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MODEL DEVELOPMENT AND PLANS IN CONTINUUM KINETIC-MHD*

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OUTLINE

STATUS AND PLANS FOR THEORETICAL MODEL DEVELOPMENT.

COMMENTS ON THE USE OF THE CEMM PLATFORM FOR TRANSPORT STUDIES.

GENERALIZED SPITZER PROBLEM WITH FOKKER-PLANCK OPERATORS IN A LOW COLLISIONALITY REGIME AND RELATED ISSUES IN THE NEOCLASSICAL THEORY OF AXISYMMETRIC EQUILIBRIA.

STATUS AND PLANS FOR THEORETICAL MODEL DEVELOPMENT

DRIFT-KINETIC CLOSURE THEORY FOR LOW-COLLISIONALITY ELECTRONS COMPLETED (TALK AT SHERWOOD CONFERENCE THIS WEEK):

- Rigorous account the electric field and consistency with the fluid system.
- First-order FLR magnetic gradient drifts and Fokker-Planck collision operators.
- Near-Maxwellian, Chapman-Enskog-like for slow dynamics. Non-Maxwellian perturbation with automatically vanishing 1, $\mathbf{v} - \mathbf{u}_e$ and $|\mathbf{v} - \mathbf{u}_e|^2$ moments.
- Compatible with the neoclassical theory in the electron banana regime. Yields neoclassical banana results for odd equilibrium closures and bootstrap current.

NUMERICAL IMPLEMENTATION IN A DRIFT-KINETIC CLOSURE MODULE FOR THE NIMROD CODE UNDER WAY (UPDATE BY E. HELD).

ON NUMERICAL IMPLEMENTATION OF THE ELECTRON DRIFT-KINETIC CLOSURE FOR EXTENDED-MHD:

- **Ideally framed as an integration project. Standard fluid and drift-kinetic interface desirable.**
- **5D+time dimensionality for data storage. 3D+time dimensionality for integration.**
- **Intrinsically implicit character of the time advance algorithm for the distribution function.**
- **Gyrophase-independent velocity coordinates in the moving reference frame of the mean flow: magnitude of the random velocity and its pitch angle relative to the local magnetic field direction, with a Legendre polynomial expansion of the pitch angle dependence.**

  Three distinct parts in drift-kinetic equation:

    Collisionless streaming: band-diagonal with Legendre-l coupled to $l+1$ and $l-1$.

    Linear Fokker-Planck collision operator: diagonal in l.

    Inhomogeneous drive: $l=0$, $l=1$ and $l=2$ components.

  Needed fluid closure moments are pure $l=1$ and $l=2$ Legendre components.

- **Less clear choice for discretization of the dependence on the magnitude of the velocity.**

ELECTRON DRIFT-KINETIC EQUATION

In polar random velocity coordinates ($v'_\parallel = v'\cos\chi,\ \ v'_\perp = v'\sin\chi$):

$$
\frac{\partial \bar{f}_{NMe}}{\partial t} + \cos\chi\left(v'\mathbf{b}\cdot\frac{\partial \bar{f}_{NMe}}{\partial \mathbf{x}} + v_{the}^2\mathbf{b}\cdot\nabla\ln n\ \frac{\partial \bar{f}_{NMe}}{\partial v'}\right) - \frac{\sin\chi}{v'}\left(v_{the}^2\mathbf{b}\cdot\nabla\ln n - \frac{v'^2}{2}\mathbf{b}\cdot\nabla\ln B\right)\frac{\partial \bar{f}_{NMe}}{\partial \chi} =
$$

$$
= \Big\{ \cos\chi\ \frac{v'}{2T_e}\left(5 - \frac{v'^2}{v_{the}^2}\right)\mathbf{b}\cdot\nabla T_e\ +\ \cos\chi\ \frac{v'}{nT_e}\ \mathbf{b}\cdot\left[\frac{2}{3}\nabla(p_{e\parallel} - p_{e\perp}) - \left(p_{e\parallel} - p_{e\perp}\right)\nabla\ln B - \mathbf{F}_e^{coll}\right]\ +
$$

$$
+\ P_2(\cos\chi)\ \frac{v'^2}{3v_{the}^2}\Big(\nabla\cdot\mathbf{u}_e - 3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}_e]\Big)\ +\ \frac{1}{3nT_e}\left(\frac{v'^2}{v_{the}^2} - 3\right)\left[\nabla\cdot(q_{e\parallel}\mathbf{b}) - G_e^{coll}\right]\ +
$$

$$
+\ \frac{1}{6eB}\left[2P_2(\cos\chi)\frac{v'^2}{v_{the}^2}\left(\frac{v'^2}{v_{the}^2} - 5\right)\ +\ \frac{v'^4}{v_{the}^4} - 10\frac{v'^2}{v_{the}^2} + 15\right](\mathbf{b}\times\boldsymbol{\kappa})\cdot\nabla T_e\ +
$$

$$
+\ \frac{1}{6eB}\left[-P_2(\cos\chi)\frac{v'^2}{v_{the}^2}\left(\frac{v'^2}{v_{the}^2} - 5\right)\ +\ \frac{v'^4}{v_{the}^4} - 10\frac{v'^2}{v_{the}^2} + 15\right](\mathbf{b}\times\nabla\ln B)\cdot\nabla T_e\ +
$$

$$
+\ P_2(\cos\chi)\ \frac{v'^2}{3eBv_{the}^2}(\mathbf{b}\times\nabla\ln n)\cdot\nabla T_e\ \Big\} f_{Me}\ +
$$

$$
+\ \langle C_{ee}[f_{Me}, f_{NMe}] + C_{ee}[f_{NMe}, f_{Me}] + C_{e\iota}^{(3)}[f_{NMe}, f_{M\iota}]\rangle_\alpha\ \ +\ \ \langle C_{e\iota}^{(3)}[f_{Me}, f_\iota]\rangle_\alpha\ .
$$

ELECTRON COLLISION OPERATORS

BASED ON THE COMPLETE LINEARIZED FOKKER-PLANCK-LANDAU OPERATORS and using the electron collision frequency definition

$$\nu_e \; \equiv \; \frac{c^4 e^4 n \ln \Lambda_e}{4\pi m_e^2 v_{the}^3} \; ,$$

The GYROPHASE AVERAGED COLLISION OPERATORS needed in the electron drift-kinetic equation are:

$$\begin{aligned} \langle C_{e\iota}^{(3)}[f_{Me}, f_\iota]\rangle_\alpha \; &= \; \nu_e v_{the} f_{Me}(v') \; \frac{j_\parallel}{en v_{th\iota}^2} \; \xi\left(\frac{v'}{v_{th\iota}}\right) \cos\chi \; + \\ &+ \; \nu_e v_{the} f_{Me}(v') \left(\frac{T_e}{T_\iota} - 1\right) \left[\frac{4\pi v_{th\iota}^2}{n} \; f_{M\iota}(v') - \frac{v'}{v_{the}^2} \; \xi\left(\frac{v'}{v_{th\iota}}\right)\right] \end{aligned}$$

where

$$\xi(x) \; = \; \frac{1}{x^2}\left[\varphi(x) - x\frac{d\varphi(x)}{dx}\right] \qquad \textbf{and} \qquad \varphi(x) \; = \; \frac{2}{(2\pi)^{1/2}} \int_0^x dt \;\; \exp(-t^2/2)$$

$\langle C_{ee}[f_{Me}, f_{NMe}] + C_{ee}[f_{NMe}, f_{Me}] + C^{(3)}_{e\iota}[f_{NMe}, f_{M\iota}]\rangle_\alpha \;=\; \mathcal{C}_e[\bar{f}_{NMe}]$ **is Legendre diagonal:**

$$\mathcal{C}_e\left[\sum_{l=0}^{\infty} f_l(v')P_l(\cos\chi)\right] \;=\; \sum_{l=0}^{\infty} P_l(\cos\chi)\;\mathcal{C}_{e,l}[f_l](v')$$

with

$$\mathcal{C}_{e,l}[f_l](v') \;=\; \frac{\nu_e v_{the}}{n} f_{Me}(v')\left\{4\pi v_{the}^2 f_l(v') \;-\; \Phi_l[f_l](v') \;+\; \frac{v'^2}{v_{the}^2}\frac{d^2\Psi_l[f_l](v')}{dv'^2}\right\} \;+$$
$$+\;\frac{\nu_e v_{the}^3}{v'^2}\frac{d}{dv'}\left\{\xi\left(\frac{v'}{v_{the}}\right)\left[v'\frac{df_l(v')}{dv'} + \frac{v'^2}{v_{the}^2}\;f_l(v')\right] \;+\; \xi\left(\frac{v'}{v_{th\iota}}\right)\left[v'\frac{df_l(v')}{dv'} + \frac{m_e v'^2}{m_\iota v_{th\iota}^2}\;f_l(v')\right]\right\} \;-$$
$$-\;\frac{\nu_e l(l+1)v_{the}^3}{2v'^3}\left[\varphi\left(\frac{v'}{v_{the}}\right) - \xi\left(\frac{v'}{v_{the}}\right) + \varphi\left(\frac{v'}{v_{th\iota}}\right) - \xi\left(\frac{v'}{v_{th\iota}}\right)\right] f_l(v')$$

and

$$\frac{1}{v'^2}\frac{d}{dv'}\left\{v'^2\frac{d\Phi_l[f_l](v')}{dv'}\right\} \;-\; \frac{l(l+1)}{v'^2}\Phi_l[f_l](v') \;=\; -4\pi f_l(v')$$

$$\frac{1}{v'^2}\frac{d}{dv'}\left\{v'^2\frac{d\Psi_l[f_l](v')}{dv'}\right\} \;-\; \frac{l(l+1)}{v'^2}\Psi_l[f_l](v') \;=\; \Phi_l[f_l](v')\;.$$

ON PLANNED THEORETICAL MODEL DEVELOPMENT WORK:

- **Derivation of the corresponding low-collisionality model for the ions.**
- **Based on the same orderings and the same mean flow reference frame formalism used for the electrons.**
- **Consistent with these same low-collisionality and mass ratio orderings, the ion theory requires a second-order drift-kinetic equation in the gyroradius expansion.**
- **Departure from conventional ion banana neoclassical theory. (Recoverable as a subset).**
- **Well established groundwork in earlier fluid and collisionless drift-kinetic publications.**

COMMENTS ON THE USE OF THE CEMM PLATFORM FOR TRANSPORT STUDIES

A CREDIBLE CONTRIBUTION IN THE TRANSPORT AREA COULD BE MADE WITH THE MOST ADVANCED CEMM SYSTEM ENVISIONED:

- **Fluid continuity, ion momentum, electron momentum and electron temperature equations.**
- **Particle-based kinetic ions contributing the full $\mathbf{P}_\iota$ tensor.**
- **Drift-kinetic electrons contributing $(p_{e\parallel} - p_{e\perp})$, $q_{e\parallel}$ and $F_{e\parallel}^{coll}$.**

FOR PROCESSES WHERE SUB-ION-LARMOR-RADIUS SCALES ARE NOT ESSENTIAL (SUCH AS FLUID-ITG TURBULENCE), A CONTINUUM FLR ION DESCRIPTION MAY BE SUFFICIENT. THIS WOULD STILL REQUIRE:

- **A slow-dynamics ion stress tensor in the fluid system.**
- **A slow-dynamics ion drift-kinetic parallel closure.**

ON THE VIABILITY OF TRANSPORT MODELS WITH REDUCED DIMENSIONALITY (2-D AXISYMMETRIC, 1-D MAGNETIC SURFACE AVERAGED):

- **These models must rely on phenomenological diffusive terms to represent the radial transport (e.g. like in the TSC code).**
- **A self-consistent, first-principle description of the radial transport at realistically low collisionality in an axisymmetric system seems very unlikely: the degeneracy of this system is such that always some quantities are left undetermined within the orders where an underlying self-consistent theory can be reasonably worked out.**
- **Rather than deriving and implementing the extraordinarily high-order theory needeed to resolve the axisymmetric degeneracies, it appears more likely that computational advances will allow to carry out 3-D, initial value simulations over transport times.**

GENERALIZED SPITZER PROBLEM WITH FOKKER-PLANCK OPERATORS IN A LOW COLLISIONALITY REGIME AND RELATED ISSUES IN THE NEOCLASSICAL THEORY OF AXISYMMETRIC EQUILIBRIA

Using the following representation for the non-Maxwellian part of the distribution function:

$$\bar{f}_{NMe}(\mathbf{x}, v', \chi) \; = \; f_{Me}^{(0)}(\psi, v') \left\{ \frac{e\big[\phi - \phi^{(1)}(\psi)\big]}{T_e^{(0)}(\psi)} \; - \; \frac{n - N^{(0)}(\psi)}{N^{(0)}(\psi)} \; - \; \left[\frac{m_e v'^2}{T_e^{(0)}(\psi)} - 3 \right] \frac{T_e - T_e^{(0)}(\psi)}{2T_e^{(0)}(\psi)} \right\} \; -$$

$$- \; f_{Me}^{(0)}(\psi, v') \left\{ \frac{m_e U_e(\psi) B}{T_e^{(0)}(\psi)} \; + \; \frac{m_e I(\psi)}{2eBT_e^{(0)}(\psi)} \left[\frac{m_e v'^2}{T_e^{(0)}(\psi)} - 5 \right] \frac{dT_e^{(0)}(\psi)}{d\psi} \right\} v' \cos\chi \; + \; h_e(\mathbf{x}, v', \chi) \; ,$$

the low-collisionality electron drift-kinetic equation in an axisymmetric equilibrium becomes

$$v' \left(\cos\chi \; \mathbf{b} \cdot \frac{\partial h_e}{\partial \mathbf{x}} \; + \; \frac{1}{2} \mathbf{b} \cdot \nabla \ln B \; \sin\chi \; \frac{\partial h_e}{\partial \chi} \right) \; - \; \mathcal{C}_e[h_e] \; = \; \mathcal{S}_e \; v' \cos\chi$$

where

$$\mathcal{S}_e \; = \; \Big\{ \frac{eV_0 I}{T_e^{(0)} B R^2} \; + \; \nu_e \left[U_\iota B \; + \; \frac{I}{eN^{(0)} B} \; \frac{d\left(2N^{(0)} T_e^{(0)}\right)}{d\psi} \right] \frac{v_{the}}{v_{th\iota}^2 v'} \; \xi \left(\frac{v'}{v_{th\iota}} \right) \; +$$

$$+ \; \frac{\nu_e m_e I}{eBT_e^{(0)}} \frac{dT_e^{(0)}}{d\psi} \; \frac{v_{the}}{v'} \left[2\varphi \left(\frac{v'}{v_{the}} \right) - 10\xi \left(\frac{v'}{v_{the}} \right) + \frac{1}{2} \varphi \left(\frac{v'}{v_{th\iota}} \right) - \frac{5 v_{the}^2}{2 v_{th\iota}^2} \xi \left(\frac{v'}{v_{th\iota}} \right) \right] \Big\} \; f_{Me}^{(0)} \; .$$

Changing variables to $(\psi, \theta, v', \lambda)$**, with** $\lambda(\psi, \theta, \chi) = \sin^2\chi \; B_{max}(\psi)/B(\psi, \theta)$**:**

$$v'_\parallel \; (\mathbf{b}\cdot\nabla\theta) \; \frac{\partial h_e}{\partial \theta} \; - \; \mathcal{C}_e[h_e] \; = \; \mathcal{S}_e \; v'_\parallel$$

where

$$v'_\parallel(\psi, \theta, v', \lambda) \; = \; \pm v'[1 - \lambda B(\psi, \theta)/B_{max}(\psi)]^{1/2} \; .$$

FOLLOWING THE STANDARD SOLUTION METHOD OF NEOCLASSICAL THEORY:

$$h_e \; = \; \sigma(v'_\parallel) H(1-\lambda) K_e(\psi, v', \lambda) \; + \; h_e^{(3)}(\psi, \theta, v', \lambda) \; = \; O(\delta_e f_{Me}) + O(\delta_e \nu_* f_{Me})$$

with

$$v'_\parallel \; (\mathbf{b}\cdot\nabla\theta) \; \frac{\partial h_e^{(3)}}{\partial \theta} \; - \; \mathcal{C}_e[\sigma H K_e] \; = \; \mathcal{S}_e \; v'_\parallel \; .$$

THIS HAS THE SOLUBILITY CONDITION THAT DETERMINES $K_e(\psi, v', \lambda)$:

$$\oint_{\psi, v', \lambda} dl \; v_\parallel'^{-1} \; \mathcal{C}_e[\sigma H K_e] \; = \; - \oint_{\psi, v', \lambda} dl \; \mathcal{S}_e \; .$$

THE SOLUTION $K_e(\psi, v', \lambda)$ OF THE ABOVE GENERALIZED SPITZER PROBLEM GIVES:

The electron poloidal flow (whence the electron contribution to the bootstrap current):

$$u_{ep} \;=\; U_e(\psi) B_p \qquad \textbf{where} \qquad U_e(\psi) \;=\; \frac{2\pi}{N^{(0)}(\psi) B_{max}(\psi)} \int_0^\infty dv' \; v'^3 \; \int_0^1 d\lambda \; K_e(\psi, v', \lambda) \; .$$

The parallel heat flux:

$$q_{e\|} = -\frac{5 N^{(0)} T_e^{(0)} I}{2eB} \; \frac{dT_e^{(0)}}{d\psi} + Q_e(\psi) B \quad \textbf{where} \quad Q_e(\psi) = \frac{\pi T_e^{(0)}}{B_{max}} \; \int_0^\infty dv' \; v'^3 \left(\frac{m_e v'^2}{T_e^{(0)}} - 5 \right) \int_0^1 d\lambda \; K_e(\psi, v', \lambda) \; .$$

The parallel collisional friction force:

$$F_{e\|}^{coll} \;=\; \frac{2 m_e \nu_e}{3 (2\pi)^{1/2}} \left(\frac{j_\|}{e} + N^{(0)} U_e B - \frac{3 N^{(0)} I}{2eB} \frac{dT_e^{(0)}}{d\psi} \right) \;-\; \frac{2\pi m_e \nu_e v_{the}^3 B}{B_{max}} \; \int_0^\infty dv' \; \int_0^1 d\lambda \; K_e(\psi, v', \lambda) \; .$$

The magnetic surface averaged neoclassical parallel viscosity:

$$-\oint_\psi dl \; (p_{e\|} - p_{e\perp}) \; \mathbf{b} \cdot \nabla \ln B \;=\; \oint_\psi dl \left(F_{e\|}^{coll} + \frac{e V_0 N^{(0)} I}{B R^2} \right) \;=\; \mu_{e1} U_e(\psi) \;+\; \mu_{e3} Q_e(\psi) \;+\; \ldots$$

SELF-CONSISTENT OHMIC AND BOOTSTRAP EQUILIBRIUM CURRENT:

The magnetic surface average of the $\nabla\zeta$ component of the electron momentum equation,

$$\oint_{\psi} \frac{dl}{B} \left(R^2 \nabla\zeta \cdot \mathbf{F}_e^{coll} \; + \; eN^{(0)} V_0 \right) \; = \; 0 \; ,$$

yields

$$U_{\iota}(\psi) \; = \; - \; \frac{3(2\pi)^{1/2} e V_0}{2 m_e \nu_e I} \; - \; \frac{\langle R^2 \rangle_{\psi}}{eI} \left(\frac{1}{2} \frac{dT_e^{(0)}}{d\psi} + \frac{2T_e^{(0)}}{N^{(0)}} \frac{dN^{(0)}}{d\psi} \right) \; + \; \frac{3(2\pi)^{3/2} v_{the}^3}{2N^{(0)} B_{max}} \; \int_0^{\infty} dv' \; \int_0^1 d\lambda \; K_e(\psi, v', \lambda) \; .$$

This, together with the previous $U_e(\psi)$, give the ohmic and bootstrap part of the current:

$$\frac{dI}{d\psi} = - \frac{3(2\pi)^{1/2} e^2 V_0 N^{(0)}}{2 m_e \nu_e I} - \frac{\langle R^2 \rangle_{\psi}}{I} \left(\frac{N^{(0)}}{2} \frac{dT_e^{(0)}}{d\psi} + 2T_e^{(0)} \frac{dN^{(0)}}{d\psi} \right) + \frac{(2\pi)^{3/2} e}{B_{max}} \int_0^{\infty} dv' \left[\frac{3 v_{the}^3}{2} - \frac{v'^3}{(2\pi)^{1/2}} \right] \int_0^1 d\lambda \; K_e.$$

Also,

$$F_{e\parallel} \; = \; - \; \frac{eV_0 N^{(0)} B}{I} \; - \; \frac{2 m_e \nu_e}{3(2\pi)^{1/2} eBI} \; \left(\langle R^2 \rangle_{\psi} B^2 - I^2 \right) \; \left(\frac{N^{(0)}}{2} \frac{dT_e^{(0)}}{d\psi} + 2T_e^{(0)} \frac{dN^{(0)}}{d\psi} \right) \; .$$

OUTSTANDING ISSUE

The generalized Spitzer problem for $K_e(\psi, v', \lambda)$:

$$\oint_{\psi,v',\lambda} dl \ v_{\|}'^{-1} \ \mathcal{C}_e[\sigma H K_e] \ = \ - \oint_{\psi,v',\lambda} dl \ \mathcal{S}_e \ ,$$

that stems from the perturbative equation

$$v_{\|}' \ (\mathbf{b} \cdot \nabla\theta) \ \frac{\partial h_e^{(3)}}{\partial \theta} \ - \ \mathcal{C}_e[\sigma H K_e] \ = \ \mathcal{S}_e \ v_{\|}' \ ,$$

is subject to the boundary conditions:

$$lim_{\lambda\to 0}\left[\lambda^{1/2}\frac{\partial K_e(\psi, v', \lambda)}{\partial \lambda}\right] = 0, \qquad K_e(\psi, v', 1) = 0 \quad \textbf{and} \quad \frac{\partial K_e(\psi, v', 1)}{\partial \lambda} = 0.$$

If no solution satisfying these boundary conditions can be found, then the original equation

$$v_{\|}' \ (\mathbf{b} \cdot \nabla\theta) \ \frac{\partial h_e}{\partial \theta} \ - \ \mathcal{C}_e[h_e] \ = \ \mathcal{S}_e \ v_{\|}'$$

must be taken into account, either globally or in a boundary layer near $\lambda = 1$.

Simplified collision operator models yield $\partial K_e(\psi, v', 1)/\partial\lambda \neq 0$, but it is not clear whether a satisfactory boundary layer solution exists that smooths this derivative jump.

SELF-CONSISTENT OHMIC AND BOOTSTRAP EQUILIBRIUM CODE

INPUT		**G-S SOLVER**		**SPITZER SOLVER**
$N(\hat{\psi})$		**In:** $2N(\hat{\psi})T(\hat{\psi})$		**In:** $N(\psi)$
$T(\hat{\psi})$	$\Rightarrow$ **guess** $I(\hat{\psi})$ $\Rightarrow$	$I(\hat{\psi})$	$\Rightarrow$ **guess** $U_\iota(\psi),\ V_0$ $\Rightarrow$	$T(\psi)$
$I(\psi_a)$	$\Uparrow$	$I(\psi_a)$	$\Uparrow$	$I(\psi)$
$I_t(\psi_a)$	$\Uparrow$	$I_t(\psi_a)$	$\Uparrow$	$U_\iota(\psi)$
.	$\Uparrow$	**Out:** $\psi(R,Z)$	$\Uparrow$	V_0
.	$\Uparrow$	ψ_a	$\Uparrow$	$B(\psi, l_B)$
.	$\Uparrow$	$B(\psi, l_B)$	$\Uparrow$	**Out:** $K_e(\psi, v', \lambda)$
.	$\Uparrow$		$\Uparrow$	$\Downarrow$
.	$\Uparrow$		$\Uparrow$	$\Downarrow$
.	$\Uparrow$		**update** $U_\iota(\psi),\ V_0$ $\Leftarrow$	**evaluate** $U_e(\psi),\ U_\iota(\psi),\ I_t(\psi_a)$
.	$\Uparrow$			$\Downarrow$
.	$\Uparrow$			$\Downarrow$
.	**update** $I(\hat{\psi})$ $\Leftarrow$ $\Leftarrow$ $\Leftarrow$	$\Leftarrow$ $\Leftarrow$ $\Leftarrow$ $\Leftarrow$ $\Leftarrow$	$\Leftarrow$ $\Leftarrow$ $\Leftarrow$ $\Leftarrow$ $\Leftarrow$	$\Leftarrow$ **evaluate** $dI(\psi)/d\psi$