MHD Basics

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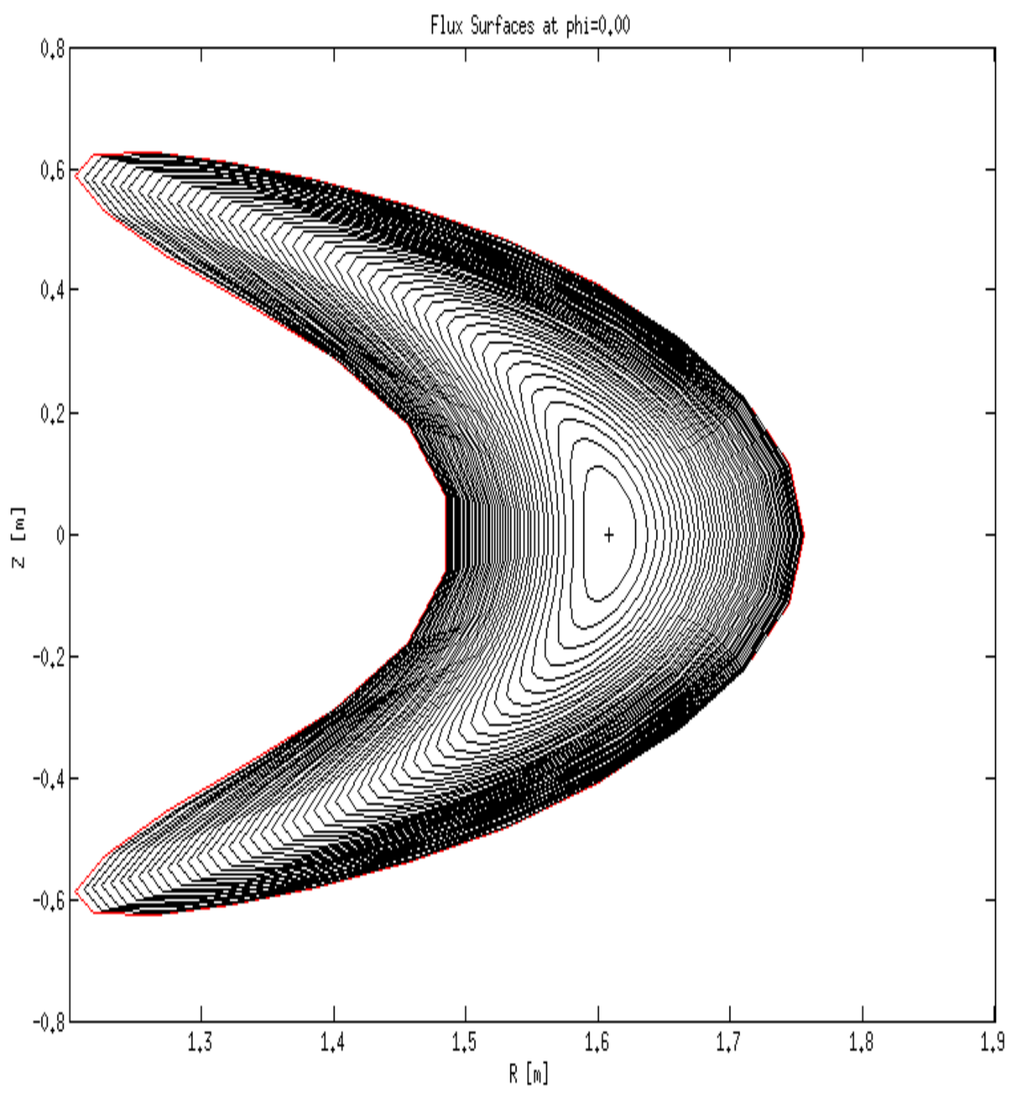
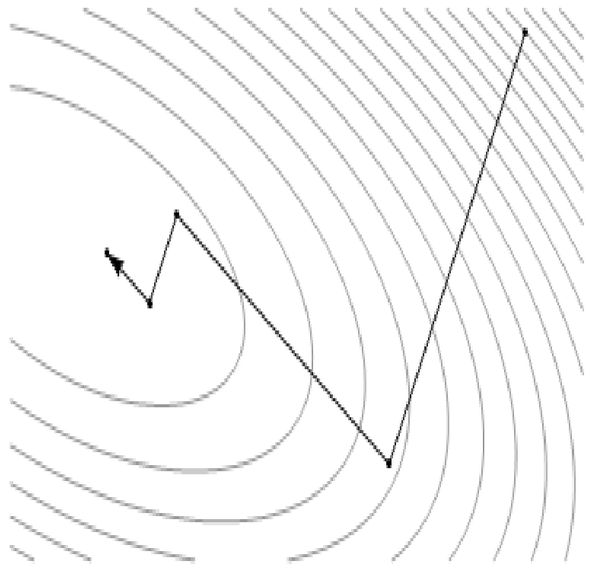
Sept 10 - Sept 21, 2012

Steep-descent moment for three-dimensional magnetohydrodynamic equilibria
Steve Hirshman
1983



NCSX flux surface



An algorithm for finding the nearest local minimum of a function presupposing that its gradient can be computed.

It is also called the gradient decent method,

starting at a point P_0, and as many times as needed,

moves from P_i to P_{i+1} by minimizing along the line

extending from P_i in the direction of $-\nabla F(P_i)$,

the local downhill gradient.

The iteration procedure is

$$F(P_{i+1}) = F(P_i) - \delta t \nabla F(P_i)$$

$$\frac{F(P_{i+1}) - F(P_i)}{\delta t} = -\nabla F(P_i)$$

$$\frac{d}{dt} F = -\nabla F$$

This is the physicist so-called Jacobi relaxation. It can be sped up by the so-called 2nd order Richardson relaxation

$$\frac{d^2}{dt^2} F + \frac{1}{\tau} \frac{d}{dt} F = -\nabla F$$

Steepest-descent moment method for 3D MHD equilibria

$X = \sum X_{mn} e^{i(m\theta - n\zeta)}$

X_{mn} is called Fourier moments

we will have eqs to solve them

The derivation of the MHD equations begins by taking the mass, momentum, and energy moments integral of Boltzmann equation in the velocity space

The most basic set of eqs is the Boltzmann eqs for each species s

$$\frac{\partial f_s(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_s(\mathbf{x},\mathbf{v},t) + \frac{e_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_s(\mathbf{x},\mathbf{v},t) = C_s$$

describing the 6D + time $(\mathbf{x},\mathbf{v},t)$ phase space probability distribution function f_s.

together with the free space Maxwell's eqs

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \qquad \text{(Faraday's law)}$$

$$\nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mu_0\mathbf{J} \qquad \text{(Ampère's law)}$$

$$\nabla\cdot\mathbf{E} = \frac{\sigma}{\epsilon_0} \qquad \text{(Gauss's law)}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \text{(Gauss's law for magnetism)}$$

It would appear that

the distribution functions for the various plasma species,

from which the constitutive relations are trivially obtained,

are determined by use of rather harmless looking 1st order PDEs.

At this stage, we might wonder why,

if the plasma dynamics is apparently so simple

when written in terms of distribution functions,

we need a fluid description of plasma dynamics at all.

It is not at all obvious that fluid theory represent an advance.

This argument is misleading for several reasons. However, by far

the most serious flaw is the view of eq for f_S as a tractable eq.

Note that this eq is easy to derive, because it is exact,

taking into account an scales from microscopic to macroscopic.

f_S follows a detailed trajectory of a single particle.

Numerical difficulties:

number of variables (1+3+3)

$\times$ dim (7)

$\times$ number density ($1.5 \times 10^{20} m^{-1}$)

$\times$ meshsize (10^3)

Time scale $10^{-8} s \sim 10^{12} s$

Space scale $10^{-5} m \sim 10 m$

plasma oscillations	$1/\omega_{pe} \sim 10^{-11} s$
Debye shielding	$\lambda_D \sim 10^{-5} m$
Coulomb collision frequency	$1/\nu \sim 10^{-7} s$
Coulomb collision length	$\lambda \sim 0.1 m$
cyclotron waves	$1/\omega_{ci} \sim 10^{-8} s$
Alfven waves	$a/C_A \sim 10^{-7} s$
compressional Alfven waves	$L_{\parallel}/C_A \sim 10^{-6} s$
sound waves	$a/C_S \sim 10^{-5.5} s$
resistivity	$> 1/\nu_e \sim 10^{-5} s$

In general, this set eqs is very difficult to solve,

due to the complexity of collision operators as well.

However, there are some situations where collions can be completely neglected.

In this case, we get the kinetic description using Vlasov eq

$$\frac{\partial f_s(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_s(\mathbf{x},\mathbf{v},t) + \frac{e_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_v f_s(\mathbf{x},\mathbf{v},t) = 0$$

It is tractable in sufficiently simple geometry.

Nevertheless,

the fluid approach has much more to offer even in the Vlasov limit.

- the fluid eq possess the key simplicity of involving fewer dimensions: $\mathbf{x}, t$
- the fluid description is intuitively appealing.

  We immediately understand the significance of fluid quantities such as density and temperature.

  Whereas the significance of distribution functions is far less obvious
- fluid variables are relatively easy to measure in experiment, whereas,

  it is extraordinary difficult to measure a distribution function.
- the kinetic approach to plasma physics is spectacularly inefficient.

  The species distribution function f_s provide vastly more information than is needed.

In order to get fluid picture of plasma, we define moments of the distribution function.

the kth moment of the distribution function $f_s(\mathbf{x}, \mathbf{v}, t)$ is written as

$$\mathbf{M}_k(\mathbf{x}, t) = \int \underbrace{\mathbf{v}\mathbf{v}\ldots\mathbf{v}}_{k} f_s(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{v}, k = 0, 1, x, \cdots$$

Integrated in the velocity phase space **v**.

The lower order moments all have names and simple physical interpretations.

number density $(\#/m^3)$: $n_s(\mathbf{x},t) = \int f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v}$

flow velocity (m/s): $\mathbf{V}_s(\mathbf{x},t) = \frac{1}{n_s}\int \mathbf{v} f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v}$

charge density (C/m^3): $\sigma = \sum_s e_s \int f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v} = \sum_s e_s n_s$

current density (A/m^2): $\mathbf{J} = \sum_s e_s \int \mathbf{v} f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v} = \sum_s e_s n_s \mathbf{V}_s$

temperature (J, eV): $T_s(\mathbf{x},t) = \frac{1}{n_s}\int \frac{m_s v_r^2}{3} f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v} = \frac{m_s v_T^2}{2}$

conductive heat flux (W/m^2): $\mathbf{q}_s(\mathbf{x},t) = \int \mathbf{v}_r \frac{m_s v_r^2}{2} f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v}$

pressure (N/m^2): $p_s = \int \frac{m_s v_r^2}{3} f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v} = n_s T_s$

pressure tensor (N/m^2): $\mathbf{P}_s = \int m \mathbf{v}_r \mathbf{v}_r f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v} = p_s \mathbf{I} + \boldsymbol{\pi}_s$

stress tensor (N/m^2): $\boldsymbol{\pi}_s = \int m_s \left(\mathbf{v}_r\mathbf{v}_r - \frac{v_r^2}{3}\mathbf{I}\right) f_s(\mathbf{x},\mathbf{v},t)d^3\mathbf{v}$

We have defined and used the relative velocity

$$\mathbf{v}_r = \mathbf{v} - \mathbf{V}_s(\mathbf{x},t)$$

In addition, we will need the lowest order velocity moments of the Coulomb collision operator C_s.

Steepest-descent moment method for 3D MHD equilibria

density conservation in collisions: $0 = \int C_s d^3\mathbf{v}$

frictional force density (N/m^3): $\mathbf{R}_s = \int m\mathbf{v} C_s d^3\mathbf{v}$

energy exchange density (W/m^3): $Q_s = \int \frac{m v_r^2}{2} C_s d^3\mathbf{v}$

The kth moment of the kinetic eq is obtained by multiplying the distribution eq

by k powers of **v** and integrate over velocity space.

$$\int \underbrace{\mathbf{vv}\ldots\mathbf{v}}_{k}\left[\frac{\partial f_s(\mathbf{x},\mathbf{v},t)}{\partial t} + \nabla\cdot(\mathbf{v}f_s(\mathbf{x},\mathbf{v},t)) + \nabla_v\cdot\left(\frac{e_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})f_s(\mathbf{x},\mathbf{v},t)\right) = C_s\right]$$

The flow term is simplified by pulling the divergence outside the velocity integral.

The acceleration term is treated by partial integration.

These 2 terms couple the kth moments to the $(k+1)$th and $(k-1)$th moments.

Making use of the collisional conservation laws,

the $0th$ order moment yields the continuity eq for species s

$$\frac{\partial}{\partial t} n_s + \nabla \cdot n_s \mathbf{V}_s = 0$$

the $1th$ order moment gives the momentum conservation eq for species s

$$\frac{\partial}{\partial t}(m_s n_s \mathbf{V}_s) + \nabla \cdot (p_s \mathbf{I} + \boldsymbol{\pi}_s + m_s n_s \mathbf{V}_s \mathbf{V}_s) - e_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) = \mathbf{R}_s$$

We rewrite it using the density eq to remove the $\frac{\partial n}{\partial t}$ contribution and $\nabla \cdot m_s n_s \mathbf{V}_s \mathbf{V}_s = m_s \mathbf{V}_s (\nabla \cdot n_s \mathbf{V}_s) + m_s n_s \mathbf{V} \cdot \nabla \mathbf{V}_s$ to obtain

$$\boldsymbol{m_s n_s \frac{d\mathbf{V}_s}{dt} = -\nabla p - \nabla \cdot \pi_s + e_s n_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) + \mathbf{R}_s}$$

the $2nd$ order moment yields the energy conservation eq for species s

$$\frac{\partial}{\partial t}\left(\frac{3}{2} p_s + \frac{1}{2} m_s n_s V_s^2\right) + \nabla \cdot \left[\mathbf{q}_s + \left(\frac{5}{2} p_s + \frac{1}{2} m_s n_s V_s^2\right) \mathbf{V}_s + \mathbf{V}_s \cdot \boldsymbol{\pi_s}\right] - e_s n_s \mathbf{E} \cdot \mathbf{V}_s - Q_s = \mathbf{V}_s \cdot \mathbf{R}_s$$

Using the dot product of the momentum eq with $\mathbf{V}_s$ to remove the $\frac{\partial V_s^2}{\partial t}$ term and using the density eq, it can be simplified to

$$\boldsymbol{\frac{3}{2} \frac{\partial p_s}{\partial t} = -\nabla \cdot (\mathbf{q}_s + \frac{5}{2} p_s \mathbf{V}_s) + \mathbf{V}_s \cdot \nabla p_s - \pi_s : \nabla \mathbf{V}_s + Q_s}$$

or

$$\frac{3}{2} \frac{dp_s}{dt} + \frac{5}{2} p_s \nabla \cdot \mathbf{V}_s = -\nabla \cdot \mathbf{q}_s - \boldsymbol{\pi}_s : \nabla \mathbf{V}_s + Q_s$$

It is often useful to write the energy eq in terms of the collisional entropy.

which is defined as

$$S_s \equiv \frac{1}{n_s}\int f_s \ln f_s d^3\mathbf{v} \simeq \ln\left(\frac{T_s^{3/2}}{n_s}\right) + const = \frac{3}{2}\ln\left(\frac{p_s}{n_s^{5/3}}\right) + const$$

An entropy eq can be obtained directly by using the density and energy eqs in the total time derivative of the entropy S_s for a given species of particles

$$n_s T_s \frac{dS_s}{dt} = \frac{3}{2} n_s \frac{dT_s}{dt} - T_s \frac{dn_s}{dt} = -\nabla\cdot\mathbf{q}_s - \boldsymbol{\pi}_s : \nabla\mathbf{V}_s + Q_s$$

The evolution of entropy in the moving fluid can be written in terms of the local time derivative of the entropy density $n_s S_s$ by making use of the density eq

$$n_s T_s \frac{dS_s}{dt} = T_s\left[\frac{d(n_s S_s)}{dt} - S_s\frac{dn_s}{dt}\right] = T_s\left[\frac{\partial(n_s S_s)}{\partial t} + \nabla\cdot n_s S_s \mathbf{V}_s\right]$$

Using this form for the rate of entropy increase and

$$\nabla\cdot\left(\frac{\mathbf{q}_s}{T_s}\right) = \frac{1}{T_s}\nabla\cdot\mathbf{q}_s - \mathbf{q}_s\cdot\nabla\ln T_s$$

We have

$$\frac{\partial n_s S_s}{\partial t} + \nabla\cdot\left(n_s S_s \mathbf{V}_s + \frac{\mathbf{q}_s}{T}\right) = \frac{1}{T_s}\left(\mathbf{q}_s\cdot\nabla\ln T_s + \boldsymbol{\pi}_s : \nabla\mathbf{V}_s - Q\right)$$

Explanation

1. According to the 1st eq, the species s density is constant along a fluid trajectory unless the fluid is non-solenoidal.
2. According to the 2nd eq, the species s flow accelerates along a fluid trajectory under the influence of the scalar pressure, the viscous stresses, the Lorentz force, and the frictional forces due to collisions with other species.
3. According to the 3rd eq, the species s energy density changes along a fluid trajectory because of the work done in compressing the fluid, viscous heating, heat flow, and the local energy gain due to collisions with other species. The electrical contribution to plasma heating has now become entirely implicit.

No amount of manipulation, or rearrangement, can cure our fluid equations of their most serious defect:

the fact that they are incomplete.

our equations relate interesting fluid quantities, such as the density, n_S, the flow velocity, $\mathbf{V}_S$, and the scalar pressure, p_S, to next higher moments, such as the viscosity tensor, $\boldsymbol{\pi}_S$, the heat flux density, $\mathbf{q}_S$, and the moments of the collision operator, $\mathbf{R}_S$ and Q_S.

They needs closures to use additional information to express the latter quantities in terms of former.

Several models

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Closure Moments

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When there is no significant entropy production on the time scale of interest, entropy is a "constant of the fluid motion".

Then we obtain the "adiabatic" (in the thermodynamic sense) eq of state for the species

$$\frac{dS_s}{dt} \equiv \frac{1}{\gamma - 1}\frac{d}{dt}\ln\frac{p_s}{n_s^\gamma} \simeq 0 \Leftrightarrow p_s \propto n_s^\gamma.$$

Here we have defined

$$\gamma = (N+2)/N$$

in which N is the number of degrees of freedom.

We have been treating the fully 3D case for which

$$N = 3, \gamma = \frac{5}{3}, \gamma - 1 = \frac{2}{3}.$$

Other eqs of state used are

$p_s \propto n_s,$	$T_s = const,$	isothermal eq of state $(\gamma = 1)$
$p_s \simeq 0,$	$T_s \simeq 0,$	cold species eq of state
$\nabla \cdot \mathbf{V}_s = 0,$	$n_s = const,$	incompressible species flow $(\gamma \to \infty)$

Consider electron and ion fluids separately.

Under the infinite conduction,
$\epsilon_0 \to 0$,
$\sigma = e(n_i - n_e) = 0$

$$n_i = n_e \equiv n$$

electron fluid eqs

$$\frac{d}{dt}n + n\nabla \cdot \mathbf{V}_e = 0$$

$$\boldsymbol{m_e n \frac{d}{dt}\mathbf{V}_e + \nabla p_e + \nabla \cdot \pi_e + en(\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) = \mathbf{R}}$$

$$\frac{3}{2}\frac{dp_e}{dt} + \frac{5}{2}p_e \nabla \cdot \mathbf{V}_e + \boldsymbol{\pi}_e : \nabla \mathbf{V}_e + \nabla \cdot \mathbf{q}_e = Q_e$$

ion fluid eqs

$$\frac{d}{dt}n + n\nabla \cdot \mathbf{V}_i = 0$$

$$\boldsymbol{m_i n \frac{d}{dt}\mathbf{V}_i + \nabla p_i + \nabla \cdot \pi_i - en(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) = -\mathbf{R}}$$

$$\frac{3}{2}\frac{dp_i}{dt} + \frac{5}{2}p_i \nabla \cdot \mathbf{V}_i + \boldsymbol{\pi}_i : \nabla \mathbf{V}_i + \nabla \cdot \mathbf{q}_i = Q_i$$

Definition of the one-fluid variables of MHD:

mass density (kg/m^3): $\rho = \sum_s m_s n_s = m_e n_e + m_i n_i \simeq m_i n$

the center-of-mass velocity (m/s): $\mathbf{V} = \frac{\sum_s m_s n_s \mathbf{V}_s}{\sum_s m_s n_s} = \frac{m_i n_i \mathbf{V}_i + m_e n_e \mathbf{V}_e}{\rho} \simeq \mathbf{V}_i$

current density (A/m^2): $\mathbf{J} = \sum_s n_s \mathbf{e}_s \mathbf{V}_s = -ne(\mathbf{V}_e - \mathbf{V}_i)$

plasma pressure (N/m^2): $p = \sum_s \left[p_s + \frac{n_s m_s}{3} V_s^2 \right] \simeq p_e + p_i$

stress tensor (N/m^2):
$$\begin{aligned} \boldsymbol{\pi} &= \sum_s \left[\boldsymbol{\pi}_s + n_s m_s \left((\mathbf{V}_s - \mathbf{V})(\mathbf{V}_s - \mathbf{V}) - \frac{1}{3}\mathbf{I}|\mathbf{V}_s - \mathbf{V}|^2 \right) \right] \\ &\simeq \boldsymbol{\pi}_e + \boldsymbol{\pi}_i \end{aligned}$$

A one fluid mass density eq is obtained by multiplying the electron and ion density eqs by their respective masses to yield

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

A one fluid momentum eq is obtained by simply adding the electron and ion momentum eqs and using the assumption of quasi neutrality in the plasma

$$\boldsymbol{\rho \frac{d\mathbf{V}}{dt} + \nabla p - \mathbf{J} \times \mathbf{B} + \nabla \cdot \pi = 0}$$

Next, a one fluid generalized Ohm's law is obtained by multiplying the electron and ion momentum eqs by $\frac{e_s}{m_s}$ and summing them to produce an eq for $\frac{\partial \mathbf{J}}{\partial t}$

$$\frac{m_e}{e^2}\frac{d}{dt}\left(\frac{\mathbf{J}_e}{n_e}\right) = \mathbf{E} + \mathbf{V} \times \mathbf{B} - \left(\frac{\mathbf{J}_\parallel}{\sigma_\parallel} + \frac{\mathbf{J}_\perp}{\sigma_\perp}\right) - \frac{\mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e}{n_e e}$$

Considering all the orderings and taking the limit of vanishing electron mass the generalized ohm's law is obtained

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \frac{1}{ne}\left[\mathbf{R}_e + \mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e\right] \simeq \eta \mathbf{J}$$

$\eta \equiv 1/\sigma_0 = m_e \nu_e / n_e e^2$ is the isotropic electrical resistivity.

Finally, the one fluid eq of state or energy eq is needed to close the hierarchy of MHD eqs. In MHD it is customary to use an isentropic eq of state

$$\begin{aligned}
\frac{d}{dt}\ln\frac{p}{\rho^\gamma} &= \frac{\gamma-1}{p}\left(p_e\frac{dS_e}{dt} + p_i\frac{dS_i}{dt}\right) \\
&\simeq \frac{\gamma-1}{p}\left(-\nabla \cdot \mathbf{q}_e - \nabla \mathbf{V}_i : \boldsymbol{\pi}_i + \eta J^2\right) \\
&\simeq (\gamma-1)\frac{\eta J^2}{p} \\
&\simeq 0 (\text{in resistivity / ideal MHD range})
\end{aligned}$$

the single fluid resistivity ($\eta \neq 0$) / ideal ($\eta \to 0$) eqs can be written as

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \rho \mathbf{V} = 0$$

$$\rho\left(\frac{\partial}{\partial t}\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla p + \mathbf{J}\times\mathbf{B}$$

$$\frac{d}{dt}\ln\frac{p}{\rho^\gamma} = (\gamma-1)\frac{\eta J^2}{p}$$

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla\times(\mathbf{V}\times\mathbf{B} - \boldsymbol{\eta}\mathbf{J})$$

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J}$$

The most important effects neglected by the ideal MHD eqs are

1. heat conductivity parallel to the magnetic field
2. viscosity
3. resistivity

   Although resistivity is numerically a small diffusion effect, it has the important consequence of allowing magnetic field topology to break.

With resistivity, the magnetic field line tends to break up into a number of thin filaments called magnetic islands which thread their way through the plasma.

Since heat flows rapidly along field lines, one of the direct effects of this island structure is to enhance transport across the plasma.

After some manipulations, it can be shown that the ideal MHD eqs yield the following conservation forms of total MHD system mass, momentum and energy density

$$\text{MHD system mass eq:} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

$$\text{MHD system momentum eq:} \quad \frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \cdot \mathbf{T} = 0$$

$$\text{MHD system energy eq:} \quad \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S} = 0$$

in which

$$\text{MHD stress tensor:} \quad \mathbf{T} \equiv \rho \mathbf{V}\mathbf{V} + \left(p + \frac{B^2}{2\mu_0} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0}$$

$$\text{MHD energy density:} \quad w \equiv \frac{\rho V^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}$$

$$\text{MHD energy flux:} \quad \mathbf{S} \equiv \left(\frac{\rho V^2}{2} + \frac{\gamma}{\gamma - 1} p \right) \mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$$

Steepest-descent moment method for 3D MHD equilibria

Take the steady state solution,

$$\frac{\partial}{\partial t} = 0$$

we get ideal MHD equilibrium eqs

$$\nabla p = \mathbf{J} \times \mathbf{B}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\nabla \cdot \mathbf{B} = 0$$

Currently there are several codes available to calculate 2D, 3D magnetic equilibria, or equilibria with magnetic island.

Equilibrium Solver:
Solve Ψ as a function of (R, Z) through iterating GS eq, such as TSC.

Inverse Equilibrium Solver:
Solve R, Z as functions of (ψ, θ), Such as Jsolver

The force density can be written as

$$\begin{aligned} f &= -\nabla p + \mathbf{J} \times \mathbf{B} \\ &= -\nabla p - \frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) \\ &= -\nabla p - \frac{B}{\mu_0}\mathbf{b} \times (\nabla \times B\mathbf{b}) \\ &= -\nabla p - \frac{B}{\mu_0}\mathbf{b} \times (\nabla B \times \mathbf{b}) - \frac{B^2}{\mu_0}\mathbf{b} \times (\nabla \times \mathbf{b}) \\ &= \underbrace{-\nabla p}_{\text{fluid pressure}} - \underbrace{\boldsymbol{\nabla}_\perp \left(\frac{B^2}{2\mu_0}\right)}_{\text{magnetic pressure}} + \underbrace{\frac{B^2}{\mu_0}\boldsymbol{\kappa}}_{\text{parallel tension}} \end{aligned}$$

Consider Frenet Coordinate System

$$\mathbf{b} = \frac{\mathbf{B}}{B} \qquad \text{magnetic field direction}$$
$$\mathbf{n} = \frac{\boldsymbol{\kappa}}{\kappa} \qquad \text{magnetic curvature direction}$$
$$\mathbf{z} = \mathbf{b} \times \mathbf{n} \qquad \text{binormal direction}$$

Then the force density can be re-written as

$$\boldsymbol{f} = -\mathbf{b}(\mathbf{b}\cdot\boldsymbol{\nabla}p) - \mathbf{n}\left[(\mathbf{n}\cdot\boldsymbol{\nabla})\left(p + \frac{B^2}{2\mu_0}\right) - \frac{B^2}{\mu_0}\boldsymbol{\kappa}\right] - \mathbf{z}(\mathbf{z}\cdot\boldsymbol{\nabla})\left(p + \frac{B^2}{2\mu_0}\right)$$

The conditions for MHD force balance equilibrium are thus

$$\text{along } \mathbf{B}: \qquad 0 = \mathbf{b}\cdot\nabla p = \frac{\partial p}{\partial l}$$
$$\text{curvature direction}: \quad \mathbf{0} = \mathbf{n}\cdot\boldsymbol{\nabla}\left(p + \frac{B^2}{2\mu_0}\right) - \frac{B^2}{\mu_0}\boldsymbol{\kappa}$$
$$\text{binormal direction}: \quad 0 = \mathbf{z}\cdot\nabla\left(p + \frac{B^2}{2\mu_0}\right)$$

Since there is no magnetic force along the magnetic filed, p must be const along magnetic field lines.

When nested magnetic flux surfaces exist, $\frac{\partial p}{\partial l}$ requires the pressure be a function of the magnetic flux

$$\mathbf{B}\cdot\nabla p = (\mathbf{B}\cdot\psi)\frac{dp}{d\psi} = 0$$

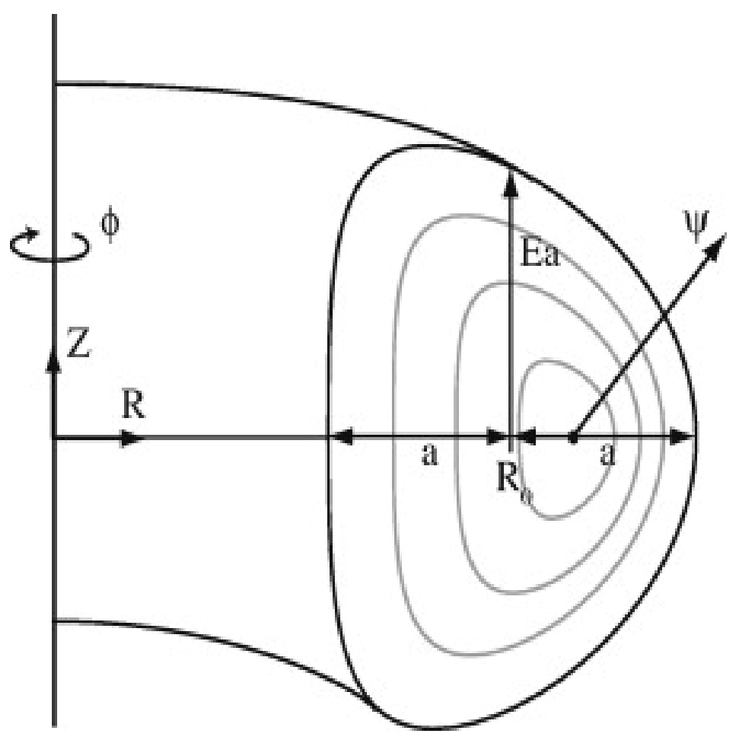
together with

$$\mathbf{J}\cdot\nabla p = \mathbf{J}\cdot\mathbf{J}\times\mathbf{B} = 0$$

we see that the vector fields **J** and **B** both lie within, and do not penetrate, magnetic flux surfaces.

Further, from $\mathbf{J}\times\mathbf{B} = \nabla p$, we know that the pressure gradient is perp to the flux surfaces.

I. Introduction
The global analysis of finite aspect ratio, high beta, 3D toroidal configurations.



$$\epsilon = \frac{R}{a}$$

The ratio of the major radius to the minor radius of toroidal plasma

low: $\epsilon \sim 1$

finite: $\epsilon \sim 3$

large: $\epsilon \geq 8$

Substitute **J** from Ampere's law into the force balance eq

$$\begin{aligned}\nabla p &= (\nabla \times \mathbf{B}) \times \mathbf{B}/\mu_0 \\ &= \left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla(B^2/2)\right]/\mu_0\end{aligned}$$

This may be rewritten as

$$\nabla(p + B^2/2\mu_0) = (\mathbf{B}\cdot\nabla)\mathbf{B}/\mu_0$$

term on the left: indicate that the magnetic field may be considered to have a "magnetic pressure" given by $B^2/2\mu_0$.

term on the right: comes form bending and parallel compression of the field, producing perpendicular and parallel forces, respectively.

The ratio of the plasma pressure to the magnetic field pressure

$$\beta = \frac{p}{B^2/2\mu_0} = \frac{\text{thermal energy density}}{\text{magnetic energy density}}$$

is the measure of confinement efficiency with which the magnetic field confines the plasma. In other words, how well does the magnetic field hold a non-uniform plasma in equilibrium.

High β is desirable for economic power balance

but is difficult to achieve because of various plasma instabilities.

In low-β plasma, the force balance is mainly a matter of different magnetic force in balance with each other.

At high-β, the magnetic field plays a minor role in the dynamics of the plasma.

Consider the single particle motion first.

A major goal is to provide an understanding of how magnetic fields confine charged particles in a fusion plasma.

Apply Newton's law in plasma physics

$$m\frac{d}{dt}\mathbf{V} = e(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \text{ (Lorentz force)}$$

$$m\frac{d}{dt}\mathbf{x} = \mathbf{V}$$

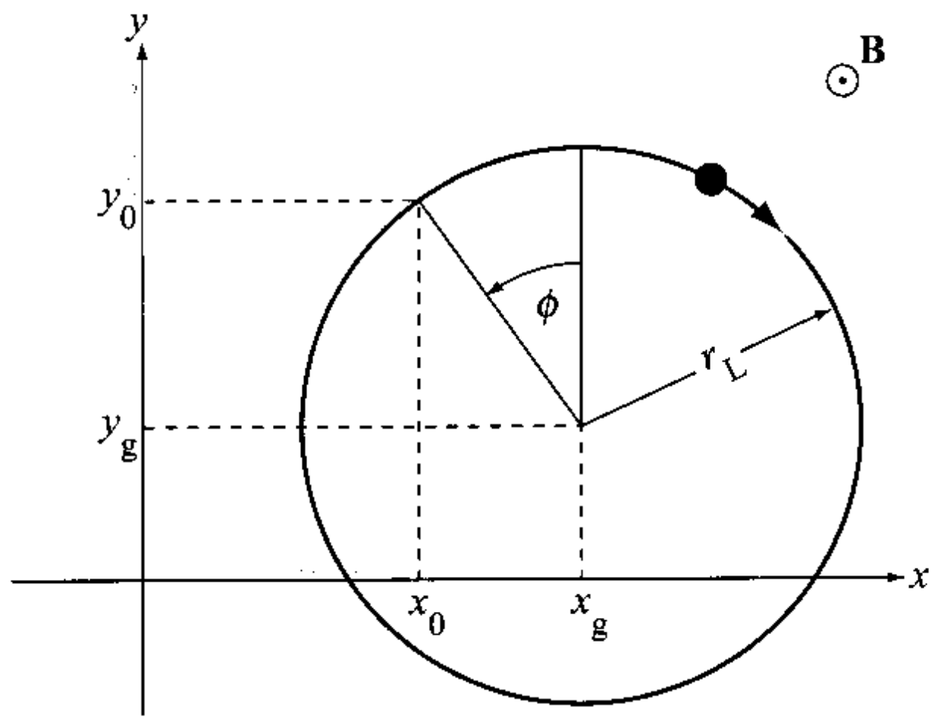
In general,

$$\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$$
$$\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$$

So this is a set of coupled nonlinear ODE.

Assuming a uniform, time independent, magnetic field, no electric field

$$\mathbf{B} = B\hat{\mathbf{z}}, B = const$$
$$\mathbf{E} = 0$$


y
B
y_0
ϕ
r_L
y_g
x
x_0
x_g

In component form, the full set of Newton's Law reduces to

$$m\frac{d}{dt}V_x = \frac{eB}{m}V_y, \qquad V_x(0) = V_{x0} = V_\perp \cos\theta$$

$$m\frac{d}{dt}V_y = -\frac{eB}{m}V_x, \qquad V_y(0) = V_{y0} = V_\perp \cos\theta$$

$$m\frac{d}{dt}V_z = 0 \qquad V_z(0) = V_{z0} = V_\parallel$$

$$\frac{d}{dt}x = V_x \qquad x(0) = x_0$$

$$\frac{d}{dt}y = V_y \qquad y(0) = y_0$$

$$\frac{d}{dt}z = V_z \qquad z(0) = z_0$$

Here

$$\omega_c = \frac{eB}{m}$$

is the gyro frequency.

$V_\perp$, $V_\parallel$ are constants representing the initial velocity.

x_0, y_0, z_0, θ, are constants representing the initial position.

1. parallel motion

$$m\frac{d}{dt}V_z = 0$$
$$\frac{d}{dt}z = V_z$$

The solutions is

$$V_z(t) = V_{||}$$
$$z(t) = z_0 + V_{||}t$$

There are no parallel force providing confinement and

particles simply proceed unimpeded.

The motion is therefore unconfined along a given magnetic line.

2. Perpendicular motion

$$m\frac{d}{dt}V_x = \frac{eB}{m}V_y, \qquad V_x(0) = V_{x0} = V_\perp \cos\theta$$

$$m\frac{d}{dt}V_y = -\frac{eB}{m}V_x, \quad V_y(0) = V_{y0} = V_\perp \cos\theta$$

Eliminating V_x gives

$$\frac{d^2}{dt^2}V_y + \omega_c^2 V_y = 0,$$

$$V_y(0) = V_\perp \sin\theta$$

$$\frac{d}{dt}V_y(0) = -\omega_c V_x(0) = -\omega_c V_\perp \cos\theta$$

It's general solution is easily found to be

$$V_y(t) = -V_\perp \sin(\omega_c t - \theta)$$

$$V_x(t) = V_\perp \cos(\omega_c t - \theta)$$

The particles rotate with an angular frequency equal to the gyro frequency.

The solution for the perpendicular motion is completed by integrating the velocity

$$\frac{d}{dt}x = V_x \quad x(0) = x_0$$
$$\frac{d}{dt}y = V_y \quad y(0) = y_0$$

yielding the particle trajectory

$$x(t) - x(0) = \int_0^t V_x(t')dt'$$
$$y(t) - y(0) = \int_0^t V_y(t')dt'$$

i.e.,

$$x(t) = x_g + r_L \sin(\omega_c t - \theta)$$
$$y(t) = y_g + r_L \cos(\omega_c t - \theta)$$

Here

$$r_L = \frac{V_\perp}{\omega_c} = \frac{mV_\perp}{eB}$$

is the gyro radius. the quantities x_g, y_g are defined as

$$x_g = x_0 + r_L \sin\theta$$
$$y_g = y_0 - r_L \cos\theta$$

So

$$(x - x_g)^2 + (y - y_g)^2 = r_L^2$$

is the orbit of the particle, which is circular.

Since the gyro radius is quite small in comparison to the minor radius a,

one can conclude that there is a good confinement perpendicular to the magnetic field.

3. Rotation direction of gyro motion

   Because the electrons and ions have opposite sign charges,

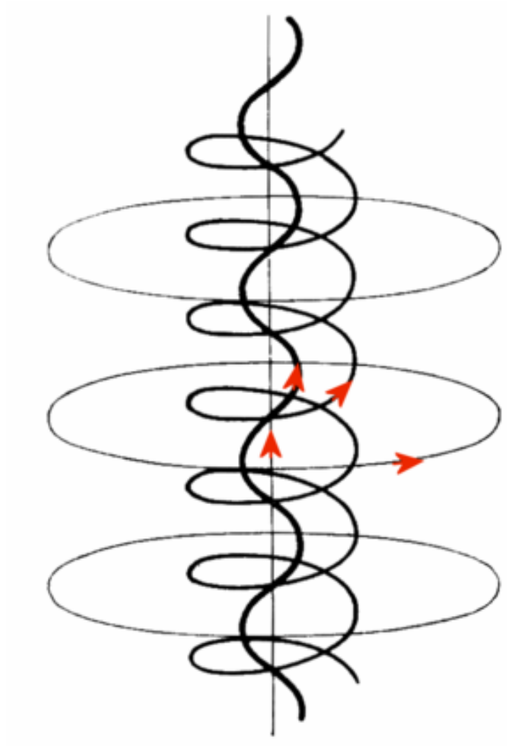
   they rotate in opposite directions.

   The magnetic field generated by the electric current of a gyrating particle always opposes the magnetic filed,

   i.e., the gyromotion is diamagnetic.

4. helical trajectory

   The combined perpendicular and parallel motions corresponds to a helical trajectory.



Particle spiral unimpeded along field lines with a small perpendicular excursion equal to the gyro radius.

This has important implications for the geometry of a magnetic fusion reactors: it must be a toroidal.

Otherwise, In a finite length linear geometry,

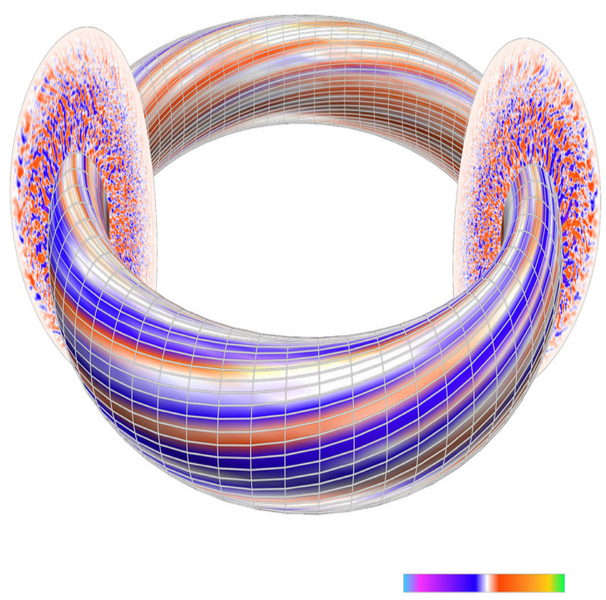
all magnetic field lines mush eventually make contact with wall as they leave the system.

The particles therefore free stream along the filed lines directory colliding with the wall in a very short time, which is an end loss.

In toroidal geometry, the field lines close upon themselves,

particles continuously spiral along the field lines and do not leave the chamber,

therefore, confined in both the parallel and perpendicular directions.



The variational formulation of MHD equilibria provides a mathematically efficient prescription for treating the truncation or closure of an approximate finite series solution of the nonlinear equilibrium eqs.

There are 2 variational formulations:

The first variation: equilibrium

The second variation: linear stability

The variational formulation of MHD equilibria provides a mathematically efficient prescription for treating the truncation or closure of an approximate finite series solution of the nonlinear equilibrium eqs.

Complex Fourier series

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

Assuming $f(x)$ is a periodic function with $T = 2\pi$,
the Fourier coefficients are then given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{-in\xi} d\xi$$

The Fourier coefficients a_n, b_n, c_n are related via

$$\begin{aligned} a_n &= c_n + c_{-n} \qquad n = 0, 1, 2, \cdots \\ b_n &= i(c_n - c_{-n}) \quad n = 1, 2, \cdots \end{aligned}$$

and

$$c_n = \begin{cases} \frac{a_n - ib_n}{2} & n > 0 \\ \frac{a_0}{2} & n = 0 \\ \frac{a_{-n} + ib_{-n}}{2} & n < 0 \end{cases}$$

Also inherent in any energy principle is an iteration scheme for obtaining the solution of this truncated set of eqs, which is based on seeking the minimum energy state.

It is often desirable to have a variational statement of a problem.

One value of such a form is that it can be cast in new variables and the principle will still hold.

The first variation determines the equilibrium: The form of w will depend on the constraints imposed during the variation.

To this end, we seek an energy functional w such that the condition

that the variation of w vanish is equivalent to the plasma equilibrium eq.

It would suffice to find a function w such that its variational is given by

$$\delta w = \int \boldsymbol{\xi} \cdot [\nabla p - \mathbf{J} \times \mathbf{B}] d^3\mathbf{x}$$

for arbitrary displacement field $\boldsymbol{\xi}(\mathbf{x})$.

If it vanishes for all $\boldsymbol{\xi}$, then the equilibrium eq is satisfied everywhere.

The second variation determines the linear stability of the system.

The first variation

Since the displacement field $\boldsymbol{\xi}(\mathbf{x})$ is the time integral of the velocity field $\mathbf{V}$,

we can obtain the ideal MHD constraints: the flux and entropy are conserved.

This the most physical of possible constraints,

Thus we examine the functional

$$w = \int \left(\frac{B^2}{2\mu_0} + \frac{p}{\gamma - 1} \right) d^3\mathbf{x}$$

subject to magnetic conservation

$$\boldsymbol{\delta}\mathbf{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B})$$

and entropy conservation

$$\boldsymbol{\delta p} = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla p} - \boldsymbol{\gamma p} \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = (\boldsymbol{\gamma} - \mathbf{1})\boldsymbol{\xi} \cdot \boldsymbol{\nabla p} - \boldsymbol{\gamma} \boldsymbol{\nabla} \cdot \boldsymbol{p\xi}$$

The 1st variation of w gives

$$\delta w = \int \left(\frac{1}{\mu_0} \mathbf{B} \cdot \delta \mathbf{B} + \frac{\delta p}{\gamma - 1} \right) d^3\mathbf{x}$$

using the constraints, it becomes

$$\delta w = \int \boldsymbol{\xi} \cdot (\nabla p - \mathbf{J} \times \mathbf{B})\, d^3\mathbf{x} - \int d\mathbf{S} \cdot \left\{ \frac{1}{\mu_0} \boldsymbol{\xi}_\perp B^2 + \frac{\gamma}{\gamma - 1} p \boldsymbol{\xi} \right\}$$

if we have the additional condition that

$$d\mathbf{S} \cdot \boldsymbol{\xi} = 0$$

on the boundary, which also follows from flux and entropy conservations,

then this is of the desired form of equilibrium eq.

This form is valid in up to 3D space $\mathbf{x} = (x, y, z)$.

VMEC Code: 3D MHD Equilibrium

Solve inverse equilibrium equation

Assuming the existence of nested flux surfaces and

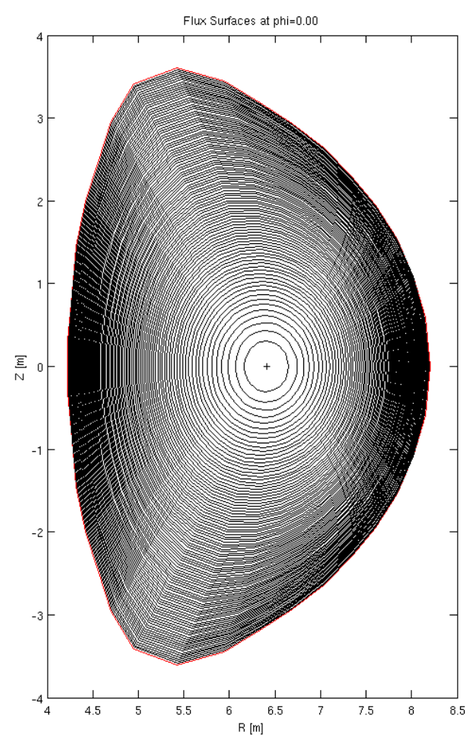
that quantities may be Fourier expanded in terms of the poloidal and toroidal coordinates,

the code uses a variational method to find a minimum in the total energy of the system.

This produces a parabolic set of equations (with second-order spatial derivatives in the radial coordinate, ψ),

which are converted to hyperbolic form through a Richardson scheme.

The VMEC code is able to rapidly solve for MHD equilibrium 3D configurations.



ITER fixed boundary flux surface

VMEC Input Namelist

```
&INDATA
 LFREEB = F
 MGRID_FILE = "none"
 DELT = 0.9
 NFP = 7
 NCURR = 0
 MPOL = 9
 NTOR = 6
 NZETA = 18
 NITER = 10000
 NSTEP = 200
 NVACSKIP = 6
 GAMMA = 0.000000E+00
 PHIEDGE = 1.13E-01
 CURTOR = 0.0
 NS_ARRAY =     9      49
 FTOL_ARRAY = 1.00e-6   1.e-11
 AM =   1.e4    -1.e4    9*0.
 AI =    6.90    0.0    0.0    3.90   0.0    0.0
         6.90    0.0    0.0    6.90   0.0
 AC =   11*0.
 RAXIS =   2.88    0.04
 ZAXIS =   0.00   -0.04
 RBC(0,0) =   2.90     ZBS(0,0) =   0.00
 RBC(0,1) =   1.00     ZBS(0,1) =   1.00
 RBC(1,1) =  -0.21     ZBS(1,1) =   0.21
 RBC(1,4) =  -0.01     ZBS(1,4) =   0.01
 RBC(1,6) =  -0.01     ZBS(1,6) =  -0.01
/
```

VMEC Output Data Format

The VMEC code outputs some runtime and diagnostic data to the screen

along with the creation of four files

(jxbout, mercier, threed1, and wout)

The data output to the screen is also indicated (in greater detail) in the threed1 file.

It is suggested that
the user redirect the output of the run to a log file.

The wout file is a text file containing data

from plotting of the final configuration.

Details of reading the 'wout' file can be found in

'LIB_OPT/vsource/readw_only_priv.f.'

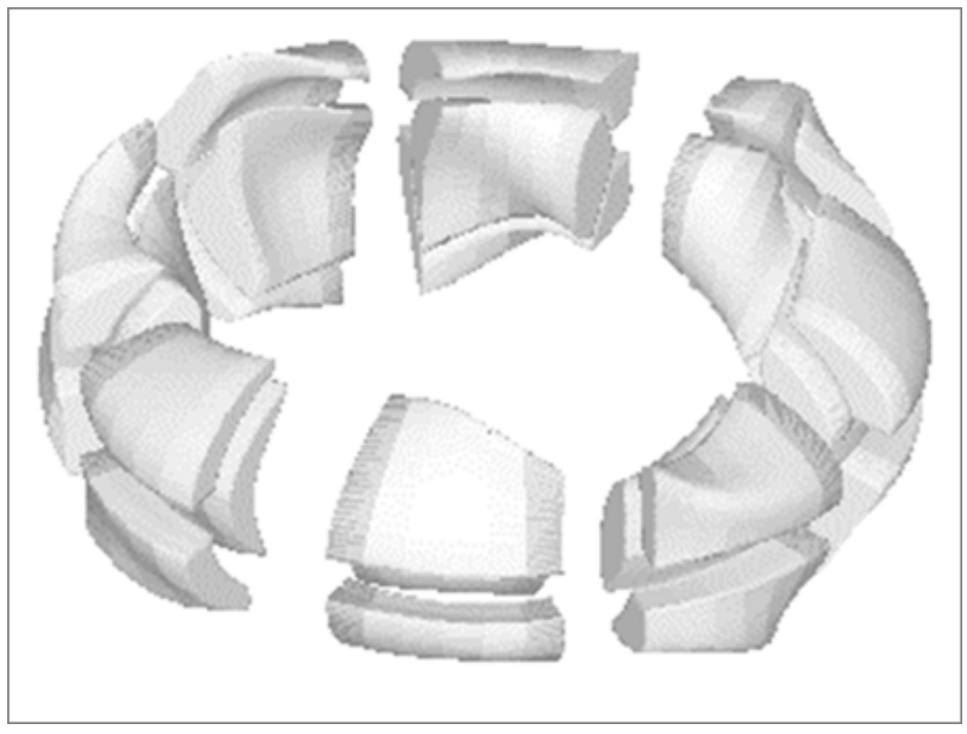
If 'LDIAGNO' was set to true in the input namelist,
a 'diagno_in' file will be created.
This file contains information regarding the outer flux surface
and currents for the DIAGNO routine
(which calculate the field at a point for magnetic reconstruction).

Array to Read VMEC Output Data

```
iNumFieldPeriod,
iNumRadialGrids,
iMaxPoloidalModeNumber,
iMaxToroidalModeNumber,
iAccumModeNumber

  iPoloidalMode = iVector1D(0,iAccumModeNumber-1);
  iToroidalMode = iVector1D(0,iAccumModeNumber-1);
  RMNC     = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  RMNS     = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  ZMNC     = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  ZMNS     = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  lmn      = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  bmn      = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  gmn      = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  bsubumn  = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  bsubvmn  = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  bsubsmn  = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  bsupumn  = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  bsupvmc  = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  bsupvms  = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  currvmc  = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  currvms  = d2D(0,iNumRadialGrids-1,0,iAccumModeNumber-1);
  iota     = d1D(0,iNumRadialGrids-1);
  mass     = d1D(0,iNumRadialGrids-1);
  Pressure = d1D(0,iNumRadialGrids-1);
  dPsids   = d1D(0,iNumRadialGrids-1);
```

```
wb1, wp, gam, pfac, iNumFieldPeriod, iNumRadialGrids,
iMaxPoloidalModeNumber, iMaxToroidalModeNumber,
iAccumModeNumber, ifsq, nit, iasym, ireconstruct

imse2, itse, nbsets, nobd, nextcur, nbfld

iPoloidalMode[:], iToroidalMode[:]

RMNC[ir][:], ZMNS[:][:], lmn[:][:]

bmn[:][:], gmn[:][:], bsubumn[:][:],
bsubvmn[:][:], bsubsmn[:][:], bsupumn[:][:]
bsupvmc[:][:], currvmc[:][:]

iota[:], mass[:], Pressure[:] dPsids[:],
buco, bvco, phijs, vol, overr, curu, curv, specw
```

```
iNumGridsPhi;
iNumCPUPhi;
iNumRadialGrids;
iNumRadialCPU;
iPoloidalSymmetry;
iNumCPUTheta;
```



Find the major radius, magnetic axis, minor radius

```
Rmajor = RMNC[0][j];

a=0.0; BphiAxis = 0.0;
i=0:iMaxTorModeNumber-1
  phi = 2.0*PI*i/((iMaxTorModeNumber+1)*iNumFieldPeriod)
  theta = 0.0;
  Raxis = 0.0; Zaxis = 0.0; Bphi = 0.0;
    Raxis += RMNC[0][:] * cos(iPolMode[:]*theta - iTorMode[:]*phi)+
             RMNS[0][:] * sin(iPolMode[:]*theta - iTorMode[:]*phi);
    Zaxis += ZMNC[0][:] * cos(iPolMode[:]*theta - iTorMode[:]*phi)+
             ZMNS[0][:] * sin(iPolMode[:]*theta - iTorMode[:]*phi);
    Bphi  += bsupvmc[0][:] * cos(iPolMode[:]*theta - iTorMode[:]*phi)+
             bsupvms[0][:] * sin(iPolMode[:]*theta - iTorMode[:]*phi);

    Bphi *= Raxis; /* this is the toroidal field */
    BphiAxis += Bphi/(1+iMaxTorModeNumber);

j=0:iMaxPolModeNumber-1
  theta = 2.0*PI*j/iMaxPolModeNumber;
  R = 0.0; Z = 0.0;
  R += RMNC[iNumRGrids-1][:]* cos(iPolMode[:]*theta - iTorMode[:]*phi)+
       RMNS[iNumRGrids-1][:]* sin(iPolMode[:]*theta - iTorMode[:]*phi);
  Z += ZMNC[iNumRGrids-1][:]* cos(iPolMode[:]*theta - iTorMode[:]*phi)+
       ZMNS[iNumRGrids-1][:]* sin(iPolMode[:]*theta - iTorMode[:]*phi);

  a += sqrt((R-Raxis)*(R-Raxis) + (Z-Zaxis)*(Z-Zaxis));

a /= (iMaxTorModeNumber+1);
a /= iMaxPolModeNumber;
AspectRatio = Rmajor/a;
```

Interpolate VMEC DATA to USER CONFIGURATION

```
sgrid[:] = pow(l2p2((double)i, &thispack, 'F'), 2.0);
vmecfit(sgrid, iNumRGrids, RMNC, iNumRGrids,
        iAccumModeNumber, iPolMode, RMNC, IntegerGrid);
vmecfit(sgrid, iNumRGrids, RMNS, iNumRGrids,
        iAccumModeNumber, iPolMode, RMNS, IntegerGrid);
vmecfit(sgrid, iNumRGrids, ZMNC, iNumRGrids,
        iAccumModeNumber, iPolMode, ZMNC, IntegerGrid);
vmecfit(sgrid, iNumRGrids, ZMNS, iNumRGrids,
        iAccumModeNumber, iPolMode, ZMNS, IntegerGrid);
vmecfit(sgrid, iNumRGrids, bsupvmc, iNumRGrids,
        iAccumModeNumber, iPolMode, bsupvmc, HalfGrid);
vmecfit(sgrid, iNumRGrids, bsupvms, iNumRGrids,
        iAccumModeNumber, iPolMode, bsupvms, HalfGrid);
vmecfit(sgrid, iNumRGrids, currvmc, iNumRGrids,
        iAccumModeNumber, iPolMode, currvmc, IntegerGrid);
vmecfit(sgrid, iNumRGrids, currvms, iNumRGrids,
        iAccumModeNumber, iPolMode, currvms, IntegerGrid);
vmecfit0(sgrid, iNumRGrids, Pressure, iNumRGrids,
         Pressure, HalfGrid);
```

Map on Real Space (R, Z)

```
  j=0:NumPhiSections-1
    phi

    i=0:NumLocalVertices-1
      mapIndexToVmecCoordinates(ir, theta)
      vmecpoint2(ir, theta, phi, r[i], z[i])

sine = sin(Data->iPolMode[:]*theta - Data->iTorMode[:]*phi);
cosine = cos(Data->iPolMode[:]*theta - Data->iTorMode[:]*phi);

R += Data->RMNC[ilocation][:]*cosine +
     Data->RMNS[ilocation][:]*sine;
Z += Data->ZMNC[ilocation][:]*cosine +
     Data->ZMNS[ilocation][:]*sine;
```

Map on Real Space (R, Z)

```
j=0:map.NumPhiSections-1
  phi

  i=0:map.NumLocalVertices-1
    mapIndexToVmecCoordinates(PolSymmetry, ir, theta)
    vmecBfield(ir, theta, phi,
               bsupvmn[i],
               currvmn[i],
               pressure[i],
               density[i],
               chi[i])
    cos = cos((Data->iPolMode[:]*theta -
              Data->iTorMode[:]*phi))
    sin = sin((Data->iPolMode[:]*theta -
              Data->iTorMode[:]*phi))

    bsupv += Data->bsupvmc[ilocation][:]*cos +
             Data->bsupvms[ilocation][:]*sin
    currv += Data->currvmc[ilocation][:]*cos +
             Data->currvms[ilocation][:]*sin

    pressure = Data->Pressure[ilocation];
```

Initialization and Perturbation Addition

`zeroFlowField`

`addPerturbation`