CEMM Challenge Problems for Advanced Numerical Algorithms (Rev 2: 8/30/03)

1. Anisotropic heat conduction:

We use a standard (R, φ, Z) cylindrical coordinate system. The solution domain is defined by: $(R - R_0)^2 + Z^2 < a^2$, where R_0 is the *major radius* and a is the *minor radius.*

Consider the anisotropic heat conduction equation:

$$\frac{\partial T}{\partial t} = \nabla \bullet \kappa_{\parallel} \frac{\vec{B}\vec{B}}{|B^2|} \bullet \nabla T + \nabla \bullet \kappa \nabla T + S(R, Z) \qquad (1.1)$$

where κ is the isotropic part of the thermal conductivity, $\kappa_{\parallel}$ (which is normally much greater than κ) is the parallel thermal conductivity, and $S(R,Z)$ is the source term. Take the magnetic field to be of the form:

$$\vec{B} = \vec{B}_0 + \nabla \times \alpha \vec{B}_0 \qquad (1.2)$$

where the axisymmetric field is given by

$$\vec{B}_0 = -\nabla \times \left(\frac{\psi}{R} \hat{\varphi} \right) + \frac{1}{R} \hat{\varphi} \qquad (1.3)$$

with

$$\begin{aligned} \psi(R,Z) &= \frac{a^2}{2R_0 q_0} \ln \left[1 + \frac{(R - R_0)^2 + Z^2}{a^2} \right] \\ &= \frac{a^2}{2R_0 q_0} \ln \left[1 + \tilde{r}^2 \right] \end{aligned} \qquad (1.4)$$

where $r^2 \equiv (R - R_0)^2 + Z^2$, $\tilde{r} \equiv r/a$, and $\hat{\varphi} \equiv R \nabla \varphi$ is a unit vector in the φ direction. Note that

$$\nabla \times \vec{B}_0 = \left[\frac{1 - \tilde{r}^2}{1 + \tilde{r}^2} + \frac{R_0}{R} \right] \frac{1}{1 + \tilde{r}^2} \frac{1}{R_0 R q_0} \hat{\varphi}. \qquad (1.5)$$

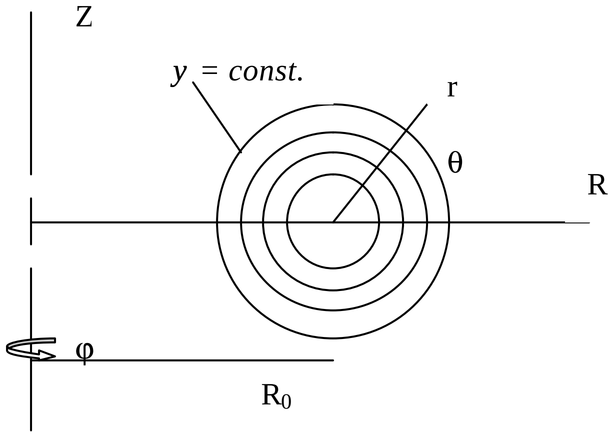
We take the function appearing in Eq. (1.2) as

$$\alpha = \frac{\alpha_{mn} a^2 \tilde{r}^m}{8R_0 q_0} \sin(m\theta - n\varphi) \qquad (1.6)$$

where θ is the angle in a polar (r, θ) coordinate system with origin at $R = R_0$, $Z = 0$ and m and n are integers. For definiteness, let (r, θ, φ) form a right-handed system. Thus, once the geometry (R_0 and a) is specified, the magnetic field is completely specified by the constants q_0, α_{mn}, m and n. We note that here that

$$\begin{aligned} \nabla \alpha \times \vec{B}_0 &= \frac{a^2 \alpha_{mn} \tilde{r}^m}{8RR_0 q_0} \left[\frac{a}{R R_0 q_0} \frac{n \tilde{r}}{1 + \tilde{r}^2} + \frac{m}{r} \right] \cos(m\theta - n\varphi) \hat{r} \\ &+ \frac{m a \alpha_{mn} \tilde{r}^{m-1}}{8R R_0 q_0} \left[\frac{a}{R_0 q_0} \frac{\tilde{r}}{1 + \tilde{r}^2} \hat{\varphi} + \hat{r} \times \hat{\varphi} \right] \end{aligned} \qquad (1.7)$$

Note: In the large aspect ratio approximation, the safety factor (or inverse winding number) at the plasma center and edge are: $q(0) \simeq q_0, \quad q(a) \simeq 2q_0$. Also, if the rational number *m/n* lies between q_0 and $2q_0$, ie, $q_0 < m/n < 2q_0$, then a *magnetic island* will be present with approximate normalized width of $(a_{mn})^{1/2}$. See Ref: [2]



(1a) Let the geometry and magnetic field be given by $R_0 = 3,\ a = 1,\ q_0 = 1,\ a_{mn}=0.$ Let the initial conditions and the source term be defined as:

$$
\begin{aligned}
T_{INITIAL}(R,Z) &= T_0\left(1-\frac{r^2}{a^2}\right) \\
S(R,Z) &= \frac{2kT_0}{a^2}\left(3-\frac{R_0}{R}\right)
\end{aligned}
\tag{1.8}
$$

Keep the boundary temperature fixed at $T=0$ and plot the central temperature $T(R_0,0)$ as $k_{||}$ is varied from 0 to a value $k_{||} >> k$ and verify to what degree it stays unchanged at T_0 as $k_{||}$ is increased to large values with ratios up to $k_{||} / k_{\wedge} = 10^8$.

(1b) As a initial condition, start with a localized Gaussian distribution:

$$
T_{INITIAL}(R,j,Z) = (2p)^{-3/2}\, T_0 e^{-\left[(R-R_1)^2+(Z-Z_1)^2+(2pR_1)^2(j-j_1)^2\right]/d^2} \tag{1.9}
$$

Let the geometry and magnetic field be given by $R_0 = 3,\ a = 1,\ q_0 = 1,\ a_{mn}=0.$ As an initial condition, use Eq. (1.8) with $R_1=3.5,\ Z_1=0,\ d=0.1$. Integrate Eq. (1.1) with parallel heat conduction only, ie, $k = 0$. Compute to what degree $<T>(r,t)$ remains unchanged after long times, where:

$$
<T>(r,t) = \int_0^{2p} dq \int_0^{2p} dj\, RT(r,q,j,t) \tag{1.10}
$$

(1c) Let the geometry and magnetic field be given by $R_0 = 3,\ a = 1,\ q_0 = 1$. Introduce a magnetic island by letting a_{32} be nonzero. (note $m=3,\ n=2$ in Eq. (1.6))
Apply the source term from (1a) only within the central section of the torus for which $\tilde{r} \le 0.1$, ie

$$
\begin{aligned}
S(R,Z) &= \frac{2kT_0}{a^2}\left(3 - \frac{R_0}{R}\right) \quad 0 \le \tilde{r} \le 0.1 \\
&= 0 \qquad\qquad\qquad\quad 0.1 < \tilde{r} \le 1.0
\end{aligned}
\tag{1.11}
$$

Start with the same initial condition as in Eq. (1.8) and solve for the steady-state temperature profile at long times. Plot the temperature as a function of $\tilde{r}$ along the ray $(j=0, q=0)$. Verify that the critical island size for which the temperature profile flattens and develops an inflection point within the island scales like $a_{mn} \sim (k_\wedge / k_{||})^{1/4}$ for ratios up to $k_{||} / k_\wedge = 10^8$.

2. Two Dimensional Tilt Mode:

The incompressible ideal MHD equations in 2D can be written in terms of 2 scalar variables: the velocity stream function f and the magnetic flux function y:

$$
\begin{aligned}
&\frac{\partial \Omega}{\partial t} + [\Omega, f] = [C, y] + m\nabla^2\Omega, \\
&\frac{\partial y}{\partial t} + [y, f] = 0, \\
&\nabla^2 f = \Omega \\
&C = \nabla^2 y
\end{aligned}
\tag{2.1}
$$

Where the two-dimensional Laplacian and commutation brackets are defined as:

$$
\begin{aligned}
\nabla^2 &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \\
[a,b] &\equiv \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}
\end{aligned}
\tag{2.2}
$$

Define the initial equilibrium state as:

$$
\begin{aligned}
y &= \begin{cases} [2/kJ_0(k)]J_1(kr)\cos q, & r < 1 \\ (1/r - r)\cos q, & r > 1 \end{cases} \\
J_1(k) &= 0.
\end{aligned}
\tag{2.3}
$$

where the computational domain is $-2 < x < 2,\ -2 < y < 2$, and (r,q) is a standard polar coordinate system such that $x = r \cos q$, etc. The boundary conditions are f = ¶y/¶$t = 0$ at the boundary of the domain. Take the viscosity coefficient to be $m=0.005$.

(2a) Give the equilibrium a small perturbation about 10^{-3} times smaller than the equilibrium and observe the growth rate and the log of the peak current density C. See Ref. [3]

3. Magnetic Reconnection in 2D (multiple time- and space-scales):

Consider the 2D, resistive MHD equations in a two-dimensional slab:

Consider the following [SI units except for temperatures, in eV] equations in Cartesian geometry *(x, y, and z)*, with *y* assumed to be a periodic direction with periodicity length $2pR_0$:

$$nM_i\left(\frac{\partial\vec{V}}{\partial t}+\vec{V}\bullet\nabla\vec{V}\right)=\vec{J}\times\vec{B}-\nabla p+n\nabla^2\vec{V} \tag{1.0}$$

$$\frac{\partial\vec{B}}{\partial t}=-\nabla\times\vec{E} \tag{1.1}$$

$$\vec{E}+\vec{V}\times\vec{B}=h\vec{J} \tag{1.2}$$

$$m_0\vec{J}=\nabla\times\vec{B} \tag{1.3}$$

$$\frac{\partial p}{\partial t}+\vec{V}\bullet\nabla p=-\tfrac{5}{3}\,p\,\nabla\bullet\vec{V}+\tfrac{2}{3}\left[hJ^2+\nabla\bullet\vec{q}+S_E\right] \tag{1.4}$$

$$\frac{\partial n}{\partial t}+\nabla\bullet\left(n\vec{V}\right)=0 \tag{1.5}$$

(3a) Apply the initial and boundary conditions of the cohelicity case described in Secttion IIB of Ref. [5].

The equations (1.0) – (1.5) are the single fluid approximation to the MHD equations. A considerable amount of literature exists on the generalization of these equations to a more complete set of Extended-MHD equations. For our initial studies, we consider the following set of 2-fluid equations [11]. These need to be extended in subsequent editions of these notes to include a more complete description of the gyro-viscous and parallel transport terms.

$$nM_i\left(\frac{\partial \vec{V}}{\partial t}+\vec{V}\bullet\nabla\vec{V}\right)=\vec{J}\times\vec{B}-\nabla p+n\nabla^2\vec{V} \tag{5.0}$$

$$\frac{\partial \vec{B}}{\partial t}=-\nabla\times\vec{E} \tag{5.1}$$

$$\vec{E}+\vec{V}\times\vec{B}=h\vec{J}+\frac{1}{ne}\left(\vec{J}\times\vec{B}-\nabla p_e\right) \tag{5.2}$$

$$m_0\vec{J}=\nabla\times\vec{B} \tag{5.3}$$

$$\frac{\partial p_i}{\partial t}+\vec{V}\bullet\nabla p_i=-\tfrac{5}{3}p_i\nabla\bullet\vec{V}+\tfrac{2}{3}\left[n\left\|\nabla\vec{V}\right\|^2+\nabla\bullet\vec{q}_i+Q_{ei}\right] \tag{5.4a}$$

$$\frac{\partial p_e}{\partial t}+\vec{V}_e\bullet\nabla p_e=-\tfrac{5}{3}p_e\nabla\bullet\vec{V}_e+\tfrac{2}{3}\left[hJ^2+\nabla\bullet\vec{q}_e-Q_{ei}+S_E\right] \tag{5.4b}$$

$$\frac{\partial n}{\partial t}+\nabla\bullet\left(n\vec{V}\right)=0 \tag{5.5}$$

These equations again assume quasi-neutrality throughout, so that $n_e = n_i = n$. The electron and ion temperatures are given by $nk_BT_e = p_e$ and $nk_BT_i = p_i = p - p_e$, where $k_B = 1.602\times 10^{-19}$ J/eV. The electron velocity is defined by $\vec{V}_e = \vec{V} - \vec{J}/ne$.

(3b) Reproduce the co-helicity case in Section V of Ref. [5].

References:

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6. C.R. Sovinec, et al, “Nonlinear Magnetohydrodynamics Simulation using High-Order Finite Elements”, submitted to J. Comp. Phys (2003)