,kl]} \;=$$

$$= \;\nu_\iota p_\iota v_{th\iota} \left[\frac{1}{2}\ b_{[j}\delta_{kl]}\ \int\int d^3\boldsymbol{\xi}\ d^3\boldsymbol{\zeta}\ \frac{9\xi_\parallel^2\zeta_\parallel - \xi^2\zeta_\parallel - 6\xi_\parallel\boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\ \hat{f}^{(0)}_\iota\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\parallel+\zeta_\parallel\Big)\ \hat{f}^{(0)}_\iota(\zeta,\ \zeta_\parallel) + \right.$$

$$\left. +\frac{9}{2} b_j b_k b_l \int\int d^3\boldsymbol{\xi}\ d^3\boldsymbol{\zeta}\ \frac{\xi^2\zeta_\parallel - 5\xi_\parallel^2\zeta_\parallel + 2\xi_\parallel\boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\ \hat{f}^{(0)}_\iota\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\parallel+\zeta_\parallel\Big)\hat{f}^{(0)}_\iota(\zeta,\ \zeta_\parallel)\right] + O(\delta\nu_\iota p_\iota v_{th\iota}). \quad (55)$$

The perpendicular stress-flux tensors, $\hat{\mathsf{Q}}_{\alpha,jkl}$, will be evaluated in their lowest significant order, $O(\delta_\alpha p_\alpha v_{th\alpha})$. To obtain this accuracy, the collisional terms $\mathsf{H}^{coll}_{\alpha,jkl} \;-\; F^{coll}_{\alpha,[j}\ \mathsf{P}_{\alpha,kl]}/(m_\alpha n)$ of Eq.(23) are needed only to $O(\nu_\alpha p_\alpha v_{th\alpha})$, as given by Eqs.(54,55). However, these expressions give a null contribution to $\hat{\mathsf{Q}}_{\alpha,jkl}$ when inserted in Eq.(29). Therefore, keeping only $O(\delta_\alpha p_\alpha v_{th\alpha})$ and allowing for the fastest flow velocities $u_\alpha \sim v_{th\iota}$, the perpendicular stress-flux tensors have just the collision-independent form[8]:

$$\hat{\mathsf{Q}}_{\alpha,jkl} \;=\; 2b_{[j}b_k q_{\alpha B\perp,l]} \;+\; \frac{1}{2}(\delta_{[jk} - b_{[j}b_k) q_{\alpha T\perp,l]} \;+\; \epsilon_{[jmn}b_k b_m \mathsf{T}_{\alpha,np}(\delta_{pl]} - b_p b_{l]}), \quad (56)$$

where $q_{\alpha B\perp,l} = \hat{\mathsf{Q}}_{\alpha,jkl}b_j b_k/2$ are the perpendicular fluxes of parallel heat given by

$$\mathbf{q}_{\iota B\perp} \;=\; \frac{1}{eB}\mathbf{b}\times\left[p_{\iota\perp}\nabla\left(\frac{p_{\iota\parallel}}{2n}\right) + \frac{p_{\iota\parallel}(p_{\iota\parallel}-p_{\iota\perp})}{n}\boldsymbol{\kappa} + 2m_\iota q_{\iota B\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota + m_\iota q_{\iota T\parallel}\mathbf{b}\times\boldsymbol{\omega}_\iota\right] \;+\; \tilde{\mathbf{q}}_{\iota B\perp}\ , \quad (57)$$

$$\mathbf{q}_{eB\perp} \;=\; -\,\frac{1}{eB}\mathbf{b}\times\left[p_{e\perp}\nabla\left(\frac{p_{e\parallel}}{2n}\right) + \frac{p_{e\parallel}(p_{e\parallel}-p_{e\perp})}{n}\boldsymbol{\kappa}\right] \;+\; \tilde{\mathbf{q}}_{eB\perp}\ , \quad (58)$$

$q_{\alpha T\perp,l} = \hat{\mathsf{Q}}_{\alpha,jkl}(\delta_{jk} - b_j b_k)/2$ are the perpendicular fluxes of perpendicular heat given by

$$\mathbf{q}_{\iota T\perp} \;=\; \frac{1}{eB}\mathbf{b}\times\left[p_{\iota\perp}\nabla\left(\frac{2p_{\iota\perp}}{n}\right) + 4m_\iota q_{\iota T\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\right] \;+\; \tilde{\mathbf{q}}_{\iota T\perp}\ , \quad (59)$$

$$\mathbf{q}_{eT\perp} \;=\; -\,\frac{1}{eB}\mathbf{b}\times\left[p_{e\perp}\nabla\left(\frac{2p_{e\perp}}{n}\right)\right] \;+\; \tilde{\mathbf{q}}_{eT\perp}\ , \quad (60)$$

and the second-rank tensors $\mathsf{T}_{\alpha,np}$ are

$$\mathsf{T}_{\iota,np} \;=\; \frac{1}{4eB}\left[m_\iota q_{\iota T\|}\frac{\partial u_{\iota,[n}}{\partial x_{p]}} \;+\; \frac{p_{\iota\perp}(p_{\iota\|}-p_{\iota\perp})}{n}\frac{\partial b_{[n}}{\partial x_{p]}}\right] \;+\; \tilde{\mathsf{T}}_{\iota,np}\ , \tag{61}$$

$$\mathsf{T}_{e,np} \;=\; -\,\frac{p_{e\perp}(p_{e\|}-p_{e\perp})}{4eBn}\frac{\partial b_{[n}}{\partial x_{p]}} \;+\; \tilde{\mathsf{T}}_{e,np}\ . \tag{62}$$

Here, $\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b}$ stands for the magnetic curvature and $\boldsymbol{\omega}_\iota = \nabla\times\mathbf{u}_\iota$ for the ion vorticity. These expressions include the closure terms

$$\tilde{\mathbf{q}}_{\alpha B\perp} = \frac{1}{e_\alpha B}\mathbf{b}\times\left[\nabla(\tilde{r}^{(0)}_{\alpha\perp}+\tilde{r}^{(0)}_{\alpha\Delta})/5 + (\tilde{r}^{(0)}_{\alpha\|}-\tilde{r}^{(0)}_{\alpha\perp}-\tilde{r}^{(0)}_{\alpha\Delta})\boldsymbol{\kappa}\right]\ , \tag{63}$$

$$\tilde{\mathbf{q}}_{\alpha T\perp} = \frac{1}{e_\alpha B}\mathbf{b}\times\left[\nabla(4\tilde{r}^{(0)}_{\alpha\perp}-\tilde{r}^{(0)}_{\alpha\Delta})/5 + \tilde{r}^{(0)}_{\alpha\Delta}\boldsymbol{\kappa}\right]\ , \tag{64}$$

and

$$\tilde{\mathsf{T}}_{\alpha,np} \;=\; \frac{\tilde{r}^{(0)}_{\alpha\Delta}}{2e_\alpha B}\,\frac{\partial b_{[n}}{\partial x_{p]}}\ , \tag{65}$$

where $\tilde{r}^{(0)}_{\alpha\perp}$, $\tilde{r}^{(0)}_{\alpha\|}$ and $\tilde{r}^{(0)}_{\alpha\Delta}$ are the zero-Larmor-radius components of the $\tilde{\mathsf{R}}_{\alpha,jklm}$ tensors that must be provided by kinetic theory:

$$\tilde{\mathsf{R}}^{(0)}_{\alpha,jklm} \;=\; (2\tilde{r}^{(0)}_{\alpha\perp}/5-\tilde{r}^{(0)}_{\alpha\Delta}/10)\;\delta_{[jk}\delta_{lm]} \;+\; \tilde{r}^{(0)}_{\alpha\Delta}\;\delta_{[jk}b_lb_{m]}/2 \;+\; (2\tilde{r}^{(0)}_{\alpha\|}-2\tilde{r}^{(0)}_{\alpha\perp}-7\tilde{r}^{(0)}_{\alpha\Delta}/2)\;b_jb_kb_lb_m\ . \tag{66}$$

These three scalars are moments of the difference between the actual zeroth-order distribution functions, $\hat{f}^{(0)}_\alpha(\xi,\ \xi_\|)$, and the two-temperature Maxwellians, $\hat{f}_{M\alpha}(\xi,\ \xi_\|)$. Therefore, they are well suited for a Landau-fluid closure approximation[3,4,7]. Specifically, they are:

$$\tilde{r}^{(0)}_{\alpha\perp} \;=\; \frac{p^2_\alpha}{4n}\int d^3\boldsymbol{\xi}\;\xi^2(\xi^2-\xi^2_\|)\;\left[\hat{f}^{(0)}_\alpha(\xi,\ \xi_\|)-\hat{f}_{M\alpha}(\xi,\ \xi_\|)\right]\ , \tag{67}$$

$$\tilde{r}^{(0)}_{\alpha\|} \;=\; \frac{p^2_\alpha}{2n}\int d^3\boldsymbol{\xi}\;\xi^2\xi^2_\|\;\left[\hat{f}^{(0)}_\alpha(\xi,\ \xi_\|)-\hat{f}_{M\alpha}(\xi,\ \xi_\|)\right]\ , \tag{68}$$

$$\tilde{r}^{(0)}_{\alpha\Delta} \;=\; \frac{p^2_\alpha}{4n}\int d^3\boldsymbol{\xi}\;(\xi^2-\xi^2_\|)(5\xi^2_\|-\xi^2)\;\left[\hat{f}^{(0)}_\alpha(\xi,\ \xi_\|)-\hat{f}_{M\alpha}(\xi,\ \xi_\|)\right]\ , \tag{69}$$

with

$$\hat{f}_{M\alpha}(\xi,\ \xi_\|) \;=\; \frac{p^{3/2}_\alpha}{(2\pi)^{3/2}p_{\alpha\perp}p^{1/2}_{\alpha\|}}\;\exp\left[-\frac{p_\alpha}{2}\left(\frac{\xi^2-\xi^2_\|}{p_{\alpha\perp}}+\frac{\xi^2_\|}{p_{\alpha\|}}\right)\right]\ . \tag{70}$$

Unlike the perpendicular parts of the stress-flux tensors, whose ordering consistent with the general hypotheses of the present work is uniquely determined as $\hat{\mathsf{Q}}_{\alpha,jkl} = O(\delta_\alpha p_\alpha v_{th\alpha})$, the ordering of the corresponding CGL parts, namely the parallel heat fluxes, requires further assumptions. The ion analysis could proceed without additional difficulty assuming the maximal ordering $\mathsf{Q}^{CGL}_{\iota,jkl} = O(p_\iota v_{th\iota})$ but for the electrons, consistent with the general evolution equations (24-27), different orderings for the parallel heat fluxes depend on more specific assumptions on the ordering of the collision frequency and the ratio between parallel and perpendicular gradient scale lengths. For the sake of conciseness, we shall carry on this work with the overall assumption $\mathsf{Q}^{CGL}_{\alpha,jkl} = O(p_\alpha u_\alpha)$, which will be possible to make compatible with Eqs.(24-27). While this amounts to little or no restriction on the ions, it could sometimes be too restrictive for the electrons. In such cases the analysis would have to be extended in a way that is specific to more precise ν_e and $L_\parallel/L_\perp$ ordering assumptions.

With the parallel heat flux orderings $\mathsf{Q}^{CGL}_{\alpha,jkl} = O(p_\alpha u_\alpha)$, the lowest significant order in the ion perpendicular stress tensor is $\hat{\mathsf{P}}_{\iota,jk} = O(\delta p_\iota u_\iota / v_{th\iota})$ which will be the maximum accuracy retained in this work. To obtain this accuracy, the collisional term $\mathsf{G}^{coll}_{\iota,jk}$ of Eq.(22) is needed only to $O(\nu_\iota p_\iota)$, as given by Eq.(53). Like in the case of the perpendicular stress-flux tensor, this expression gives a null contribution to $\hat{\mathsf{P}}_{\iota,jk}$ when inserted in Eq.(28). Therefore, keeping $O(\delta p_\iota u_\iota / v_{th\iota})$, the ion perpendicular stress tensor is given by the following expression which does not depend explicitly on collisions, i.e. the so called gyroviscous stress:

$$\hat{\mathsf{P}}_{\iota,jk} \;=\; \frac{1}{4}\, \epsilon_{[jlm} b_l \; \mathsf{K}^{gyr}_{\iota,mn} \, \Big(\delta_{nk]} + 3 b_n b_{k]} \Big) \tag{71}$$

with

$$\mathsf{K}^{gyr}_{\iota,mn} \;=\; \frac{m_\iota}{eB}\left[\frac{\partial \mathsf{P}^{CGL}_{\iota,mn}}{\partial t} + \frac{\partial}{\partial x_p}\left(\mathsf{P}^{CGL}_{\iota,mn} u_{\iota,p} + \mathsf{Q}^{CGL}_{\iota,mnp} + \hat{\mathsf{Q}}_{\iota,mnp} \right) + \mathsf{P}^{CGL}_{\iota,[mp} \frac{\partial u_{\iota,n]}}{\partial x_p} \right] \;=\; O\!\left(\frac{\delta p_\iota u_\iota}{v_{th\iota}} \right) . \tag{72}$$

Following similar considerations, the electron perpendicular stress tensor turns out to be $\hat{\mathsf{P}}_{e,jk} = O(\delta_e p_e u_e / v_{the}) \lesssim O(\delta^3 p_e)$, which will always be negligible for our purposes.

Finally, (52,53) imply that the collisional heat exchange rates $g^{coll}_\alpha = \mathsf{G}^{coll}_{\alpha,jj}/2$ are $g^{coll}_\alpha = O(\delta_\alpha \nu_\alpha p_\alpha)$ which will be negligible within the maximum accuracy, $O(\delta v_{th\iota} p_\alpha / L)$, to be retained in our mean pressure evolution equations (24). In summary, the only collisional terms that will play a role in our low-collisionality fluid systems are the friction force and the scalars $g^{coll}_{B\alpha}$, h^{coll}_{α} and $h^{coll}_{B\alpha}$ of Eqs.(25-27).

V Two-fluid system for fast dynamics.

In this section, the results obtained so far will be further specialized to the fast dynamics ordering $\partial/\partial t \sim u_\alpha/L \sim v_{th\iota}/L$. This will yield a two-fluid system for plasma evolution on the sonic time scale with first-order FLR corrections. Here, no distinction will be made between parallel and perpendicular length scales ($L \sim L_\perp \sim L_\|$) and the plasma "beta" will be taken as order unity ($p_\alpha \sim B^2$).

In our single-ion quasineutral plasma we have always:

$$n_e \ = \ n_\iota \ = \ n \ , \tag{73}$$

$$\frac{\partial n}{\partial t} \ + \ \nabla \cdot (n\mathbf{u}_\iota) \ = \ 0 \ , \tag{74}$$

$$\mathbf{u}_e \ = \ \mathbf{u}_\iota \ - \ \frac{1}{en}\mathbf{j} \ , \tag{75}$$

$$\mathbf{j} \ = \ \nabla \times \mathbf{B} \tag{76}$$

and

$$\frac{\partial \mathbf{B}}{\partial t} \ + \ \nabla \times \mathbf{E} \ = \ 0 \ . \tag{77}$$

All the other fluid equations will be expanded keeping the first FLR corrections of order δ beyond the lowest-order or zero-Larmor-radius terms.

The electric field is obtained from the electron momentum equation. Keeping $O(v_{th\iota}B)+O(\delta v_{th\iota}B)$, we get a generalized Ohm's law of the form:

$$\mathbf{E} \ = \ -\mathbf{u}_\iota \times \mathbf{B} \ + \ \frac{1}{en}\left(\mathbf{j} \times \mathbf{B} - \nabla \cdot \mathsf{\mathbf{P}}^{CGL}_e\right) \ , \tag{78}$$

where

$$\nabla \cdot \mathsf{\mathbf{P}}^{CGL}_\alpha \ = \ \nabla p_{\alpha\perp} \ + \ (\mathbf{B}\cdot\nabla)\left(\frac{p_{\alpha\|} - p_{\alpha\perp}}{B^2}\ \mathbf{B}\right) \ . \tag{79}$$

The parallel component of the electric field begins in $O(\delta v_{th\iota}B)$ and is available to the accuracy of $O(\delta v_{th\iota}B) + O(\delta^2 v_{th\iota}B)$:

$$\mathbf{b}\cdot\mathbf{E} \;=\; \frac{1}{en}\Big[-\mathbf{b}\cdot\nabla p_{e\|} \;+\; (p_{e\|}-p_{e\perp})\mathbf{b}\cdot\nabla(\ln B) \;+\; F^{coll}_{e\|}\Big]\,, \tag{80}$$

where the parallel friction force is needed only in its lowest significant order,

$$F^{coll}_{e\|} \;=\; -\;\frac{\nu_e p_e}{v_{the}}\int d^3\boldsymbol{\xi}\;\frac{\xi_\|}{\xi^3}\hat{f}^{(0)}_e(\xi,\xi_\|) \;=\; O\!\left(\frac{\nu_e p_e}{v_{the}}\right)\,. \tag{81}$$

Here it is worth pointing out that in the general case of three-dimensional geometry, anisotropic pressures and independent dynamical evolution of pressures and density, the term $-\nabla\cdot\mathsf{P}^{CGL}_e/(en)$ cannot be represented as the gradient of a global scalar and yields a non-vanishing contribution to the parallel electric field, hence it is the largest term to break the magnetic frozen-in law and allow magnetic reconnection.

The ion flow velocity is obtained from the sum of the ion and electron momentum equations. Keeping $O(m_\iota n v^2_{th\iota}/L) + O(\delta m_\iota n v^2_{th\iota}/L)$, we get:

$$m_\iota n\left[\frac{\partial\mathbf{u}_\iota}{\partial t} + (\mathbf{u}_\iota\cdot\nabla)\mathbf{u}_\iota\right] \;+\; \nabla\cdot\left(\mathsf{P}^{CGL}_e + \mathsf{P}^{CGL}_\iota + \hat{\mathsf{P}}_\iota\right) \;-\; \mathbf{j}\times\mathbf{B} \;=\; 0\,. \tag{82}$$

Here, the ion perpendicular stress tensor is needed in its lowest significant order, $\hat{\mathsf{P}}_{\iota,jk} = O(\delta p_\iota)$. Keeping this accuracy and using the fast dynamics ordering in Eqs.(71,72), we get the fast dynamics gyroviscous tensor:

$$\hat{\mathsf{P}}_{\iota,jk} \;=\; \frac{1}{4}\;\epsilon_{[jlm}b_l\;\mathsf{K}^{(1)}_{\iota,mn}\;\left(\delta_{nk]} + 3b_n b_{k]}\right) \tag{83}$$

with

$$\mathsf{K}^{(1)}_{\iota,mn} = \frac{m_\iota}{eB}\left\{p_{\iota\perp}\frac{\partial u_{\iota,n]}}{\partial x_{[m}} + \frac{\partial(q_{\iota T\|}b_{n]})}{\partial x_{[m}} + b_{[m}\left[(2q_{\iota B\|} - 3q_{\iota T\|})\kappa_{n]} + 2(p_{\iota\|} - p_{\iota\perp})\;b_p\frac{\partial u_{\iota,n]}}{\partial x_p}\right]\right\} = O(\delta p_\iota). \tag{84}$$

The divergence of this gyroviscous stress tensor, in coordinate-free vector form for general magnetic geometry and general flows, is given in Ref.21.

The remaining equations in the two-fluid system are the evolution equations for the CGL variables, obtained by expanding (24-27). Keeping $O(p_\iota v_{th\iota}/L) + O(\delta p_\iota v_{th\iota}/L)$, the ion pressure equations become:

$$\frac{3}{2}\left[\frac{\partial p_\iota}{\partial t} + \nabla\cdot(p_\iota \mathbf{u}_\iota)\right] \;+\; p_\iota \nabla\cdot\mathbf{u}_\iota \;+\; (p_{\iota\|} - p_{\iota\perp})\Big\{\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big] - \nabla\cdot\mathbf{u}_\iota/3\Big\} \;+\; \nabla\cdot(q_{\iota\|}\mathbf{b}) \;+$$

$$+\; \hat{\mathsf{P}}_\iota : (\nabla\mathbf{u}_\iota) \;+\; \nabla\cdot\mathbf{q}_{\iota\perp} \;=\; 0 \tag{85}$$

and

$$\frac{1}{2}\left[\frac{\partial p_{\iota\|}}{\partial t} + \nabla\cdot(p_{\iota\|}\mathbf{u}_\iota)\right] \;+\; p_{\iota\|}\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big] \;+\; \nabla\cdot(q_{\iota B\|}\mathbf{b}) \;+\; q_{\iota T\|}\mathbf{b}\cdot\nabla(\ln\ B) \;+$$

$$+\; \mathbf{b}\cdot\hat{\mathsf{P}}_\iota\cdot(\mathbf{b}\times\boldsymbol{\omega}_\iota) \;+\; \nabla\cdot\mathbf{q}_{\iota B\perp} \;-\; \mathbf{b}\cdot\hat{\mathsf{Q}}_\iota : (\nabla\mathbf{b}) \;-\; g_{\iota B}^{coll} \;=\; 0\ . \tag{86}$$

The first four terms of each of these equations constitute the classic CGL collisionless, zero-Larmor-radius result[22]. The first-order, collision-independent FLR corrections[8,23] are represented by the terms involving the perpendicular stress tensor $\hat{\mathsf{P}}_\iota$ as given by (83,84) and the perpendicular stress-flux tensor $\hat{\mathsf{Q}}_\iota$ as given by (56,57,59,61,63-65). Accordingly,

$$\hat{\mathsf{P}}_\iota : (\nabla\mathbf{u}_\iota) \;=\; \mathbf{b}\cdot\hat{\mathsf{P}}_\iota\cdot\Big[2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota + \mathbf{b}\times\boldsymbol{\omega}_\iota\Big] \;+\; q_{\iota T\|}\sigma_\iota\ , \tag{87}$$

where the vector $\mathbf{b}\cdot\hat{\mathsf{P}}_\iota$ is

$$\mathbf{b}\cdot\hat{\mathsf{P}}_\iota \;=\; \frac{m_\iota}{eB}\,\mathbf{b}\times\Big[2p_{\iota\|}(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota + p_{\iota\perp}\mathbf{b}\times\boldsymbol{\omega}_\iota + \nabla q_{\iota T\|} + 2(q_{\iota B\|} - q_{\iota T\|})\boldsymbol{\kappa}\Big] \tag{88}$$

and the scalar σ_ι is

$$\sigma_\iota \;=\; \frac{m_\iota}{4eB}\epsilon_{jkl}b_j\left(\frac{\partial b_k}{\partial x_m} + \frac{\partial b_m}{\partial x_k}\right)(\delta_{mn} - b_m b_n)\left(\frac{\partial u_{\iota,l}}{\partial x_n} + \frac{\partial u_{\iota,n}}{\partial x_l}\right)\ , \tag{89}$$

the total perpendicular heat flux vector is $\mathbf{q}_{\iota\perp} = \mathbf{q}_{\iota B\perp} + \mathbf{q}_{\iota T\perp}$, with $\mathbf{q}_{\iota B\perp}$ and $\mathbf{q}_{\iota T\perp}$ as given by (57) and (59), and

$$\mathbf{b}\cdot\hat{\mathsf{Q}}_\iota : (\nabla\mathbf{b}) \;=\; 2\mathbf{q}_{\iota B\perp}\cdot\boldsymbol{\kappa} \;-\; q_{\iota T\|}\sigma_\iota\ . \tag{90}$$

As discussed before, the collisional contributions to the perpendicular viscosity and the perpendicular heat fluxes, as well as the collisional heat exchange term in (85), are negligible within our orderings and the only collisional term that needs to be retained here is $g_{\iota B}^{coll}$ in (86). Sometimes it is useful to consider the linear combination between (85) and (86) that gives the evolution of the ion pressure anisotropy:

$$\frac{\partial(p_{\iota\|}-p_{\iota\perp})}{\partial t} \;+\; \nabla\cdot\Big[(p_{\iota\|}-p_{\iota\perp})\mathbf{u}_\iota\Big] \;+\; (p_{\iota\|}-p_{\iota\perp})\Big\{\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big]+\nabla\cdot\mathbf{u}_\iota/3\Big\}\;+$$

$$+\;p_\iota\Big\{3\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big]-\nabla\cdot\mathbf{u}_\iota\Big\} \;+\; \nabla\cdot\Big[(3q_{\iota B\|}-q_{\iota\|})\mathbf{b}\Big] \;+\; 3q_{\iota T\|}\mathbf{b}\cdot\nabla(\ln\ B)\;+$$

$$+\;3\mathbf{b}\cdot\hat{\mathbf{P}}_\iota\cdot(\mathbf{b}\times\boldsymbol{\omega}_\iota) \;-\; \hat{\mathbf{P}}_\iota:(\nabla\mathbf{u}_\iota) \;+\; \nabla\cdot(3\mathbf{q}_{\iota B\perp}-\mathbf{q}_{\iota\perp}) \;-\; 3\mathbf{b}\cdot\hat{\mathbf{Q}}_\iota:(\nabla\mathbf{b}) \;-\; 3g^{coll}_{\iota B} \;=\; 0\;. \qquad (91)$$

Given our low-collisionality ordering $\nu_\iota \lesssim \delta^2\Omega_{c\iota} \sim \delta v_{th\iota}/L$, the collisional term $g^{coll}_{\iota B}$ is needed only in its lowest significant order, as derived from (53):

$$g^{coll}_{\iota B} \;=\; \nu_\iota p_\iota \int d^3\boldsymbol{\xi}\;\frac{\xi^2-3\xi_\|^2}{2\xi^3}\int d^3\boldsymbol{\zeta}\;\;\hat{f}^{(0)}_\iota\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\;\xi_\|+\zeta_\|\Big)\hat{f}^{(0)}_\iota(\zeta,\;\zeta_\|) \;=\; O(\nu_\iota p_\iota)\;. \qquad (92)$$

The angular dependence of the $(\xi^2-3\xi_\|^2)$ factor is the $l=2, m=0$ spherical harmonic. Hence, the $g^{coll}_{\iota B}$ moment samples the anisotropic and even along the direction of the magnetic field part of the distribution function, which also yields the pressure anisotropy. We may then use the estimate $g^{coll}_{\iota B} \sim \nu_\iota(p_{\iota\|}-p_{\iota\perp}) \lesssim \delta(p_{\iota\|}-p_{\iota\perp})v_{th\iota}/L$ (with a negative multiplier of order unity). The zero-Larmor-radius part of (91) contains the following piece independent of the pressure anisotropy,

$$D_\iota \;=\; p_\iota\Big\{3\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big]-\nabla\cdot\mathbf{u}_\iota\Big\} \;+\; \nabla\cdot\Big[(3q_{\iota B\|}-q_{\iota\|})\mathbf{b}\Big] \;+\; 3q_{\iota T\|}\mathbf{b}\cdot\nabla(\ln\ B)\;, \qquad (93)$$

which in the present sonic flow ordering and with the exception of some special configurations such as certain quasi-equilibria with closed magnetic surfaces is $D_\iota \sim p_\iota v_{th\iota}/L$. Therefore, collisions cannot in general force $(p_{\iota\|}-p_{\iota\perp})$ to be much smaller than p_ι and, in order to balance D_ι, our strong anisotropy ordering $(p_{\iota\|}-p_{\iota\perp}) \sim p_\iota$ must be retained. The same argument holds for the electrons in the sonic flow ordering, where a similarly defined D_e is $D_e \sim p_e v_{th\iota}/L$ and $g^{coll}_{eB} \sim \nu_e(p_{e\|}-p_{e\perp}) \lesssim (p_{e\|}-p_{e\perp})v_{th\iota}/L$.

The electron pressure equations are similar to the ion ones, only lacking a number of terms which are negligible due to the small electron mass. Keeping $O(p_e v_{th\iota}/L)+O(\delta p_e v_{th\iota}/L)$, we get:

$$\frac{3}{2}\left[\frac{\partial p_e}{\partial t}+\nabla\cdot(p_e\mathbf{u}_e)\right] + p_e\nabla\cdot\mathbf{u}_e + (p_{e\|}-p_{e\perp})\Big\{\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\Big]-\nabla\cdot\mathbf{u}_e/3\Big\} + \nabla\cdot(q_{e\|}\mathbf{b}+\mathbf{q}_{e\perp}) = 0 \quad (94)$$

and

$$\frac{1}{2}\left[\frac{\partial p_{e\|}}{\partial t}+\nabla\cdot(p_{e\|}\mathbf{u}_e)\right]+p_{e\|}\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\Big]+\nabla\cdot(q_{eB\|}\mathbf{b}+\mathbf{q}_{eB\perp})+q_{eT\|}\mathbf{b}\cdot\nabla(\ln\ B)-2\mathbf{q}_{eB\perp}\cdot\boldsymbol{\kappa}-g_{eB}^{coll} = 0 \quad (95)$$

or, alternatively,

$$\frac{\partial(p_{e\|}-p_{e\perp})}{\partial t} + \nabla\cdot\Big[(p_{e\|}-p_{e\perp})\mathbf{u}_e\Big] + (p_{e\|}-p_{e\perp})\Big\{\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\Big]+\nabla\cdot\mathbf{u}_e/3\Big\} +$$

$$+ p_e\Big\{3\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\Big]-\nabla\cdot\mathbf{u}_e\Big\} + \nabla\cdot\Big[(3q_{eB\|}-q_{e\|})\mathbf{b}\Big] + 3q_{eT\|}\mathbf{b}\cdot\nabla(\ln\ B) +$$

$$+ \nabla\cdot(3\mathbf{q}_{eB\perp}-\mathbf{q}_{e\perp}) - 6\mathbf{q}_{eB\perp}\cdot\boldsymbol{\kappa} - 3g_{eB}^{coll} = 0\ . \quad (96)$$

Here, the total electron perpendicular heat flux vector is $\mathbf{q}_{e\perp}=\mathbf{q}_{eB\perp}+\mathbf{q}_{eT\perp}$, with $\mathbf{q}_{eB\perp}$ and $\mathbf{q}_{eT\perp}$ as given by (58) and (60), and the collisional term g_{eB}^{coll} is as derived from (52):

$$g_{eB}^{coll} = \nu_e p_e\int d^3\boldsymbol{\xi}\ \frac{\xi^2-3\xi_\|^2}{2\xi^3}\left[\hat{f}_e^{(0)}(\xi,\ \xi_\|)+\int d^3\boldsymbol{\zeta}\ \ \hat{f}_e^{(0)}\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\|+\zeta_\|\Big)\hat{f}_e^{(0)}(\zeta,\ \zeta_\|)\right]\sim\nu_e(p_{e\|}-p_{e\perp}).(97)$$

Considering the parallel heat flux evolution equations for the ions and keeping the accuracy of $O(p_\iota v_{th\iota}^2/L)+O(\delta p_\iota v_{th\iota}^2/L)$, we get:

$$\frac{\partial q_{\iota\|}}{\partial t}+\nabla\cdot(q_{\iota\|}\mathbf{u}_\iota)+q_{\iota\|}\nabla\cdot\mathbf{u}_\iota+q_{\iota B\|}\Big\{3\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big]-\nabla\cdot\mathbf{u}_\iota\Big\}+$$

$$+\frac{p_{\iota\|}}{m_\iota}\mathbf{b}\cdot\nabla\left(\frac{3p_{\iota\|}+2p_{\iota\perp}}{2n}\right)-\frac{p_{\iota\perp}(p_{\iota\|}-p_{\iota\perp})}{m_\iota n}\mathbf{b}\cdot\nabla(\ln B)+\frac{1}{m_\iota}\Big[\mathbf{b}\cdot\nabla\tilde{r}_{\iota\|}^{(0)}-(\tilde{r}_{\iota\|}^{(0)}-\tilde{r}_{\iota\perp}^{(0)})\mathbf{b}\cdot\nabla(\ln B)\Big]+$$

$$+\frac{1}{m_\iota}\mathbf{b}\cdot\hat{\mathbf{P}}_\iota\cdot\left[\nabla\left(\frac{3p_{\iota\|}+2p_{\iota\perp}}{2n}\right)-\frac{2(p_{\iota\|}-p_{\iota\perp})}{n}\boldsymbol{\kappa}\right]+\left(\frac{p_{\iota\|}-2p_{\iota\perp}}{m_\iota n}\right)\hat{\mathbf{P}}_\iota:(\nabla\mathbf{b})+\left(\frac{p_{\iota\perp}}{m_\iota}\right)\nabla\cdot\left(\frac{1}{n}\mathbf{b}\cdot\hat{\mathbf{P}}_\iota\right)+$$

$$+\mathbf{q}_{\iota\perp}\cdot(\mathbf{b}\times\boldsymbol{\omega}_\iota)+\ \mathbf{b}\cdot\hat{\mathbf{Q}}_\iota:(\nabla\mathbf{u}_\iota)+\frac{\tilde{s}_\iota^{(1)}}{2m_\iota}+\left(\frac{3p_{\iota\|}+2p_{\iota\perp}}{2m_\iota n}\right)F_{\iota\|}^{coll}-\ h_\iota^{coll} = 0 \quad (98)$$

and

$$\frac{\partial q_{\iota B\|}}{\partial t} \ + \ \nabla \cdot (q_{\iota B\|}\mathbf{u}_\iota) \ + \ 3q_{\iota B\|}\mathbf{b} \cdot \Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big] \ + \ \frac{3p_{\iota\|}}{2m_\iota}\mathbf{b}\cdot\nabla\left(\frac{p_{\iota\|}}{n}\right) +$$

$$+ \ \frac{1}{m_\iota}\Big[\mathbf{b}\cdot\nabla(\tilde{r}^{(0)}_{\iota\|} - 2\tilde{r}^{(0)}_{\iota\perp}/5 - 2\tilde{r}^{(0)}_{\iota\Delta}/5) - (\tilde{r}^{(0)}_{\iota\|} - \tilde{r}^{(0)}_{\iota\perp} - \tilde{r}^{(0)}_{\iota\Delta})\mathbf{b}\cdot\nabla(\ln B)\Big] +$$

$$+ \ \frac{3}{2m_\iota}\mathbf{b}\cdot\hat{\mathsf{P}}_\iota\cdot\left[\nabla\left(\frac{p_{\iota\|}}{n}\right) - \frac{2p_{\iota\|}}{n}\boldsymbol{\kappa}\right] \ + \ 3\mathbf{q}_{\iota B\perp}\cdot(\mathbf{b}\times\boldsymbol{\omega}_\iota) \ + \ \frac{\tilde{s}^{(1)}_{\iota B}}{2m_\iota} \ + \ \frac{3p_{\iota\|}}{2m_\iota n}F^{coll}_{\iota\|} \ - \ h^{coll}_{\iota B} \ = \ 0 \ . \quad (99)$$

In the collision-independent parts of these FLR parallel heat flux equations[8], two additional scalars involving the perpendicular stress and stress-flux tensors are

$$\hat{\mathsf{P}}_\iota : (\nabla\mathbf{b}) \ = \ \mathbf{b}\cdot\hat{\mathsf{P}}_\iota\cdot\boldsymbol{\kappa} \ - \ p_{\iota\perp}\sigma_\iota \quad (100)$$

and

$$\mathbf{b}\cdot\hat{\mathbf{Q}}_\iota : (\nabla\mathbf{u}_\iota) \ = \ 2\mathbf{q}_{\iota B\perp}\cdot\Big[2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota + \mathbf{b}\times\boldsymbol{\omega}_\iota\Big] \ + \ \frac{1}{m_\iota}\left[\frac{p_{\iota\perp}(p_{\iota\|} - p_{\iota\perp})}{n} + 2\tilde{r}^{(0)}_{\iota\Delta}\right]\sigma_\iota \ . \quad (101)$$

Also, two FLR closure terms appear in (98-99):

$$\tilde{s}^{(1)}_\iota \ = \ b_j\frac{\partial\tilde{\mathsf{R}}^{(1)}_{\iota,jkll}}{\partial x_k} \quad (102)$$

and

$$\tilde{s}^{(1)}_{\iota B} \ = \ b_jb_kb_l\frac{\partial\tilde{\mathsf{R}}^{(1)}_{\iota,jklm}}{\partial x_m} \ , \quad (103)$$

where $\tilde{\mathsf{R}}^{(1)}_{\iota,jklm} = O(\delta m_\iota^2 n v_{th\iota}^2)$ is the first-order, FLR part of the $\tilde{\mathsf{R}}_{\iota,jklm}$ tensor defined in (18). These are the only closure terms in our analysis that require knowledge of the first-order part of the ion distribution function, $\hat{f}^{(1)}_\iota(\boldsymbol{\xi}) = O(\delta)$, and are therefore the most difficult to evaluate. If these two terms could be neglected, then all the required closure terms, including the collisional ones, could be obtained from the lowest-order distribution functions $\hat{f}^{(0)}_\alpha(\xi, \xi_\|) = O(1)$ obeying zero-Larmor-radius drift-kinetic equations.

Again, as a consequence of our low-collisionality ordering $\nu_\iota \lesssim \delta^2\Omega_{c\iota} \sim \delta v_{th\iota}/L$, the collisional terms in (98-99) are needed only in their lowest significant order, $\left[h_\iota^{coll} - (3p_{\iota\|} + 2p_{\iota\perp})F_{\iota\|}^{coll}/(2m_\iota n)\right] \sim \left[h_{\iota B}^{coll} - 3p_{\iota\|}F_{\iota\|}^{coll}/(2m_\iota n)\right] \sim \nu_\iota p_\iota v_{th\iota} \lesssim \delta p_\iota v_{th\iota}^2/L$, as derived from (55):

$$h_\iota^{coll} - \left(\frac{3p_{\iota\|} + 2p_{\iota\perp}}{2m_\iota n}\right)F_{\iota\|}^{coll} = \nu_\iota p_\iota v_{th\iota} \int\int d^3\boldsymbol{\xi}\ d^3\boldsymbol{\zeta}\ \frac{\xi^2\zeta_\| - 3\xi_\|\boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\ \hat{f}_\iota^{(0)}\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\| + \zeta_\|\Big)\ \hat{f}_\iota^{(0)}(\zeta,\ \zeta_\|) \tag{104}$$

and

$$h_{\iota B}^{coll} - \frac{3p_{\iota\|}}{2m_\iota n}F_{\iota\|}^{coll} = \frac{3}{2}\nu_\iota p_\iota v_{th\iota} \int\int d^3\boldsymbol{\xi}\ d^3\boldsymbol{\zeta}\ \frac{\xi^2\zeta_\| - 3\xi_\|^2\zeta_\|}{\xi^3}\ \hat{f}_\iota^{(0)}\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\| + \zeta_\|\Big)\ \hat{f}_\iota^{(0)}(\zeta,\ \zeta_\|)\ . \tag{105}$$

Finally, in the parallel heat flux evolution equations for the electrons, the leading terms are $O(p_e v_{the}^2/L)$. Therefore, the maximum accuracy that can consistently be kept there is $O(p_e v_{the}^2/L) + O(\delta p_e v_{the}^2/L)$. Neglecting higher-order terms under our ordering scheme, those equations yield the following time-independent constraints that do not involve explicitly the parallel heat fluxes:

$$\frac{p_{e\|}}{m_e}\mathbf{b}\cdot\nabla\left(\frac{3p_{e\|} + 2p_{e\perp}}{2n}\right) - \frac{p_{e\perp}(p_{e\|} - p_{e\perp})}{m_e n}\mathbf{b}\cdot\nabla(\ln B) + \frac{1}{m_e}\left[\mathbf{b}\cdot\nabla\tilde{r}_{e\|}^{(0)} - (\tilde{r}_{e\|}^{(0)} - \tilde{r}_{e\perp}^{(0)})\mathbf{b}\cdot\nabla(\ln B)\right] +$$

$$+ \left(\frac{3p_{e\|} + 2p_{e\perp}}{2m_e n}\right)F_{e\|}^{coll} - h_e^{coll} = 0 \tag{106}$$

and

$$\frac{3p_{e\|}}{2m_e}\mathbf{b}\cdot\nabla\left(\frac{p_{e\|}}{n}\right) + \frac{1}{m_e}\left[\mathbf{b}\cdot\nabla(\tilde{r}_{e\|}^{(0)} - 2\tilde{r}_{e\perp}^{(0)}/5 - 2\tilde{r}_{e\Delta}^{(0)}/5) - (\tilde{r}_{e\|}^{(0)} - \tilde{r}_{e\perp}^{(0)} - \tilde{r}_{e\Delta}^{(0)})\mathbf{b}\cdot\nabla(\ln B)\right] +$$

$$+ \frac{3p_{e\|}}{2m_e n}F_{e\|}^{coll} - h_{eB}^{coll} = 0\ . \tag{107}$$

Accordingly, the electron parallel heat fluxes are determined implicitly by the condition that the solutions of the dynamical electron pressure evolution equations (94,95) be compatible with these time-independent constraints. The latter, whose collisionless limit was obtained in Ref.24 using the gyrofluid formalism and in Ref.8 using the present fluid moment formalism, provide an improvement over the adiabatic electron response model. The collisional terms that need be retained in this case are $\left[h_e^{coll} - (3p_{e\|} + 2p_{e\perp})F_{e\|}^{coll}/(2m_en)\right] \sim \left[h_{eB}^{coll} - 3p_{e\|}F_{e\|}^{coll}/(2m_en)\right] \sim \nu_e p_e v_{the} \lesssim \delta p_e v_{the}^2/L$, as derived from (54):

$$h_e^{coll} - \left(\frac{3p_{e\|} + 2p_{e\perp}}{2m_en}\right)F_{e\|}^{coll} = \nu_e p_e v_{the}\left[\int\int d^3\boldsymbol{\xi}\ d^3\boldsymbol{\zeta}\ \frac{\xi^2\zeta_\| - 3\xi_\|\boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\ \hat{f}_e^{(0)}\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\| + \zeta_\|\Big)\ \hat{f}_e^{(0)}(\zeta,\ \zeta_\|) - \right.$$

$$\left. - \frac{1}{2}\int d^3\boldsymbol{\xi}\ \frac{\xi_\|}{\xi}\ \hat{f}_e^{(0)}(\xi,\ \xi_\|)\ + \left(\frac{3p_{e\|} + 2p_{e\perp}}{2p_e}\right)\int d^3\boldsymbol{\xi}\ \frac{\xi_\|}{\xi^3}\ \hat{f}_e^{(0)}(\xi,\ \xi_\|)\ \right] \tag{108}$$

and

$$h_{eB}^{coll} - \frac{3p_{e\|}}{2m_en}F_{e\|}^{coll} = \nu_e p_e v_{the}\left[\frac{3}{2}\int\int d^3\boldsymbol{\xi}\ d^3\boldsymbol{\zeta}\ \frac{\xi^2\zeta_\| - 3\xi_\|^2\zeta_\|}{\xi^3}\ \hat{f}_e^{(0)}\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\ \xi_\| + \zeta_\|\Big)\ \hat{f}_e^{(0)}(\zeta,\ \zeta_\|) + \right.$$

$$\left. + \frac{3}{2}\int d^3\boldsymbol{\xi}\ \frac{\xi^2\xi_\| - 2\xi_\|^3}{\xi^3}\ \hat{f}_e^{(0)}(\xi,\ \xi_\|)\ + \frac{3p_{e\|}}{2p_e}\int d^3\boldsymbol{\xi}\ \frac{\xi_\|}{\xi^3}\ \hat{f}_e^{(0)}(\xi,\ \xi_\|)\ \right]. \tag{109}$$

Summarizing, our fast dynamics FLR two-fluid system comprises the continuity equation (74) to evolve the particle density, the constitutive relation (75) for the electron flow velocity, Ampere's and Faraday's laws (76,77) for the current and the magnetic field, the generalized Ohm's law (78,80) for the electric field, the momentum equation (82) for the ion flow velocity, the evolution equations (85,86,94,95) for the anisotropic ion and electron pressures, the evolution equations (98,99) for the ion parallel heat fluxes and the implicit constraints (106,107) for the electron parallel heat fluxes. Explicit representations are given for the "perpendicular" parts of the stress and stress-flux tensors involved in this system. The unspecified closure variables that must be provided by kinetic theory are the irreducible fourth rank moments $\tilde{r}_{\alpha\perp}^{(0)},\ \tilde{r}_{\alpha\|}^{(0)},\ \tilde{r}_{\alpha\Delta}^{(0)},\ \tilde{s}_\iota^{(1)}$ and $\tilde{s}_{\iota B}^{(1)}$ (67,68,69,102,103), and the collisional moments $F_{e\|}^{coll},\ g_{\alpha B}^{coll},\ h_\alpha^{coll}$ and $h_{\alpha B}^{coll}$ (81,92,97,104,105,108,109). Of these, only $\tilde{s}_\iota^{(1)}$ and $\tilde{s}_{\iota B}^{(1)}$ require knowledge of the FLR part of any distribution function, all the others being derived from the lowest-order distribution functions $\hat{f}_\alpha^{(0)}(\xi, \xi_\|)$ obeying zero-Larmor-radius drift-kinetic equations. The problematic terms $\tilde{s}_\iota^{(1)}$ and $\tilde{s}_{\iota B}^{(1)}$ contribute only to the FLR corrections to the ion parallel heat fluxes which in turn only enter the theory either acted upon by parallel gradient operators or multiplied by magnetic gradient factors. A plausible truncation scheme would therefore be to neglect $\tilde{s}_\iota^{(1)}$ and

$\tilde{s}_{\iota B}^{(1)}$ (or even perhaps all the first-order terms including the diamagnetic and collisional ones) in the ion parallel heat flux equations (98,99). This results in a very inclusive and "almost fully consistent" FLR fluid-kinetic hybrid model, whose kinetic side is required to provide only zero-Larmor-radius drift-kinetic solutions to evaluate the remaining fluid closure variables.

VI Reduced two-fluid system for slow dynamics.

A widely used ordering for slow dynamics on the diamagnetic drift scale is $\partial/\partial t \sim u_\alpha/L \sim \delta v_{th\iota}/L$, meaning that the flow velocities and time derivatives are taken to be comparable to the diamagnetic drift velocities and frequencies respectively. There is a difficulty specific to this ordering which has to do with the fact that, to obtain the parallel flow velocities in their leading order, $u_{\alpha\|} = O(\delta v_{th\iota})$, the parallel components of $\nabla \cdot \mathbf{P}_\alpha^{CGL}$ in the momentum equations must be known to second order accuracy: $\mathbf{b} \cdot (\nabla \cdot \mathbf{P}_\alpha^{CGL}) = O(\delta^2 m_\alpha n v_{th\alpha}^2/L)$. The fluid equations cannot provide second-order-accurate CGL pressures, $\mathbf{P}_\alpha^{CGL} = O(m_\alpha n v_{th\alpha}^2) + O(\delta^2 m_\alpha n v_{th\alpha}^2)$, since this would require knowing the parallel velocities and the heat fluxes to third order accuracy, namely $u_{\alpha\|} = O(\delta v_{th\iota}) + O(\delta^3 v_{th\iota})$ and $\mathbf{q}_\alpha = O(\delta v_{th\iota} m_\alpha n v_{th\alpha}^2) + O(\delta^3 v_{th\iota} m_\alpha n v_{th\alpha}^2)$. One way of avoiding this difficulty is to assume the small-parallel-gradient subsidiary ordering $\mathbf{b} \cdot \nabla \sim \delta^2/L$ which allows a lowest-significant-order formulation where the parallel velocities decouple from the perpendicular dynamics and only lowest-order pressures, $\mathbf{P}_\alpha^{CGL} = O(m_\alpha n v_{th\alpha}^2)$, are needed.

The adoption of small-parallel-gradient orderings leads naturally to the so-called "reduced systems" characterized by separate parallel and perpendicular length scales with a subsidiary expansion parameter $\epsilon \sim L_\perp/L_\| \sim k_\|/k_\perp \ll 1$ besides $\delta \sim \rho_\iota/L_\perp \ll 1$, weakly inhomogeneous magnetic fields such that $\nabla \mathbf{B} \sim B/L_\|$ and elimination of the fast magnetosonic (compressional Alfven) wave[9−17]. This approach will be followed here and, based on our general fluid equations, we shall derive the corresponding slow dynamics reduced system, taking $\epsilon \sim \delta^2$ as in the prototypical reduced system for dynamics on the diamagnetic drift scale[13]. Specifically we shall assume a slow dynamics ordering with diamagnetic drift scale particle and heat flows:

$$\partial/\partial t \sim u_\alpha/L_\perp \sim \delta^2\Omega_{c\iota} \ , \tag{110}$$

$$u_{\alpha\|} \sim \mathbf{u}_{\alpha\perp} \sim \delta v_{th\iota} \ , \tag{111}$$

$$q_{\alpha\|} \sim \mathbf{q}_{\alpha\perp} \sim u_\alpha p_\alpha \sim \delta v_{th\iota} p_\alpha \ , \tag{112}$$

and a small-parallel-gradient ordering in a toroidal background geometry with inverse aspect ratio ϵ of the order of δ^2 and weakly inhomogeneous, mainly toroidal magnetic field:

$$\epsilon \sim (R - R_0)/R_0 \sim L_\perp/L_\| \sim \delta^2 \ , \tag{113}$$

$$k_\| \sim \mathbf{b}\cdot\nabla \sim \mathbf{e}_\zeta\cdot\nabla \sim 1/R_0 \sim \epsilon k_\perp \ , \tag{114}$$

$$\mathbf{B} = B_0\mathbf{e}_\zeta + \mathbf{B}_1 \ , \tag{115}$$

$$\mathbf{B}_1 \sim \epsilon B_0 \ , \tag{116}$$

where R_0 and B_0 are constants and $\mathbf{e}_\zeta$ is the azimuthal unit vector of the (R, ζ, Z) cylindrical coordinate system. In addition, the plasma "beta" will be taken as $O(\epsilon)$ and the ion and electron pressures will be assumed to be comparable with arbitrary anisotropies:

$$p_\iota \sim p_e \sim (p_{\iota\|} - p_{\iota\perp}) \sim (p_{e\|} - p_{e\perp}) \sim \epsilon B_0^2 \ . \tag{117}$$

From the above orderings and its divergence-free condition, it follows that the magnetic field can be represented as

$$\mathbf{B} \ = \ (B_0 + B_{1\zeta})\,\mathbf{e}_\zeta \ - \ \mathbf{e}_\zeta\times\nabla\psi \ + \ O(\epsilon^2 B_0) \tag{118}$$

and, from Ampere's law, the current density is:

$$\mathbf{j} \ = \ \frac{B_0}{R_0}\,\mathbf{e}_Z \ - \ \mathbf{e}_\zeta\times\nabla B_{1\zeta} \ - \ \nabla_\perp^2\psi\,\mathbf{e}_\zeta \ + \ O(\epsilon^2 B_0/L_\perp) \ , \tag{119}$$

where we have adopted the notation $\nabla_\perp^2 f = \nabla_\perp\cdot(\nabla_\perp f)$, with $\nabla_\perp f = (\partial f/\partial R)\mathbf{e}_R + (\partial f/\partial Z)\mathbf{e}_Z$ and $\nabla_\perp\cdot\mathbf{h} = \partial(\mathbf{e}_R\cdot\mathbf{h})/\partial R + \partial(\mathbf{e}_Z\cdot\mathbf{h})/\partial Z$.

Bringing the magnetic field representation (118) along with the orderings (110,111,113,114) to Faraday's law, we obtain for the electric field

$$\mathbf{E} \;=\; -\nabla_\perp \Phi \;+\; O(\epsilon\delta v_{th\iota} B_0) \;, \tag{120}$$

where $\Phi = O(\delta v_{th\iota} B_0 L_\perp)$ is the electric potential. The parallel component of the electric field begins in $O(\epsilon\delta v_{th\iota} B_0)$ and is

$$\mathbf{b}\cdot\mathbf{E} \;=\; -\frac{\partial \psi}{\partial t} \;-\; \nabla_\parallel \Phi \;+\; O(\epsilon^2 \delta v_{th\iota} B_0) \;, \tag{121}$$

where we use the notation $\nabla_\parallel f = R_0^{-1}\partial f/\partial\zeta - B_0^{-1}[\psi, f]$ and $[g,f] = \partial f/\partial R\, \partial g/\partial Z - \partial g/\partial R\, \partial f/\partial Z$. Then, keeping the leading order accuracy of $O(\epsilon\delta v_{th\iota} B_0)$, the parallel component of the electron momentum equation or generalized Ohm's law yields:

$$\frac{\partial \psi}{\partial t} \;+\; \nabla_\parallel \Phi \;-\; \frac{1}{en}\nabla_\parallel\, p_{e\parallel} \;+\; \frac{1}{en}\, F_{e\parallel}^{coll} \;=\; 0 \;, \tag{122}$$

where the collisional friction force needs to be kept only in its lowest-order form given by Eq.(81).

The sum of the ion and electron momentum equations yields a time-independent quasi-equilibrium condition in its leading order, $O(p_\alpha/L_\perp) = O(\epsilon B_0^2/L_\perp)$:

$$\nabla(p_{\iota\perp} + p_{e\perp} + B^2/2) \;-\; B^2 \boldsymbol{\kappa} \;=\; O(\epsilon^2 B_0^2/L_\perp) \;. \tag{123}$$

Moreover, for our weakly inhomogeneous magnetic field in large-aspect-ratio toroidal geometry, we have

$$B^2\boldsymbol{\kappa} \;=\; -\,\frac{B_0^2}{R_0}\,\mathbf{e}_R \;+\; O(\epsilon^2 B_0^2/L_\perp) \;=\; -\,B_0^2\,\nabla\!\left(\frac{R-R_0}{R_0}\right) \;+\; O(\epsilon^2 B_0^2/L_\perp) \;. \tag{124}$$

Therefore, Eq.(123) can be integrated to obtain

$$B_{1\zeta} \;=\; -B_0\!\left(\frac{R-R_0}{R_0}\right) \;-\; \frac{1}{B_0}\,(p_{\iota\perp} + p_{e\perp}) \;+\; O(\epsilon^2 B_0) \;. \tag{125}$$

This time-independent relation for the toroidal component of the magnetic field removes the fast magnetosonic wave from the system. The first term of this formula takes into account the $R^{-1} = R_0^{-1}[1 - (R-R_0)/R_0] + O(\epsilon^2 R_0^{-1})$ spatial dependence of the vacuum field that was not included in the constant B_0 term of (118) and the cross product of $\mathbf{e}_\zeta$ with its gradient cancels the first term in the expression (119) for the current density which becomes:

$$\mathbf{j} \;=\; \frac{1}{B_0}\mathbf{e}_\zeta \times \nabla(p_{\iota\perp} + p_{e\perp}) \;-\; \nabla_\perp^2 \psi\; \mathbf{e}_\zeta \;+\; O(\epsilon^2 B_0^2/L_\perp) \tag{126}$$

or, recalling that the magnetic unit vector is $\mathbf{b} = \mathbf{e}_\zeta + O(\epsilon)$,

$$\mathbf{j} \;=\; \frac{1}{B_0}\mathbf{b}\times\nabla(p_{\iota\perp}+p_{e\perp}) \;-\; \nabla_\perp^2\psi\;\mathbf{b} \;+\; O(\epsilon^2 B_0^2/L_\perp)\;. \tag{127}$$

Taking the cross product of the ion momentum equation with the magnetic unit vector, we obtain the expression for the ion flow velocity:

$$\mathbf{u}_\iota \;=\; u_{\iota\|}\;\mathbf{b} \;+\; \frac{1}{B_0}\;\mathbf{b}\times\left(\nabla\Phi+\frac{1}{en}\nabla p_{\iota\perp}\right) \;+\; O(\epsilon\delta v_{th\iota}) \tag{128}$$

and, from the ion velocity and the current:

$$\mathbf{u}_e \;=\; \mathbf{u}_\iota \;-\; \frac{1}{en}\,\mathbf{j} \;=\; \left(u_{\iota\|}+\frac{1}{en}\nabla_\perp^2\psi\right)\mathbf{b} \;+\; \frac{1}{B_0}\;\mathbf{b}\times\left(\nabla\Phi-\frac{1}{en}\nabla p_{e\perp}\right) \;+\; O(\epsilon\delta v_{th\iota})\;. \tag{129}$$

Accordingly, the divergence of the particle fluxes is

$$\nabla\cdot(n\mathbf{u}_\iota) \;=\; \nabla\cdot(n\mathbf{u}_e) \;=\; B_0^{-1}\;[\Phi,n] \;+\; O(\epsilon\delta n v_{th\iota}/L_\perp)\;. \tag{130}$$

Thus, introducing the fluid time derivative associated with an advection by the leading-order form of the $\mathbf{E}\times\mathbf{B}$ drift, $d'f/dt = \partial f/\partial t + B_0^{-1}[\Phi, f]$, the lowest-significant-order continuity equation can be written as:

$$\frac{d'n}{dt} \;=\; 0\;. \tag{131}$$

To complete the slow dynamics reduced system, there remains to obtain the evolution equations for the ion and electron parallel and perpendicular pressures, the ion parallel velocity and the electric potential. These will be derived from the evolution equations for the CGL stress tensors (24,25), the parallel component of the total (sum of ion and electron) momentum equation (11) and the vorticity equation obtained by taking the parallel component of the curl of the total momentum equation. In their lowest significant orders, these equations require knowledge of the ion and electron perpendicular heat fluxes and the ion gyroviscosity. The parallel heat fluxes are either acted upon by parallel gradient operators or multiplied by magnetic gradient factors and do not contribute to this leading-order system as a consequence of the assumptions (110-116). Bringing these orderings to Eqs.(57-60) and keeping only the required accuracy of $O(\delta p_\alpha v_{th\iota})$, the ion and electron perpendicular heat fluxes become:

$$\mathbf{q}_{\alpha B\perp} \;=\; \frac{p_{\alpha\perp}}{2e_\alpha B}\;\mathbf{b}\times\nabla\left(\frac{p_{\alpha\|}}{n}\right) \;+\; \frac{1}{5e_\alpha B}\;\mathbf{b}\times\nabla(\tilde{r}^{(0)}_{\alpha\perp}+\tilde{r}^{(0)}_{\alpha\Delta}) \tag{132}$$

and

$$\mathbf{q}_{\alpha T\perp} \;=\; \frac{2p_{\alpha\perp}}{e_\alpha B}\;\mathbf{b}\times\nabla\left(\frac{p_{\alpha\perp}}{n}\right) \;+\; \frac{1}{5e_\alpha B}\;\mathbf{b}\times\nabla(4\tilde{r}^{(0)}_{\alpha\perp}-\tilde{r}^{(0)}_{\alpha\Delta})\;. \tag{133}$$

Therefore, keeping only $O(\delta p_\alpha v_{th\iota}/L_\perp)$, we get:

$$\nabla\cdot\mathbf{q}_{\alpha B\perp} \;=\; -\;\frac{1}{2e_\alpha B_0}\;[p_{\alpha\perp}, p_{\alpha\|}/n] \tag{134}$$

and

$$\nabla\cdot\mathbf{q}_{\alpha T\perp} \;=\; -\;\frac{2p_{\alpha\perp}}{e_\alpha B_0}\;[p_{\alpha\perp}, 1/n]\;. \tag{135}$$

Similarly, using (110-116) and keeping the required accuracy of $\hat{\mathsf{P}}_{\iota,jk} = O(\delta^2 p_\iota)$, the ion gyroviscous stress tensor (71,72) becomes:

$$\hat{\mathsf{P}}_{\iota,jk} \;=\; \frac{1}{4}\;\epsilon_{[jlm}b_l\;\mathsf{K}^{(2)}_{\iota,mn}\;\Big(\delta_{nk]}+3b_nb_{k]}\Big) \tag{136}$$

with

$$\mathsf{K}^{(2)}_{\iota,mn} \;=\; \frac{m_\iota}{eB}\left[p_{\iota\perp}\frac{\partial u_{\iota,n]}}{\partial x_{[m}} \;+\; \frac{\partial(q_{\iota T\|}b_{n]})}{\partial x_{[m}} \;+\; \left(\frac{p_{\iota\|}-p_{\iota\perp}}{en^2B}\right)b_{[m}\frac{\partial n}{\partial x_p}\frac{\partial p_{\iota\perp}}{\partial x_q}\epsilon_{pqn]} \;+\; \frac{\partial\hat{\mathsf{Q}}_{\iota,mnp}}{\partial x_p}\right]\;. \tag{137}$$

The divergence of this tensor can be evaluated following the analysis of Ref.21. Making use of the present orderings (110-116) and keeping only $O(\delta^2 p_\iota/L_\perp)$, it reduces to

$$\nabla\cdot\hat{\mathbf{P}}_\iota \;=\; -\;\frac{m_\iota}{eB_0}\Big[(\mathbf{b}\times\nabla p_{\iota\perp})\cdot\nabla\Big]\mathbf{u}_\iota \;-\; \nabla\times\left[\left(\frac{m_\iota p_{\iota\perp}}{2eB_0}\nabla\cdot\mathbf{u}_\iota+\frac{m_\iota}{4eB_0}\nabla\cdot\mathbf{q}_{\iota T\perp}\right)\mathbf{b}\;\right] \;-\; \nabla\chi^\dagger_\iota\;, \tag{138}$$

where the last term, $\nabla\chi^\dagger_\iota$, can be ignored since its contribution to $\mathbf{b}\cdot(\nabla\cdot\hat{\mathbf{P}}_\iota)$ and $\mathbf{b}\cdot[\nabla\times(\nabla\cdot\hat{\mathbf{P}}_\iota)]$ will be negligible. Substituting for $\nabla\cdot\mathbf{u}_\iota$ from (128) and for $\nabla\cdot\mathbf{q}_{\iota T\perp}$ from (135), always keeping only $O(\delta^2 p_\iota/L_\perp)$, we get:

$$\nabla\cdot\hat{\mathbf{P}}_\iota \;=\; -\;\frac{m_\iota}{eB_0}\Big[(\mathbf{b}\times\nabla p_{\iota\perp})\cdot\nabla\Big]\mathbf{u}_\iota \;-\; \nabla\chi^\dagger_\iota\;. \tag{139}$$

Notice that, like in the case of the parallel heat fluxes, the contributions of the closure variables $\tilde{r}^{(0)}_{\alpha\|}$, $\tilde{r}^{(0)}_{\alpha\perp}$ and $\tilde{r}^{(0)}_{\alpha\Delta}$ to the divergences of the heat flux vectors and to the ion gyroviscous force are one order in ϵ higher than the leading terms to be retained in the pressure evolution equations, the parallel momentum equation and the vorticity equation. As the result, the final reduced two-fluid system will be closed except for the collisional terms.

Considering the pressure evolution equations (24,25), their leading terms under the assumptions (110-116) are $O(\delta m_\iota n v_{th\iota}^3/L_\perp)$. Keeping this lowest-significant-order accuracy, we get

$$\frac{1}{2}\left[\frac{\partial p_{\alpha\|}}{\partial t} \; + \; \nabla\cdot(p_{\alpha\|}\mathbf{u}_\alpha)\right] \; + \; \nabla\cdot\mathbf{q}_{\alpha B\perp} \; - \; g_{\alpha B}^{coll} \; = \; 0 \tag{140}$$

and

$$\frac{\partial p_{\alpha\perp}}{\partial t} \; + \; \nabla\cdot(p_{\alpha\perp}\mathbf{u}_\alpha) \; + \; p_\alpha\nabla\cdot\mathbf{u}_\alpha \; + \; \nabla\cdot\mathbf{q}_{\alpha T\perp} \; + \; g_{\alpha B}^{coll} \; = \; 0 \; , \tag{141}$$

where the collisional exchange terms $g_{\alpha B}^{coll}$ are needed only in their lowest-order form given by Eqs.(92) and (97). Using now our expressions (128,129,134,135) for the particle and heat flows, the pressure equations reduce to

$$\frac{1}{2}\frac{d' p_{\alpha\|}}{dt} \; - \; g_{\alpha B}^{coll} \; = \; 0 \tag{142}$$

and

$$\frac{d' p_{\alpha\perp}}{dt} \; + \; \left(\frac{p_{\alpha\|} - p_{\alpha\perp}}{3 e_\alpha B_0 n^2}\right) [n, p_{\alpha\perp}] \; + \; g_{\alpha B}^{coll} \; = \; 0 \; . \tag{143}$$

In its leading order, $O(\delta^2 m_\iota n v_{th\iota}^2/L_\perp)$, the parallel component of the total momentum equation yields

$$m_\iota n\left(\frac{\partial u_{\iota\|}}{\partial t} + \mathbf{u}_\iota\cdot\nabla u_{\iota\|}\right) \; + \; \mathbf{b}\cdot\nabla(p_{\iota\|} + p_{e\|}) \; + \; \mathbf{b}\cdot(\nabla\cdot\hat{\mathbf{P}}_\iota) \; = \; 0 \tag{144}$$

and, substituting for $\mathbf{u}_\iota$ (128) and $\nabla\cdot\hat{\mathbf{P}}_\iota$ (139), we obtain within this accuracy

$$\frac{d' u_{\iota\|}}{dt} \; + \; \frac{1}{m_\iota n} \; \nabla_\|(p_{\iota\|} + p_{e\|}) \; = \; 0 \; . \tag{145}$$

Finally, we consider the vorticity equation that will determine the evolution of the electric potential. In its leading order, $O(\delta^2 m_\iota n v_{th\iota}^2/L_\perp^2)$, the parallel component of the curl of the total momentum equation yields

$$m_\iota n\left(\frac{\partial}{\partial t}+\mathbf{u}_\iota\cdot\nabla\right)\Big[\mathbf{b}\cdot(\nabla\times\mathbf{u}_\iota)+\nabla(\ln n)\cdot(\mathbf{u}_\iota\times\mathbf{b})\Big]\;+$$

$$+\;m_\iota\nabla\cdot\Big[n\;(\nabla\cdot\mathbf{u}_\iota)\;\mathbf{u}_\iota\times\mathbf{b}\Big]\;+\;\frac{m_\iota}{2}(\nabla n\times\mathbf{b})\cdot\nabla(u_\iota^2-u_{\iota\|}^2)\;+$$

$$+\;\mathbf{b}\cdot\Big[\nabla\times(\nabla\cdot\hat{\mathbf{P}}_\iota)\Big]\;+\;(\boldsymbol{\kappa}\times\mathbf{b})\cdot\nabla(p_{\iota\|}+p_{\iota\perp}+p_{e\|}+p_{e\perp})\;-\;B_0\;\mathbf{b}\cdot\nabla(\mathbf{b}\cdot\mathbf{j})\;=\;0\;. \qquad (146)$$

Substituting for $\mathbf{u}_\iota$ (128), $\nabla\cdot\hat{\mathbf{P}}_\iota$ (139), $\boldsymbol{\kappa}$ (124) and $\mathbf{j}$ (127), introducing the auxiliary variable

$$W\;=\;\frac{B_0}{n}\;\mathbf{b}\cdot\Big[\nabla\times(n\mathbf{u}_\iota)\Big]\;=\;B_0\;\mathbf{b}\cdot(\nabla\times\mathbf{u}_\iota)\;+\;\frac{1}{n}\nabla_\perp n\cdot\left(\nabla_\perp\Phi+\frac{1}{en}\nabla_\perp p_{\iota\perp}\right)\;+\;O(\epsilon\delta B_0 v_{th\iota}/L_\perp) \qquad (147)$$

and keeping only the leading order accuracy of $O(\delta^2 v_{th\iota}^2/L_\perp^2)$ after division by $m_\iota n$ we obtain

$$\frac{d'W}{dt}\;+\;\frac{1}{2B_0 n}\;[|\nabla_\perp\Phi|^2,n]\;+\;\frac{1}{eB_0 n}\;[\nabla_\perp\Phi;\nabla_\perp p_{\iota\perp}]\;+$$

$$+\;\frac{B_0}{m_\iota R_0 n}\;[R,(p_{\iota\|}+p_{\iota\perp}+p_{e\|}+p_{e\perp})]\;+\;\frac{B_0^2}{m_\iota n}\;\nabla_\|(\nabla_\perp^2\psi)\;=\;0\;, \qquad (148)$$

where we have used the notation $[\nabla_\perp g;\nabla_\perp f]=[\partial g/\partial R,\partial f/\partial R]+[\partial g/\partial Z,\partial f/\partial Z]$. The relationship between the generalized parallel vorticity W and the electric potential Φ in their leading orders, follows from the definition (147) and the expression (128) for the ion flow velocity:

$$\frac{1}{n}\nabla_\perp\cdot(n\nabla_\perp\Phi)\;=\;W\;-\;\frac{1}{en}\;\nabla_\perp^2 p_{\iota\perp}\;. \qquad (149)$$

In summary, our diamagnetic-drift-scale reduced two-fluid analysis yields a coupled dynamical system for the seven scalar fields ψ, n, $p_{\alpha\|}$, $p_{\alpha\perp}$ and Φ. The evolution equations for these variables are the parallel Ohm's law (122), the continuity equation (131), the parallel and perpendicular pressure equations for each species (142,143) and the vorticity equation (148) along with the time-independent elliptic equation (149) for Φ in terms on n, $p_{\iota\perp}$ and the auxiliary variable W. All these equations are in their lowest significant order and all the terms in each of them are comparable under our ordering assumptions (110-116). The ion and electron flow velocities can be determined once this primary seven-field system has been solved, by integrating the decoupled parallel momentum equation (145) for $u_{\iota\|}$, relating the parallel current to the perpendicular Laplacian of ψ and recalling Eqs.(128,129) which specify the perpendicular components of the flows as sums of $\mathbf{E}\times\mathbf{B}$ plus diamagnetic drifts. Unlike the fast dynamics system in general magnetic geometry considered in Section V, this slow dynamics system for small-parallel-gradient and large-aspect-ratio geometry with weak magnetic inhomogeneity admits a consistent isotropic-pressure limit, provided the collisional terms $g_{\alpha B}^{coll}$ can be justified to vanish with isotropic pressures. The corresponding five-field model follows by setting $p_{\alpha\|} = p_{\alpha\perp}$ and $g_{\alpha B}^{coll} = 0$.

It should be emphasized that, in deriving the present reduced two-fluid system, no special assumptions have been made on the density or temperature gradients or the amplitude of the density or temperature fluctuations. Rather, the maximal orderings $\nabla_\perp(\ln n) \sim \nabla_\perp(\ln p_{\alpha\|}) \sim \nabla_\perp(\ln p_{\alpha\perp}) \sim L_\perp^{-1}$ have been implied and no distinction between equilibrium and fluctuating parts has ever been made. This represents a significant improvement over previous reduced systems for diamagnetic-drift-scale dynamics [13−17] and our approach shows without ambiguity the proper way of including the diamagnetic effects for arbitrary density and anisotropic temperatures, a subject on which there is no general agreement in the literature. In particular, noting that the first two terms of the parallel Ohm's law (122) can be rewritten as $\partial\psi/\partial t + \nabla_\|\Phi = d'\psi/dt + R_0^{-1}\partial\Phi/\partial\zeta$, we see that all the dynamical fields (ψ, n, $p_{\alpha\|}$, $p_{\alpha\perp}$, W and $u_{\iota\|}$) evolve with the d'/dt derivative, i.e. they are advected by the leading-order $\mathbf{E}\times\mathbf{B}$ drift. This is the manifestation of the "diamagnetic cancellations" which are prominent in the case of weakly inhomogeneous magnetic field and small parallel gradients under consideration here. (These cancellations are only partial and not very useful in practice for general magnetic geometries and parallel gradients such as in the case considered in Section V.) Notice also the density dependence in the elliptic operator acting on Φ in Eq.(149) and the novel $[|\nabla_\perp\Phi|^2, n]$ term (a cubic nonlinearity) in the vorticity equation (148). The main physical effects missing in this seven-field reduced system are those associated with the parallel dynamics. They are "ordered out" by the assumptions (110-116), which make all the terms involving parallel particle or heat flows negligible and allow the leading-order, seven-field system to be closed except for the collisional terms. The assumption on the electron parallel heat flux is probably the most restrictive one. The ordering $\epsilon \sim \delta^2$ is also restrictive, although it is standard in the diamagnetic-drift-scale fluid approach[13] and may be difficult to relax whithout having to abandon the fluid framework and being forced to carry out a kinetic evaluation of the CGL pressures in a consistent slow dynamics analysis. In any case, while other orderings might be devised to bring parallel physics terms into the reduced system, the perpendicular physics especially the diamagnetic effects associated with general density and temperature gradients should remain in the form given here.

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