First let's visit the coordinate vectors in curved coordinate systems.

reciprocal sets of vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\mathbf{A} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{b} = \mathbf{C} \cdot \mathbf{c} = 1$$

$$\mathbf{A} \cdot \mathbf{b} = \mathbf{A} \cdot \mathbf{c} = \mathbf{B} \cdot \mathbf{a} = \mathbf{B} \cdot \mathbf{c} = \mathbf{C} \cdot \mathbf{a} = \mathbf{C} \cdot \mathbf{b} = 0$$

nonzero triple products $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ and $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

Find vectors of one set $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in terms of its reciprocal $\mathbf{A}, \mathbf{B}, \mathbf{C}$:

$$\mathbf{a} = \frac{\mathbf{B} \times \mathbf{C}}{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})}, \mathbf{b} = \frac{\mathbf{C} \times \mathbf{A}}{\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})}, \mathbf{c} = \frac{\mathbf{A} \times \mathbf{B}}{\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})},$$

any vector $\mathbf{W}$ can be written as

$$\mathbf{W} = (\mathbf{W} \cdot \mathbf{a})\mathbf{A} + (\mathbf{W} \cdot \mathbf{b})\mathbf{B} + (\mathbf{W} \cdot \mathbf{c})\mathbf{C}$$

or

$$\mathbf{W} = (\mathbf{W} \cdot \mathbf{A})\mathbf{a} + (\mathbf{W} \cdot \mathbf{B})\mathbf{b} + (\mathbf{W} \cdot \mathbf{C})\mathbf{c}$$

The parallel component of a vector **V** w.r.t. vector **B**

$$V_{\|} = \frac{\mathbf{V}\cdot\mathbf{B}}{B} = \mathbf{V}\cdot\mathbf{b}$$

b being the unit vector along **B**.

The perpendicular component of vector **V** wrt **B**

$$\mathbf{V}_{\perp} = -\mathbf{b}\times(\mathbf{b}\times\mathbf{V})$$

Thus

$$\begin{aligned}\mathbf{V} &= V_{\|}\mathbf{b} + \mathbf{V}_{\perp}\\ &= \mathbf{b}(\mathbf{b}\cdot\mathbf{V}) - \mathbf{b}\times(\mathbf{b}\times\mathbf{V})\end{aligned}$$

Next, we build the unit coordinate vectors in curvilinear coordinates

3D position vector can be parametrized by u^1, u^2, u^3:

$$\mathbf{R}(u^1, u^2, u^3) : \begin{cases} x = x(u^1, u^2, u^3) \\ y = y(u^1, u^2, u^3) \\ z = z(u^1, u^2, u^3) \end{cases}$$

one-to-one transformation, invertible

$$\begin{cases} u^1 = u^1(x, y, z) \\ u^2 = u^2(x, y, z) \\ u^3 = u^3(x, y, z) \end{cases}$$

if the determinant of transformation is nonzero

$$\begin{pmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^2} & \frac{\partial x}{\partial u^3} \\ \frac{\partial y}{\partial u^1} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^3} \\ \frac{\partial z}{\partial u^1} & \frac{\partial z}{\partial u^2} & \frac{\partial z}{\partial u^3} \end{pmatrix}$$

$\Longrightarrow u^1, u^2, u^3$ are coordinate, curvilinear.

Then we define the coordinate surfaces and coordinate curves

eqs of coordinate surfaces are

$$\begin{cases} u^1 = c^1 & (u^2, u^3 \text{ vary}) \\ u^2 = c^2 & (u^3, u^1 \text{ vary}) \\ u^3 = c^3 & (u^1, u^2 \text{ vary}) \end{cases}$$

c^1, c^2, c^3 are arbitrary constant.

eqs of coordinate curves are

$$\begin{cases} u^2 = c^2, \quad u^3 = c^3 & (u^1 \text{ vary}) \\ u^3 = c^3, \quad u^1 = c^1 & (u^2 \text{ vary}) \\ u^1 = c^1, \quad u^2 = c^2 & (u^3 \text{ vary}) \end{cases}$$

II. equilibrium eq in flux coordinates

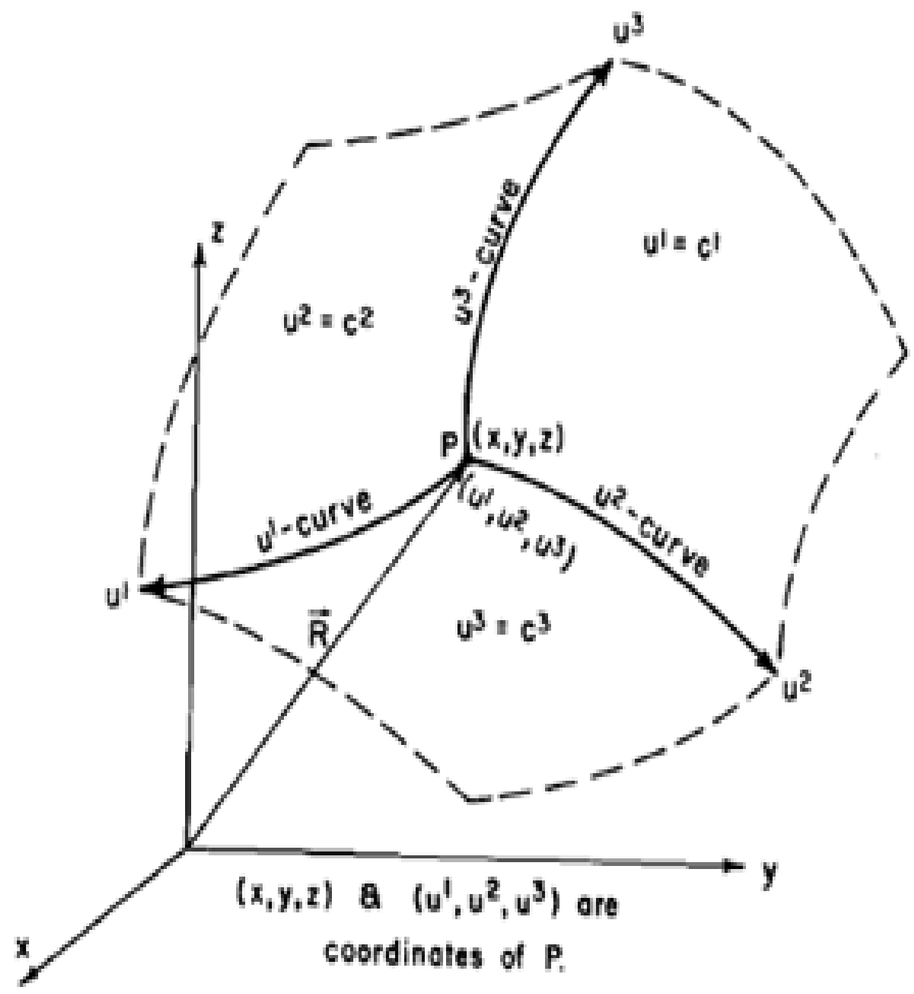


Fig. 2.1. General curvilinear coordinate system with coordinates (u^1, u^2, u^3). The coordinate curves and coordinate surfaces are represented with respect to a Cartesian system with coordinates (x, y, z)

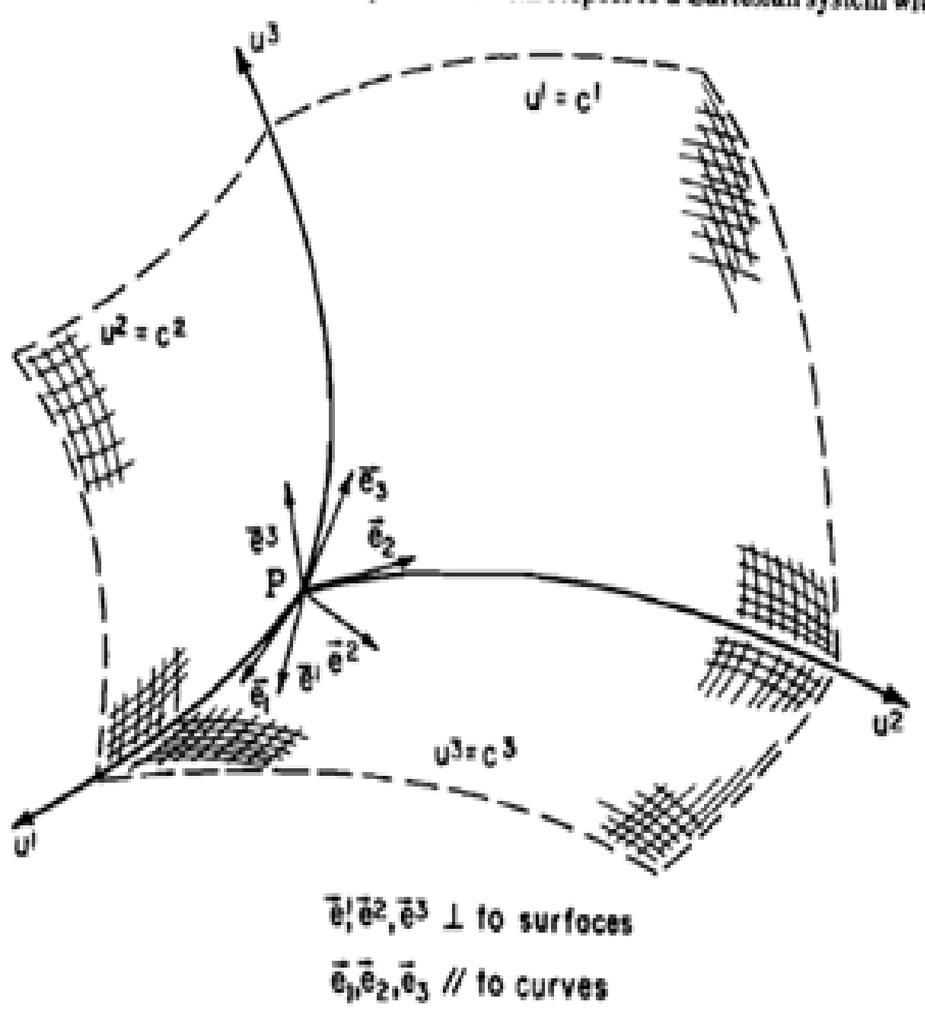


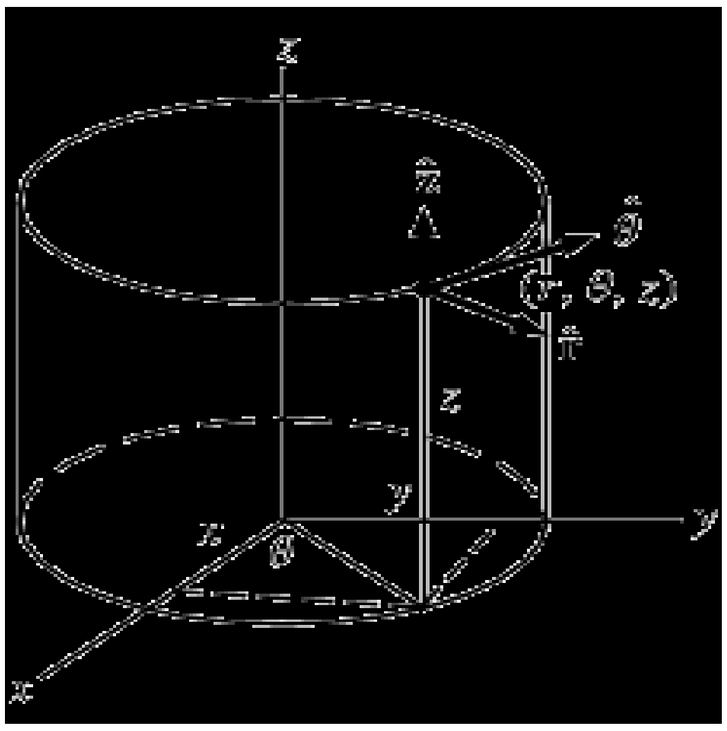
Fig. 2.2. Basis vectors e^i and e_j at point P. The contravariant basis vectors $e^i = \nabla u^i$ are perpendicular to the constant coordinate surfaces; the covariant basis vectors $e_j = \partial R/\partial u^j$ are parallel to the coordinate curves

II. equilibrium eq in flux coordinates

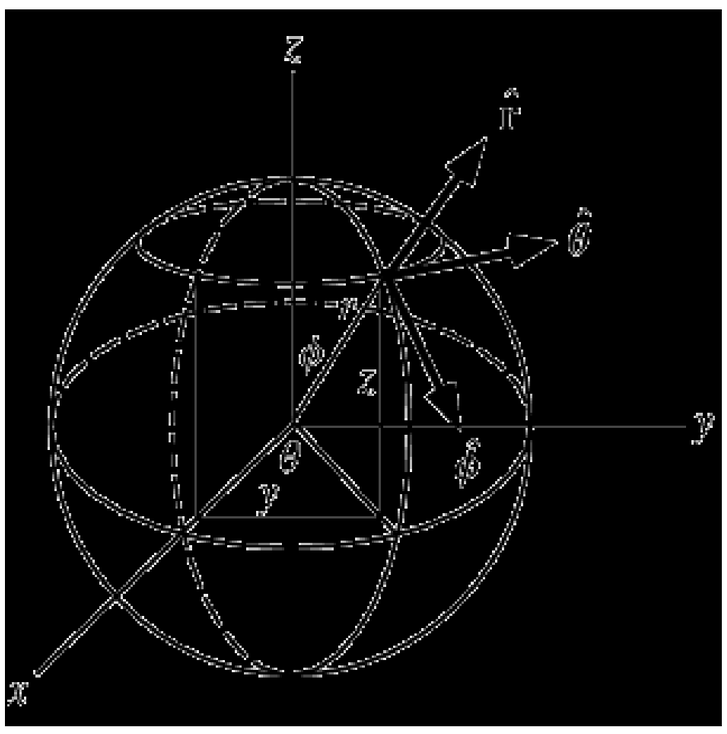
orthogonal: if the coordinate curves intersect at right angle

examples

cylindrical



spherical systems.



Then we construct the tangent-Basis vectors

define a set of vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of fixed length,

pointing in the positive direction of the coordinate curves u^1, u^2, u^3

$$\mathbf{e}_1 = \frac{\partial \mathbf{R}}{\partial u^1}; \mathbf{e}_2 = \frac{\partial \mathbf{R}}{\partial u^2}; \mathbf{e}_3 = \frac{\partial \mathbf{R}}{\partial u^3};$$

$\Longrightarrow$ define the tangent vectors at point **P**.

- linearly independent;
- triple product $\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 \neq 0$
- local, varies form point to point

infinitesimal position vector **R**

$$d\mathbf{R} = \sum_{j=1}^{3} \frac{\partial \mathbf{R}}{\partial u^j} du^j = \mathbf{e}_j du^j$$

Reciprocal-Basis Vectors

Take coordinate differential

$$du^i = \nabla u^i \cdot d\mathbf{R} = \nabla u^i \cdot \mathbf{e}_j du^j$$

which can only hold if, and only if

$$\nabla u^i \cdot \mathbf{e}_j = \delta^i_j \text{ (Kronecker delta)}$$

$\Longrightarrow \nabla u^i$ and $\mathbf{e}_j$ form reciprocal sets of vectors

define the reciprocal-basis vectors

$$\mathbf{e}^i = \nabla u^i$$

Obviously

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$$

Summary:

$\mathbf{e}_j$: tangent to the u^j coordinate curve

$\mathbf{e}^i$: perpendicular to the $u^i = c^i$ coordinate surface

Covariant components of a vector

using subscript $\equiv ()_i$

$$
\begin{aligned}
\mathbf{D} &= \left(\mathbf{D}\cdot\frac{\partial\mathbf{R}}{\partial u^1}\right)\nabla u^1 + \left(\mathbf{D}\cdot\frac{\partial\mathbf{R}}{\partial u^2}\right)\nabla u^2 + \left(\mathbf{D}\cdot\frac{\partial\mathbf{R}}{\partial u^3}\right)\nabla u^3 \\
&= (\mathbf{D}\cdot\mathbf{e}_1)\,\mathbf{e}^1 + (\mathbf{D}\cdot\mathbf{e}_2)\,\mathbf{e}^2 + (\mathbf{D}\cdot\mathbf{e}_3)\,\mathbf{e}^3 \\
&= D_1\mathbf{e}^1 + D_2\mathbf{e}^2 + D_3\mathbf{e}^3 \\
&= D_i\mathbf{e}^i \text{ with } D_i \equiv \mathbf{D}\cdot\mathbf{e}_i
\end{aligned}
$$

Contravariant components of a vector

using superscript $\equiv ()^i$

$$
\begin{aligned}
\mathbf{D} &= \left(\mathbf{D}\cdot\nabla u^1\right)\frac{\partial\mathbf{R}}{\partial u^1} + \left(\mathbf{D}\cdot\nabla u^2\right)\frac{\partial\mathbf{R}}{\partial u^2} + \left(\mathbf{D}\cdot\nabla u^3\right)\frac{\partial\mathbf{R}}{\partial u^3} \\
&= \left(\mathbf{D}\cdot\mathbf{e}^1\right)\mathbf{e}_1 + \left(\mathbf{D}\cdot\mathbf{e}^2\right)\mathbf{e}_2 + \left(\mathbf{D}\cdot\mathbf{e}^3\right)\mathbf{e}_3 \\
&= D^1\mathbf{e}_1 + D^2\mathbf{e}_2 + D^3\mathbf{e}_3 \\
&= D^i\mathbf{e}_i \text{ with } D^i \equiv \mathbf{D}\cdot\mathbf{e}^i
\end{aligned}
$$

Both covariant and contravariant representations contain all the information needed and are equally valid.

They are related through metric coefficients.

$$d\mathbf{R} = \sum_{j=1}^{3} \frac{\partial \mathbf{R}}{\partial u^j} du^j = \mathbf{e}_j du^j$$

We can also write $d\mathbf{R}$ as

$$d\mathbf{R} = (d\mathbf{R})_i \mathbf{e}^i = (d\mathbf{R})_i \nabla u^i$$

with

$$(d\mathbf{R})_i = d\mathbf{R} \cdot \mathbf{e}_i = du^j \mathbf{e}_j \cdot \mathbf{e}_i = \mathbf{e}_i \cdot \mathbf{e}_j du^j \equiv g_{ij} du^j$$

Write the basis in its covariant component

$$\mathbf{e}_i = (\mathbf{e}_i)_j \mathbf{e}^j = (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}^j = g_{ij}\mathbf{e}^j$$

Write the basis in its contravariant component

$$\mathbf{e}^i = (\mathbf{e}^i)^j \mathbf{e}_j = (\mathbf{e}^i \cdot \mathbf{e}^j)\mathbf{e}_j = g^{ij}\mathbf{e}_j$$

The metric coefficients g_{ij} and g^{ij}

form the backbone of general curvilinear coordinate systems

- determine the differential arc length along a curve
- allow to switch back and forth between the covariant and contravariant components of a vector
- provide a means to calculate the dot and cross products of two vectors

Their definition

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial \mathbf{R}}{\partial u^i} \frac{\partial \mathbf{R}}{\partial u^j}$$

$$g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j = \nabla u^i \cdot \nabla u^j$$

$\Longrightarrow$ symmetric:

$$g_{ij} = g_{ji}, \qquad g^{ij} = g^{ji}$$

Orthogonal coordinate system:

$$g_{ij} = 0 \text{ and } g^{ij} = 0 \text{ if } i \neq j$$

II. equilibrium eq in flux coordinates

Changing between covariant and contravariant components

$$D_i = \mathbf{D} \cdot \mathbf{e}_i = D^j \mathbf{e}_j \cdot \mathbf{e}_i = D^j g_{ji} = D^j g_{ij}$$

$$D^i = \mathbf{D} \cdot \mathbf{e}^i = D_j \mathbf{e}^j \cdot \mathbf{e}^i = D_j g^{ji} = D^j g^{ij}$$

Since

$$\mathbf{e}_i = g_{ij}\mathbf{e}^j \qquad \mathbf{e}^i = g^{ij}\mathbf{e}_j$$

Kronecker delta δ^i_j can be obtained through

$$\delta^i_j = \mathbf{e}_j \cdot \mathbf{e}^i = g_{jk}\mathbf{e}^k \cdot \mathbf{e}^i = g_{jk}g^{ki} = g^{ik}g_{kj}$$

matrix of metric tensor g^{ij} and g_{ij}

$$\left[g^{ij}\right]\left[g_{ij}\right] = I \text{ (identity matrix)}$$

so

$$\left[g^{ij}\right] = \left[g_{ij}\right]^{-1}, \qquad \left[g_{ij}\right] = \left[g^{ij}\right]^{-1}$$

determinant

$$g = det\left[g_{ij}\right] \qquad g^{-1} = det\left[g_{ij}\right]$$

The Jacobian of the Curvilinear Coordinate System

Jacobian of (x, y, z) wrt u^i

$$J \equiv J(u^1, u^2, u^3) = \frac{\partial(x,y,z)}{\partial(u^1,u^2,u^3)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^2} & \frac{\partial x}{\partial u^3} \\ \frac{\partial y}{\partial u^1} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^3} \\ \frac{\partial z}{\partial u^1} & \frac{\partial z}{\partial u^2} & \frac{\partial z}{\partial u^3} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^1} \\ \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^2} \\ \frac{\partial z}{\partial u^3} & \frac{\partial z}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix}$$

$$= \frac{\partial \mathbf{R}}{\partial u^1} \cdot \frac{\partial \mathbf{R}}{\partial u^2} \times \frac{\partial \mathbf{R}}{\partial u^3} = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3$$

The inverse transformation Jacobian

$$\mathcal{J} \equiv J(x, y, z) = \frac{\partial(u^1,u^2,u^3)}{\partial(x,y,z)}$$

$$= \begin{vmatrix} \frac{\partial u^1}{\partial x} & \frac{\partial u^1}{\partial y} & \frac{\partial u^1}{\partial z} \\ \frac{\partial u^2}{\partial x} & \frac{\partial u^2}{\partial y} & \frac{\partial u^2}{\partial z} \\ \frac{\partial u^3}{\partial x} & \frac{\partial u^3}{\partial y} & \frac{\partial u^3}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial u^1}{\partial x} & \frac{\partial u^2}{\partial x} & \frac{\partial u^3}{\partial x} \\ \frac{\partial u^1}{\partial y} & \frac{\partial u^2}{\partial y} & \frac{\partial u^3}{\partial y} \\ \frac{\partial u^1}{\partial z} & \frac{\partial u^2}{\partial z} & \frac{\partial u^3}{\partial z} \end{vmatrix}$$

$$= \nabla u^1 \cdot \nabla u^2 \times \nabla u^3 = \mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3$$

J and $\mathcal{J}$ are inverse of each other:

$$J = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3, \qquad \mathcal{J} = \mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3$$

It can be proved from

$$\begin{aligned}
\mathcal{J} &= \frac{(\mathbf{e}_2 \times \mathbf{e}_3) \cdot [(\mathbf{e}_3 \times \mathbf{e}_1) \times (\mathbf{e}_1 \times \mathbf{e}_2)]}{[\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3]^3} \\
&= \frac{(\mathbf{e}_2 \times \mathbf{e}_3)}{J^3} \cdot [\mathbf{e}_1(\mathbf{e}_2 \cdot \mathbf{e}_3 \times \mathbf{e}_1) - \mathbf{e}_2(\mathbf{e}_1 \cdot \times \mathbf{e}_3 \times \mathbf{e}_1)] \\
&= \frac{[\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3][\mathbf{e}_2 \cdot \mathbf{e}_3 \times \mathbf{e}_1)]}{J^3} \\
&= \frac{J^2}{J^3} \\
&= J^{-1}
\end{aligned}$$

The basis vectors can be rewritten as for $i = 1, 2, 3$

$$\mathbf{e}^i \equiv \nabla u^i = \frac{1}{J}\left(\frac{\partial \mathbf{R}}{\partial u^j} \times \frac{\partial \mathbf{R}}{\partial u^k}\right) = \frac{1}{J}(\mathbf{e}_j \times \mathbf{e}_k)$$

and

$$\mathbf{e}_i \equiv \frac{\partial \mathbf{R}}{\partial u^i} = J(\nabla u^j \times \nabla u^k) = J(\mathbf{e}^j \times \mathbf{e}^k)$$

The Relationship between J and g

We have defined

$$g = det[g_{ij}]$$

with

$$g_{ij} = \frac{\partial \mathbf{R}}{\partial u^i} \cdot \frac{\partial \mathbf{R}}{\partial u^j} = \frac{\partial x}{\partial u^i}\frac{\partial x}{\partial u^j} + \frac{\partial y}{\partial u^i}\frac{\partial y}{\partial u^j} + \frac{\partial z}{\partial u^i}\frac{\partial z}{\partial u^j}$$

which is

$$\begin{aligned} g &= det[g_{ij}] = \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^1} \\ \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^2} \\ \frac{\partial z}{\partial u^3} & \frac{\partial z}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^2} & \frac{\partial x}{\partial u^3} \\ \frac{\partial y}{\partial u^1} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^3} \\ \frac{\partial z}{\partial u^1} & \frac{\partial z}{\partial u^2} & \frac{\partial z}{\partial u^3} \end{vmatrix} \\ &= \left(\frac{\partial \mathbf{R}}{\partial u^1} \cdot \frac{\partial \mathbf{R}}{\partial u^2} \times \frac{\partial \mathbf{R}}{\partial u^3}\right)^2 \\ &= J^2 \end{aligned}$$

So J is related to g by

$$J = \sqrt{g}$$

From now on we will use $\sqrt{g}$ as Jacobian throughout this lecture and J will be reserved for electric current density.

The Dot Product in Curvilinear Coordinates

Two vectors **A** and **B** can be written in terms of their contravariant or covariant components

$$\mathbf{A} = A^i \mathbf{e}_i = A_i \mathbf{e}^i$$
$$\mathbf{B} = B^j \mathbf{e}_j = B_j \mathbf{e}^j$$

So

$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= A^i \mathbf{e}_i \cdot \mathbf{B}_j \mathbf{e}^j &= A^i \mathbf{B}_j \mathbf{e}_i \cdot \mathbf{e}^j &= A^i \mathbf{B}_j \delta_i^j \\
&= A_i \mathbf{e}^i \cdot \mathbf{B}^j \mathbf{e}_j &= A_i \mathbf{B}^j \mathbf{e}^i \cdot \mathbf{e}_j &= A_i \mathbf{B}^j \delta_j^i \\
&= A^i \mathbf{e}_i \cdot \mathbf{B}^j \mathbf{e}_j &= A^i \mathbf{B}^j \mathbf{e}_i \cdot \mathbf{e}_j &= A^i \mathbf{B}^j g_{ij} \\
&= A_i \mathbf{e}^i \cdot \mathbf{B}_j \mathbf{e}^j &= A_i \mathbf{B}_j \mathbf{e}^i \cdot \mathbf{e}^j &= A_i \mathbf{B}_j g^{ij}
\end{aligned}$$

In summary

$$\mathbf{A}\cdot\mathbf{B} = A^i \mathbf{B}_j = A_i \mathbf{B}^j = g_{ij} A^i \mathbf{B}^j = g^{ij} A_i \mathbf{B}_j$$

The magnitude of a vector

$$A = \sqrt{\mathbf{A}\cdot\mathbf{A}} = \sqrt{A^i A_i} = \sqrt{A_i A^i} = \sqrt{g_{ij} A^i A^i} = \sqrt{g^{ij} A_i A_i}$$

The Cross Product in Curvilinear Coordinates

$$\mathbf{A} \times \mathbf{B} = A^i B^j \mathbf{e}_i \times \mathbf{e}_j = A_i B_j \mathbf{e}^i \times \mathbf{e}^j$$

$$= \sqrt{g} \sum_k (A^i B^j - A^j B^i) \mathbf{e}^k = \frac{1}{\sqrt{g}} \sum_k (A_i B_j - A_j B_i) \mathbf{e}_k$$

We also have

$$(\mathbf{A} \times \mathbf{B})_k = g_{ki} (\mathbf{A} \times \mathbf{B})^i$$

and

$$(\mathbf{A} \times \mathbf{B})^k = g^{ki} (\mathbf{A} \times \mathbf{B})_i$$

The differential arc length $dl(i)$

$$dl = |d\mathbf{R}| = \sqrt{d\mathbf{R} \cdot d\mathbf{R}}$$

since

$$d\mathbf{R} = \frac{\partial \mathbf{R}}{\partial u^1} du^1 + \frac{\partial \mathbf{R}}{\partial u^2} du^2 + \frac{\partial \mathbf{R}}{\partial u^3} du^3 = \frac{\partial \mathbf{R}}{\partial u^j} du^j = \mathbf{e}_j du^j$$

the square of the arc length can be expressed as

$$(dl)^2 = \sum_{i,j} du^i du^j \mathbf{e}_i \cdot \mathbf{e}_j = g_{ij} du^i du^j$$

For each i

$$d\mathbf{R}(i) = \frac{\partial \mathbf{R}}{\partial u^i} du^i = \mathbf{e}_i du^i = \sqrt{g} \nabla u^j \times \nabla u^k du^i$$

and

$$dl(i) = \sqrt{g} \left| \nabla u^j \times \nabla u^k \right| du^i$$

i, j, k cycle $1, 2, 3$ and no summation is used.

The Differential Area Element $d\mathbf{S}(i)$

surface area vector

$$
\begin{aligned}
d\mathbf{S}(\text{ in } u^i = \text{constant}) &\equiv d\mathbf{S}(i) \\
&= d\mathbf{R}(j) \times d\mathbf{R}(k) \\
&= \frac{\partial \mathbf{R}}{\partial u^j} \times \frac{\partial \mathbf{R}}{\partial u^k} du^j du^k, \text{ no summation over } j \text{ or } k \\
&= \mathbf{e}_j \times \mathbf{e}_k du^j du^k, \text{ no summation over } j \text{ or } k. \\
&= J \nabla u^i du^j du^k
\end{aligned}
$$

surface area element

$$
\begin{aligned}
dS(i) &= \sqrt{d\mathbf{S}(i) \cdot d\mathbf{S}(i)} \\
&= \sqrt{(\mathbf{e}_j \times \mathbf{e}_k) \cdot (\mathbf{e}_j \times \mathbf{e}_k)} du^j du^k \\
&= \sqrt{(\mathbf{e}_j \cdot \mathbf{e}_j)(\mathbf{e}_k \cdot \mathbf{e}_k) - (\mathbf{e}_j \cdot \mathbf{e}_k)(\mathbf{e}_j \cdot \mathbf{e}_k)} du^j du^k \text{ no summation over} j \text{or} k. \\
&= \sqrt{(g_{jj} g_{kk} - g_{jk} g_{jk}} du^j du^k \text{ no summation over} j \text{or} k.
\end{aligned}
$$

The Differential Volume Element d^3R

d^3R : differential-volume determined by the position vector **R** infinitesimal in all three coordinates

$d\mathcal{V}$: volume element, infinitesimal in only one direction, i.e., radial direction

$$\begin{aligned} d^3R &= d\mathbf{R}(1) \cdot d\mathbf{R}(2) \times d\mathbf{R}(3) \\ &= du^1 du^2 du^3 \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) \\ &= \sqrt{g} du^1 du^2 du^3 \end{aligned}$$

This result is positive

because be definition $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a right-handed system.

The Del Operator ∇

is defined via the relationship

$$d\Phi = \nabla\Phi \cdot \mathbf{dR}$$

Expansion of $d\Phi$ as

$$d\Phi = \frac{\partial\Phi}{\partial u^i} du^i = \frac{\partial\Phi}{\partial u^i}\mathbf{e}^i \cdot \mathbf{e}_j du^j = \frac{\partial\Phi}{\partial u^i}\nabla u^i \cdot \mathbf{e}_j du^j = \nabla\Phi \cdot d\mathbf{R}$$
$$(\delta^i_j = \mathbf{e}^i \cdot \mathbf{e}_j)$$

So we have

$$\nabla \equiv \nabla u^i \frac{\partial}{\partial u^i} \equiv \mathbf{e}^i \frac{\partial}{\partial u^i}$$

The Gradient Operator ∇

Operate ∇ on a scalar function $\Phi(u^1, u^2, u^3)$

$$\nabla\Phi \equiv \nabla\Phi = \nabla u^i \frac{\partial \Phi}{\partial u^i} = \frac{\partial \Phi}{\partial u^i}\mathbf{e}^i$$

Or the con variant component can be given as

$$(\nabla\Phi)_i = \frac{\partial \Phi}{\partial u^i}$$

The Divergence Operator $\nabla\cdot$

Making use of the properties

$$\nabla \times \nabla\Phi = 0$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Because the vector

$$\frac{\mathbf{e}_j}{sqrtg} = (\nabla u^j \times \nabla u^k) = \nabla \times (u^j \nabla u^k)$$

is the curl of another vector, its divergence is zero

$$\nabla \cdot (\frac{\mathbf{e}_j}{\sqrt{g}}) = \nabla \cdot \Big(\nabla \times (u^j \nabla u^k)\Big) = 0$$

we can take the divergence of a vector **D**

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \nabla \cdot \Big(D^j \mathbf{e}_j\Big) = \nabla \cdot \left(\sqrt{g} D^j \frac{\mathbf{e}_j}{\sqrt{g}}\right) \\
&= \sqrt{g} D^j \nabla \cdot \left(\frac{\mathbf{e}_j}{\sqrt{g}}\right) + \frac{\mathbf{e}_j}{\sqrt{g}} \cdot \nabla(\sqrt{g} D^j) \\
&= \frac{\mathbf{e}_j}{\sqrt{g}} \cdot \nabla(\sqrt{g} D^j) \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i}(\sqrt{g} D^i) \quad (\nabla(\sqrt{g} D^j) = \frac{\partial(\sqrt{g} D^j)}{\partial u^i} \mathbf{e}^i)
\end{aligned}$$

The Curl Operator $\nabla\times$

from

$$\mathbf{e}^i \times \mathbf{e}^j = \frac{\mathbf{e}_k}{\sqrt{g}} \qquad \nabla \times \nabla u^j = 0$$

$$\nabla \times \mathbf{D} = \nabla \times (D_j \mathbf{e}^j) = D_j(\nabla \times \mathbf{e}^j) + \nabla D_j \times \mathbf{e}^j = \frac{\partial D_j}{\partial u^i}\mathbf{e}^i \times \mathbf{e}^j$$

$$i = 1 : \frac{\partial D_j}{\partial u^1}\mathbf{e}^1 \times \mathbf{e}^j \quad \mathbf{e}^1 \times \mathbf{e}^1 \quad \mathbf{e}^1 \times \mathbf{e}^2 \quad \mathbf{e}^1 \times \mathbf{e}^3$$

$$i = 2 : \frac{\partial D_j}{\partial u^2}\mathbf{e}^2 \times \mathbf{e}^j \quad \mathbf{e}^2 \times \mathbf{e}^1 \quad \mathbf{e}^2 \times \mathbf{e}^2 \quad \mathbf{e}^2 \times \mathbf{e}^3$$

$$i = 3 : \frac{\partial D_j}{\partial u^3}\mathbf{e}^3 \times \mathbf{e}^j \quad \mathbf{e}^3 \times \mathbf{e}^1 \quad \mathbf{e}^3 \times \mathbf{e}^2 \quad \mathbf{e}^3 \times \mathbf{e}^3$$

Thus

$$\nabla \times \mathbf{D} = \frac{1}{\sqrt{g}} \sum_k \left(\frac{\partial D_j}{\partial u^i} - \frac{\partial D_i}{\partial u^j}\right)\mathbf{e}_k$$

Differentiation of Vector Components and Basis Vectors

Consider vector field $\mathbf{W} = \mathbf{W}(\mathbf{R}(u^1, u^2, u^3))$

$$\mathbf{W} = W^i \mathbf{e}_i$$

or

$$\mathbf{W} = W_i \mathbf{e}^i$$

W^i, W_i, $\mathbf{e}_i$, $\mathbf{e}^i$, are position dependent.

The partial derivatives $\frac{\partial \mathbf{W}}{\partial u^k}$ follows

$$\frac{\partial \mathbf{W}}{\partial u^k} = \frac{\partial W^i}{\partial u^k}\mathbf{e}_i + W^i \frac{\partial \mathbf{e}_i}{\partial u^k}$$

or

$$\frac{\partial \mathbf{W}}{\partial u^k} = \frac{\partial W_i}{\partial u^k}\mathbf{e}^i + W_i \frac{\partial \mathbf{e}^i}{\partial u^k}$$

OR

$$\frac{\partial \mathbf{W}}{\partial u^k} = \left(\frac{\partial \mathbf{W}}{\partial u^k}\right)^j \mathbf{e}_j = \left(\frac{\partial \mathbf{W}}{\partial u^k} \cdot \mathbf{e}^j\right) \mathbf{e}_j$$

or

$$\frac{\partial \mathbf{W}}{\partial u^k} = \left(\frac{\partial \mathbf{W}}{\partial u^k}\right)_j \mathbf{e}^j = \left(\frac{\partial \mathbf{W}}{\partial u^k} \cdot \mathbf{e}_j\right) \mathbf{e}^j$$

Combine the above 4 expressions
the partial derivative of **W** can be written in 4 different ways

The contravariant forms

$$\left(\frac{\partial \mathbf{W}}{\partial u^k}\right)^j = \frac{\partial W^i}{\partial u^k}\mathbf{e}_i \cdot \mathbf{e}^j + W^i \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}^j = \frac{\partial W^j}{\partial u^k} + W^i \left\{ \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}^j \right\}$$

$$\left(\frac{\partial \mathbf{W}}{\partial u^k}\right)^j = \frac{\partial W_i}{\partial u^k}\mathbf{e}^i \cdot \mathbf{e}^j + W_i \frac{\partial \mathbf{e}^i}{\partial u^k} \cdot \mathbf{e}^j$$

The covariant forms

$$\left(\frac{\partial \mathbf{W}}{\partial u^k}\right)_j = \frac{\partial W^i}{\partial u^k}\mathbf{e}_i \cdot \mathbf{e}_j + W^i \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}_j$$

$$\left(\frac{\partial \mathbf{W}}{\partial u^k}\right)_j = \frac{\partial W_i}{\partial u^k}\mathbf{e}^i \cdot \mathbf{e}_j + W_i \frac{\partial \mathbf{e}^i}{\partial u^k} \cdot \mathbf{e}_j = \frac{\partial W_j}{\partial u^k} + W_i \left\{ \frac{\partial \mathbf{e}^i}{\partial u^k} \cdot \mathbf{e}_j \right\}$$

The absolute differential $d\mathbf{W}$ is given by

$$d\mathbf{W} = \frac{\partial \mathbf{W}}{\partial u^k} du^k$$

Using partial derivatives, its jth contravariant and covariant components are

$$\begin{aligned}
\delta W^j &\equiv (dW)^j = \left(\frac{\partial \mathbf{W}}{\partial u^k}\right)^j du^k = \frac{\partial W^j}{\partial u^k} du^k + W^i \left\{ \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}^j \right\} du^k \\
&= dW^j + W^i \left\{ \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}^j \right\} du^k
\end{aligned}$$

$$\begin{aligned}
\delta W_j &\equiv (dW)_j = \left(\frac{\partial \mathbf{W}}{\partial u^k}\right)_j du^k = \frac{\partial W_j}{\partial u^k} du^k + W_i \left\{ \frac{\partial \mathbf{e}^i}{\partial u^k} \cdot \mathbf{e}_j \right\} du^k \\
&= dW_j + W_i \left\{ \frac{\partial \mathbf{e}^i}{\partial u^k} \cdot \mathbf{e}_j \right\} du^k
\end{aligned}$$

Expression Basis Derivatives in terms of Metric Coefficients

The information about the chosen basis is largely contained in the metric coefficient g_{ij}.

The vector $\frac{\partial \mathbf{e}_i}{\partial u^k}$ can be expanded in two ways along $\mathbf{e}_j$ or $\mathbf{e}^j$

$$\frac{\partial \mathbf{e}_i}{\partial u^k} = \left\{ \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}^j \right\} \mathbf{e}_j \equiv (\frac{\partial \mathbf{e}_i}{\partial u^k})^j \mathbf{e}_j$$

$$\frac{\partial \mathbf{e}_i}{\partial u^k} = \left\{ \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}_j \right\} \mathbf{e}^j \equiv (\frac{\partial \mathbf{e}_i}{\partial u^k})_j \mathbf{e}^j$$

Using the properties

$$(\mathbf{e}^j = g^{jn} \mathbf{e}_n) \qquad (\mathbf{e}_j = g_{jn} \mathbf{e}^n)$$

we have

$$\left\{ \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}^j \right\} = g^{jn} \left[\frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}_n \right]$$

$$\left\{ \frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}_j \right\} = g_{jn} \left[\frac{\partial \mathbf{e}_i}{\partial u^k} \cdot \mathbf{e}^n \right]$$

Since

$$\frac{\partial \mathbf{e}_i}{\partial u^k} = \frac{\partial}{\partial u^k}\frac{\partial \mathbf{R}}{\partial u^i} = \frac{\partial}{\partial u^i}\frac{\partial \mathbf{R}}{\partial u^k} = \frac{\partial \mathbf{e}_k}{\partial u^i}$$

We have

$$\begin{aligned}
\left\{\frac{\partial \mathbf{e}_i}{\partial u^k}\cdot \mathbf{e}^j\right\} &= g^{jn}\frac{1}{2}\left[\mathbf{e}_n\cdot\frac{\partial \mathbf{e}_i}{\partial u^k} + \mathbf{e}_n\cdot\frac{\partial \mathbf{e}_k}{\partial u^i}\right] \\
&= g^{jn}\frac{1}{2}\left[\frac{\partial}{\partial u^k}(\mathbf{e}_n\cdot\mathbf{e}_i) + \frac{\partial}{\partial u^i}(\mathbf{e}_n\cdot\mathbf{e}_k) - \mathbf{e}_i\cdot\frac{\partial \mathbf{e}_n}{\partial u^k} - \mathbf{e}_k\cdot\frac{\partial \mathbf{e}_n}{\partial u^i}\right] \\
&= g^{jn}\frac{1}{2}\left[\frac{\partial g_{ni}}{\partial u^k} + \frac{\partial g_{nk}}{\partial u^i} - \frac{\partial \mathbf{e}_i\cdot\mathbf{e}_k}{\partial u^n}\right] \\
&= g^{jn}\frac{1}{2}\left[\frac{\partial g_{ni}}{\partial u^k} + \frac{\partial g_{nk}}{\partial u^i} - \frac{\partial g_{ik}}{\partial u^n}\right]
\end{aligned}$$

From $\mathbf{e}_j \cdot \mathbf{e}^i = \delta^i_j$

we have

$$\frac{\partial}{\partial u^k}(\mathbf{e}_j \cdot \mathbf{e}^i) = 0$$

and

$$\left\{ \mathbf{e}_j \cdot \frac{\partial \mathbf{e}^i}{\partial u^k} \right\} = \left\{ \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial u^k} \right\}$$

thus

$$\left\{ \frac{\partial \mathbf{e}^i}{\partial u^k} \cdot \mathbf{e}_j \right\} = -g^{in} \frac{1}{2} \left[\frac{\partial g_{nj}}{\partial u^k} + \frac{\partial g_{nk}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^n} \right]$$

The magnetic field line curvature vector κ

Denote the unit tangent direction of magnetic field line as $\mathbf{b} = \frac{\mathbf{B}}{B}$

$\mathbf{b} \cdot \mathbf{b} = 1$ yields $\mathbf{b} \cdot \frac{\partial \mathbf{b}}{\partial l} = 0$

Hence $\frac{\partial \mathbf{b}}{\partial l}$ is perpendicular to $\mathbf{b}$. Define

$$\frac{d\mathbf{b}}{dl}|_{\text{along } \mathbf{b}} \equiv \kappa \mathbf{n} = \frac{1}{R_c}\mathbf{n} = \boldsymbol{\kappa}$$

$\mathbf{n}$ is the principle normal to the field line.

The vector $\boldsymbol{\kappa}$ is called the curvature vector.

The quantity $R_c = 1/\kappa$ is the local radius of curvature.

The directional derivative is equivalent to

$$\mathbf{b} \cdot \nabla = \frac{\partial}{\partial l} = \frac{d}{dl}|_{\text{along } \mathbf{b}}$$

So the curvature can be written in the form

$$\boldsymbol{\kappa} = (\mathbf{b} \cdot \nabla)\mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{b}$$

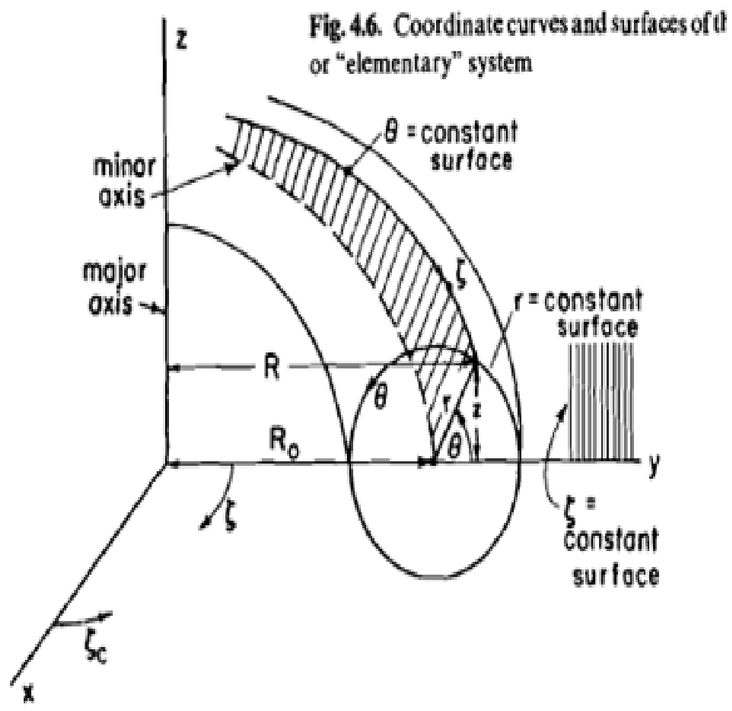
Since $\mathbf{b} \times (\nabla \times \mathbf{b}) = \nabla \frac{b^2}{2} - \mathbf{b} \cdot \nabla \mathbf{b} = -\mathbf{b} \cdot \nabla \mathbf{b}$

The curvature can also be written =as

$$\boldsymbol{\kappa} = -\mathbf{b} \times (\nabla \times \mathbf{b})$$

Elementary Toroidal Coordinates

Circles upon intersection with vertical and horizontal planes



Cylindrical system (R, ζ_c, z)

Elementary system $(u^1, u^2, u^3) = (r, \theta_e, \zeta_e)$

The relationship between (x, y, z) and (r, θ_e, ζ_e)

$$x = R_0 + r\cos\theta\sin\zeta$$
$$y = R_0 + r\cos\theta\cos\zeta$$
$$z = r\sin\theta$$

R_0: major radius
r: minor radius
θ: poloidal angle, encircles the minor axis
ζ: toroidal angle, encircles the major axis
poloidal plane: $\zeta =$ constant
toroidal plane: $\theta =$ constant

Generalized Toroidal Coordinates

Choose magnetic surfaces as the coordinate surfaces.

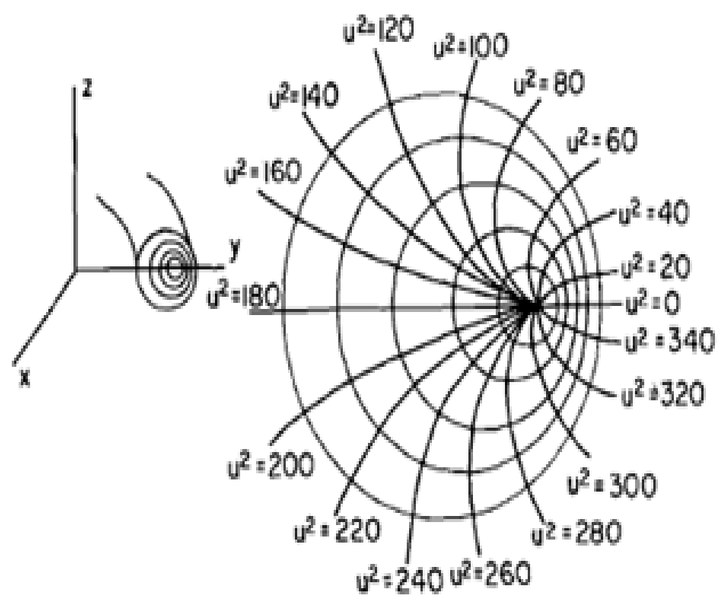


Fig. 4.8. u^2 = constant surfaces which are perpendicular to the Shafran simple tokamak

Generalize the elementary coordinate (r, θ_e, ζ_e) to

$r \to \psi = const$:	magnetic surface label
	magnetic flux enclosed by the surface
	volume
	pressure
$\theta_e \to \theta_g$:	generalized poloidal angle, based on magnetic axis
	θ_g is multi-valued, changes by 2π after each complete circuit.
$\zeta_e \to \zeta_g$:	generalized toroidal angle
	$\zeta_g = const$ makes poloidal cross section.
	ζ_g is multi-valued, changes by 2π after each complete circuit.

Now $(u^1, u^2, u^3) = (\psi, \theta_g, \zeta_g)$

Flux Coordinates

θ_e and ζ_e can be deformed by adding periodic functions

$$\theta = \theta_g + f(\rho, \theta, \zeta)$$
$$\zeta = \zeta_g + g(\rho, \theta, \zeta)$$

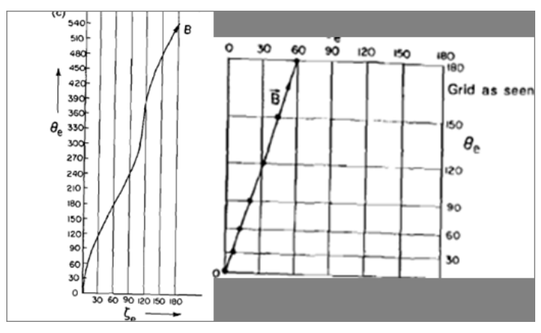
with period 2π

so that a field **B** line appear as a straight line, i.e.,

$$\frac{\theta}{\zeta} = const$$

Further deformation can even makes the current density **J** line straight.

Now $(u^1, u^2, u^3) = (\psi, \theta, \zeta)$



The covariant form of magnetic field line in flux coordinates

$$\mathbf{B} = B_\psi \mathbf{e}^\psi + B_\theta \mathbf{e}^\theta + B_\zeta \mathbf{e}^\zeta, \text{ with } B_i = \mathbf{B} \cdot \mathbf{e}_i$$

and the contravariant form becomes

$$\mathbf{B} = B^\psi \mathbf{e}_\psi + B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta, \text{ with } B^i = \mathbf{B} \cdot \mathbf{e}^i$$

In which

$$B^{\psi} = 0$$

$$B^{\theta} = \frac{1}{2\pi\sqrt{g}}\dot{\Psi}^{r}_{pol}$$

$$B^{\zeta} = \frac{1}{2\pi\sqrt{g}}\dot{\Psi}_{tol}$$

and

$$\frac{B^{\theta}}{B^{\zeta}} = \frac{\dot{\Psi}^{r}_{pol}}{\dot{\Psi}_{tol}} = \text{ is a function of flux}$$

From the equation of a field line

$$\frac{d\theta}{d\zeta} = \frac{B^{\theta}}{B^{\zeta}} = \frac{\dot{\Psi}^{r}_{pol}}{\dot{\Psi}_{tol}} = \iota(\psi)$$

we see that the slop of the line is indeed constant on the flux surface, and equal to the rotational transform ι.

This guarantees that the magnetic filed line is straight on the flux surfaces.

Symmetric Flux Coordinates in a Tokamak

Elementary system $(u^1, u^2, u^3) = (\psi, \theta_O, \zeta_O)$

ζ_O is the symmetric angle and

$\nabla\psi \cdot \nabla\zeta_O = 0$

$\nabla\theta_O \cdot \nabla\zeta_O = 0$

$\frac{\partial}{\partial\zeta_O} = 0$

$\mathbf{e}_\zeta \propto \nabla\zeta_O,$

$\mathbf{e}_{\theta_O} \cdot \mathbf{e}_{\zeta_O} = 0,$

$|\nabla\zeta_O| = \frac{1}{R},$

R is the distance from the major axis to a particular point on the surface

The magnetic field can be written into the poloidal and toroidal components.

$$\mathbf{B} = q(\nabla\psi \times \nabla\zeta_O) + (\nabla\zeta_O \times \nabla\psi)$$

which is the form used most in the tokamak literature.

q is the safety factor.

From the above orthogonality, it follows that

$\nabla\psi \times \nabla\theta_O = k\nabla\zeta_O, k$ is the proportionality factor.

From the Jacobian

$$\boldsymbol{\sqrt{g}_O = (\nabla\zeta_O \cdot \nabla\psi \times \nabla\theta_O)^{-1} = \frac{1}{k|\nabla\zeta_O|^2}}$$

we can solve for k, as g_O is known

$$k = \frac{1}{\sqrt{g}_O|\nabla\zeta_O|^2}$$

and the toroidal field

$$\mathbf{B}_{\boldsymbol{T}} = \boldsymbol{q(\nabla\Psi \times \nabla\theta_O) = \frac{q\nabla\zeta_O}{\sqrt{g_O}|\nabla\zeta_O|^2} = \frac{qR^2}{\sqrt{g_O}}\nabla\zeta_O}$$

Defining

$$I = \frac{qR^2}{\sqrt{g_O}}$$

it can be proved from Ampere's law that I is a flux surface function

$$I = I(\Psi)$$

and effectively measures the total poloidal current outside the flux surface $\psi = const$.

The eqs describing MHD equilibrium of a static (no fluid flow), isotropic plasma are the force balance eq and Ampere's and Gauss laws.

Equilibrium with no flow

$$\nabla p = \mathbf{J} \times \mathbf{B}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\nabla \cdot \mathbf{B} = 0$$

Equilibrium with toroidal flow

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{J} \times \mathbf{B}$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\nabla \cdot \mathbf{B} = 0$$

When an anisotropy is present in a system, a particular form of pressure tensor

$$\mathbf{P} = p_\perp \mathbf{I} + (p_\parallel - p_\perp)\frac{\mathbf{BB}}{B^2}$$

needs to be prescribed.

$p_\parallel$ and $p_\perp$ are scalar functions representing the pressure parallel and perpendicular to the magnetic field direction.

The equilibrium equation with this form for the pressure tensor becomes

$$\nabla \cdot \mathbf{P} = \mathbf{J} \times \mathbf{B}$$

$$\mathbf{F} \equiv -\mathbf{J} \times \mathbf{B} + \nabla p$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\nabla \cdot \mathbf{B} = 0$$

$p = p(\psi)$ is the pressure
ψ is the radial coordinate labeling a magnetic flux surface.
The quantity $\mathbf{F}$ is the residual MHD force, which must vanish in equilibrium.

Existence of Nested Flux Surfaces

An important concept is the flux surface,
a surface that is traced out by a series of magnetic field lines, closed.

On such surface **B** is everywhere perpendicular to its normal.

The flux-freezing constraint guarantees such surfaces exists
(to be demonstrated after the flux quantity is introduced).

magnetic field-lines embedded in an MHD plasma can never break and reconnect: MHD forbids any change in topology of the field-lines.

The force balance equation implies that p is constant on these surfaces ($\mathbf{B} \cdot \nabla p = 0$)

The current density lines lie on these surfaces as well ($\mathbf{J} \cdot \nabla p = 0$)

In 3D geometry
the existence of flux surfaces (nested or not) is not guaranteed.
but one can follow the field to demonstrate the existence of suitable surfaces.

$$\mathbf{F} \equiv -\mathbf{J} \times \mathbf{B} + \nabla p$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\nabla \cdot \mathbf{B} = 0$$

$p = p(\psi)$ is the pressure
ψ is the radial coordinate labeling a magnetic flux surface.
The quantity $\mathbf{F}$ is the residual MHD force, which must vanish in equilibrium.

Two types of nested surfaces:
rational surfaces
irrational surfaces

Between the surfaces traced out by non-closing field lines, there exist surfaces that contain field lines closing upon themselves after one or several transit around the machine.

$\Longrightarrow$ Rational surfaces.
because the ratio of the number of poloidal transits (the short way around the torus) to the number of toroidal transits (the long way around) of a field line is a rational number.

The other surfaces are called irrational surfaces.

Since the set of real number is dense, every irrational number can be approximated infinitesimally closely by a rational number and vice versa.

The same holds for magnetic surfaces.

That is
every rational surface can be approximated very accurately by a nearby irrational surface, and the other way around.

For the nested toroidal flux surface geometry considered here, flux coordinate angles θ and ζ many be introduced.
θ is the poloidal angle (the short way around the magnetic axis)
ζ is the toroidal angle (the long way around the torus).

Assuming the flux surfaces have toroidal topology,

Each toroidal surface ψ encloses a volume $\mathcal{V}(\psi)$.

The surface corresponding to an infinitesimal volume $\mathcal{V}$ is essentially a line

that corresponds to the toroidal axis

called magnetic axis

It is a degenerated magnetic surface formed by a field line closing upon itself after one toroidal transit.

It is a line around which the magnetic surfaces are nested.

For the nested toroidal flux surface geometry considered here, flux coordinate angles θ and ζ many be introduced.
θ is the poloidal angle (the short way around the magnetic axis)
ζ is the toroidal angle (the long way around the torus).

The ideal MHD Ohm's law, $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$

implies that the magnetic flux through any closed contour in the plasma, is a conserved quantity. To verify, consider the magnetic flux,

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

through a contour, Γ, which is co-moving with the plasma and spans surface $\mathbf{S}$. The time rate of change of Ψ is made up of two parts.

1st, there is the part due to the time variation of $\mathbf{B}$ over the surface $\mathbf{S}$. This can be written

$$\left(\frac{\partial \Psi}{\partial t}\right)_1 = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} \text{ Faraday law}$$

2nd, there is the part due to the motion of Γ. If $d\mathbf{l}$ is an element of Γ then $\mathbf{V} \times d\mathbf{l}$ is the area swept out by $d\mathbf{l}$ per unit time. Hence, the flux crossing this area is $\mathbf{B} \cdot \mathbf{V} \times d\mathbf{l}$, and

$$\left(\frac{\partial \Psi}{\partial t}\right)_2 = \int_\Gamma \mathbf{B} \cdot \mathbf{V} \times d\mathbf{l} = \int_\Gamma \mathbf{B} \times \mathbf{V} \cdot d\mathbf{l} = \int_S \nabla \times (\mathbf{B} \times \mathbf{V}) \cdot d\mathbf{S} \text{ Stokes's theorem}$$

Hence, the total time rate of change of Ψ is given by

$$\frac{d\Psi}{dt} = -\int_S \nabla \times (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot d\mathbf{S} = 0 \text{ Ohm's law}$$

This implies that Ψ remains constant in time for any arbitrary contour. This, in turn, implies that magnetic field-lines must move with the plasma. In other words, the field-lines are frozen into the plasma and the surfaces have to be nested.

The condition

$$\mathbf{B} \cdot \nabla p = 0$$

and

$$\nabla \cdot \mathbf{B} = 0$$

cab be satisfied by writing B in <u>Contravariant form</u> as follows:

$$\begin{aligned} \mathbf{B} &= \nabla\zeta \times \nabla\chi + \nabla\Phi \times \nabla\theta^\star \\ &= B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta \end{aligned}$$

It satisfies

$$\mathbf{B} \cdot \nabla p = p'_\psi \mathbf{B} \cdot \nabla\psi = p'_\psi (B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta) \cdot \nabla\psi = 0$$

and

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \nabla \cdot (\nabla\zeta \times \nabla\chi + \nabla\Phi \times \nabla\theta^\star) \\ &= \nabla\chi \cdot \nabla \times \nabla\zeta - \nabla\zeta \cdot \nabla \times \nabla\chi + \nabla\theta^\star \cdot \nabla \times \nabla\Phi - \nabla\Phi \cdot \nabla \times \nabla\theta^\star \\ &= 0 \end{aligned}$$

$2\pi\chi(\psi)$ is the poloidal magnetic flux
$2\pi\Phi(\psi)$ is the toroidal magnetic flux
between the magnetic surface label ψ and the magnetic axis ($\psi = 0$, where $\nabla\chi = 0$).

The flux Ψ through an arbitrary surface **S** is defined as

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

Generally two magnetic fluxes can be defined from two corresponding surfaces.

The poloidal flux is defined by $\psi = \int_{S_p} \mathbf{B} \cdot \mathbf{n} dS$

S_p : a ring-shaped ribbon stretched between the magnetic axis and the flux surface ψ.

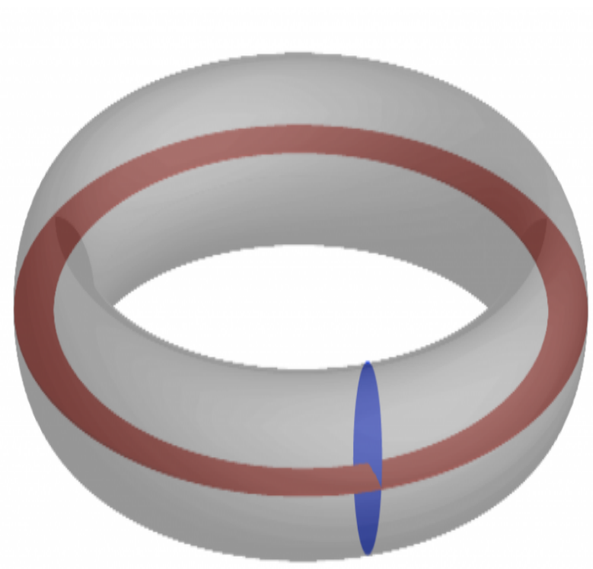
(Complementarity, S_p can be taken to be a surface spanning the central hole of the torus.)

Likewise,

the toroidal flux is defined by $\phi = \int_{S_t} \mathbf{B} \cdot \mathbf{n} dS$

S_t : a poloidal section of the flux surface.

It is natural to use ψ or ϕ to label the flux surfaces.



$2\pi\chi(\psi)$ is the poloidal magnetic flux
$2\pi\Phi(\psi)$ is the toroidal magnetic flux
between the magnetic surface label ψ and the magnetic axis ($\psi = 0$, where $\nabla\chi = 0$).

Consider the integral and apply Gauss's theorem

$$\int d^3\mathbf{x}\mathbf{B}\cdot\nabla\zeta = \int d^3\mathbf{x}\nabla\cdot(\mathbf{B}\zeta) = \oint \mathbf{B}\zeta\cdot d\mathbf{S}$$

$$\int d^3\mathbf{x}\mathbf{B}\cdot\nabla\theta = \int d^3\mathbf{x}\nabla\cdot(\mathbf{B}\theta) = \oint \mathbf{B}\theta\cdot d\mathbf{S}$$

We choose the $\zeta = 0$ surface to cut the torus open and consider the volume enclosed by the surfaces S_{torus}, $S_{\zeta=0}$, and $S_{\zeta=2\pi}$.

$$\oint \mathbf{B}\zeta\cdot d\mathbf{S} = \int_{S_{torus}} \zeta\mathbf{B}\cdot d\mathbf{S} + \int_{S_{\zeta=2\pi}} \zeta\mathbf{B}\cdot d\mathbf{S} + \int_{S_{\zeta=0}} \zeta\mathbf{B}\cdot d\mathbf{S}$$

The 1st one vanishes by definition: **B** is tangent everywhere to the magnetic surface, $d\mathbf{S} \perp \mathbf{B}$.
$\zeta = 0$ on the surface $S_{\zeta=0}$;
$\zeta = 2\pi$ on the surface $S_{\zeta=2\pi}$.

$$\int d^3\mathbf{x}\mathbf{B}\cdot\nabla\zeta = 2\pi\int_{S_{\zeta=2\pi}} \zeta\mathbf{B}\cdot d\mathbf{S}$$

$\mathbf{S}_{2\pi}$ or $\mathbf{S}_0$ is what we previously called toroidal flux surface, $\mathbf{S}_{tor}$
So

$$\Psi_{tor} = \int_{s_{tor}} \mathbf{B}\cdot d\mathbf{S} = \frac{1}{2\pi}\int_V d^3\mathbf{x}\mathbf{B}\cdot\nabla\zeta$$

$2\pi\chi(\psi)$ is the poloidal magnetic flux
$2\pi\Phi(\psi)$ is the toroidal magnetic flux
between the magnetic surface label ψ and the magnetic axis ($\psi = 0$, where $\nabla\chi = 0$).

Similarly,

We choose the $\theta = 0$ surface to cut the long way torus open and consider
the volume enclosed by the surfaces S_{torus}, $S_{\theta=0}$, and $S_{\theta=2\pi}$.

$$\oint \mathbf{B}\theta \cdot d\mathbf{S} = \int_{S_{torus}} \theta\mathbf{B} \cdot d\mathbf{S} + \int_{S_{\theta=2\pi}} \theta\mathbf{B} \cdot d\mathbf{S} + \int_{S_{\theta=0}} \theta\mathbf{B} \cdot d\mathbf{S}$$

The 1st one vanishes by definition: **B** is tangent everywhere to the magnetic surface, $d\mathbf{S} \perp \mathbf{B}$.
$\theta = 0$ on the surface $S_{\theta=0}$;
$\theta = 2\pi$ on the surface $S_{\theta=2\pi}$.

$$\int d^3\mathbf{x}\mathbf{B} \cdot \nabla\theta = 2\pi \int_{S_{\theta=2\pi}} \theta\mathbf{B} \cdot d\mathbf{S}$$

$\mathbf{S}_{2\pi}$ or $\mathbf{S}_0$ is what we previously called toroidal flux surface, $\mathbf{S}_{pol}$
So

$$\Psi_{pol} = \frac{1}{2\pi} \int_V d^3\mathbf{x}\mathbf{B} \cdot \nabla\theta$$

$2\pi\chi(\psi)$ **is the poloidal magnetic flux**
$2\pi\Phi(\psi)$ **is the toroidal magnetic flux**
between the magnetic surface label ψ **and the magnetic axis (**$\psi = 0$**, where** $\nabla\chi = 0$**).**

Following

$$\mathbf{B} = \chi'_\psi \nabla\zeta \times \nabla\psi + \phi'_\psi \nabla\psi \times (\nabla\theta + \lambda'_\psi \nabla\psi + \lambda'_\theta \nabla\theta + \lambda'_\zeta \nabla\zeta)$$

we have

$$\mathbf{B} \cdot \nabla\zeta = \Phi'_\psi (\nabla\psi \times (\nabla\theta + \lambda'_\theta \nabla\theta) \cdot \nabla\zeta = \sqrt{g}^{-1}(\Phi'_\psi + \lambda'_\theta)$$

Hence

$$\begin{aligned}
\Psi_{tor} &= \frac{1}{2\pi} \int \mathbf{B} \cdot \nabla\zeta \sqrt{g} d\psi d\theta d\psi \\
&= \frac{1}{2\pi} \int \sqrt{g}^{-1} (\Phi'_\psi + \lambda'_\theta) \sqrt{g} d\psi d\theta d\psi \\
&= \frac{1}{2\pi} 2\pi 2\pi \Phi \\
&= 2\pi\Phi
\end{aligned}$$

Use has been of that λ is periodic in θ and ζ.

Similarly

$$\begin{aligned}
\mathbf{B} \cdot \nabla\theta &= \chi'_\psi (\nabla\zeta \times \nabla\chi \cdot \nabla\theta + \Phi'_\psi (\nabla\psi \times (\nabla\theta + \lambda'_\psi \nabla\psi + \lambda'_\theta \nabla\theta + \lambda'_\zeta \nabla\zeta) \cdot \nabla\theta \\
&= \sqrt{g}^{-1} \chi'_\psi + \Phi'_\psi (\nabla\psi \times (\lambda'_\zeta \nabla\zeta) \cdot \nabla\theta
\end{aligned}$$

$$\begin{aligned}
\Psi_{pol} &= \frac{1}{2\pi} \int \mathbf{B} \cdot \nabla\theta \sqrt{g} d\psi d\zeta d\psi \\
&= \frac{1}{2\pi} \int \sqrt{g}^{-1} \chi'_\psi \sqrt{g} d\psi d\zeta d\psi \\
&= \frac{1}{2\pi} 2\pi 2\pi \chi \\
&= 2\pi\chi
\end{aligned}$$

$$\theta^\star = \theta + \lambda(\psi, \theta, \zeta)$$

is the poloidal angle that makes the magnetic field lines straight.
<u>the local rotational number</u> $\frac{\mathbf{B}\cdot\nabla\theta^\star}{\mathbf{B}\cdot\nabla\zeta}$ is a function of ψ alone in the $(\psi, \theta^\star, \zeta)$ coordinate system. and λ is a periodic function of θ and ζ with zero average over a magnetic surface $\int\int d\theta d\zeta \lambda = 0$

From previous slides, the rotational transform is defined as

$$\frac{\iota}{2\pi} = \frac{m}{n}$$

n: the mean number of toroidal transits

m: the mean number of poloidal transits

A rational surface,
on which the field line does not cover a surface but constitutes a 1D structure,
has ι equal to a rational number;
on an irrational surface,
on which the field line covers the surface entirely ι is irrational.

when **B** is straight, it follows from the definition of ι that

$$\iota = \frac{d\theta}{d\zeta}$$

Using field line eq (to be introduced later),
ι can be written in an equivalent and more useful expression

$$\frac{\iota}{2\pi} = \frac{d\Psi_{pol}}{d\Psi_{tor}}$$

Ψ_{pol} : the poloidal magnetic flux
Ψ_{tor} : the toroidal magnetic flux.

$$\theta^\star = \theta + \lambda(\psi, \theta, \zeta)$$

is the poloidal angle that makes the magnetic field lines straight.
<u>the local rotational number</u> $\frac{\mathbf{B}\cdot\nabla\theta^\star}{\mathbf{B}\cdot\nabla\zeta}$ is a function of ψ alone in the $(\psi, \theta^\star, \zeta)$ coordinate system. and λ is a periodic function of θ and ζ with zero average over a magnetic surface $\int\int d\theta d\zeta \lambda = 0$

In tokamak research, the quantity $q = 2\pi/\iota$, "safety factor", is preferred

In an axisymmetric tokamak with circular cross section flux surfaces, the eqs of a field line are, approximately:

$$\frac{r d\theta_e}{B_P} = \frac{R d\zeta_e}{B_T}$$

$B_P \equiv (\mathbf{B} \cdot \theta_e)$ poloidal field
$B_T \equiv (\mathbf{B} \cdot \zeta_e)$ toroidal field

Thus

$$q \simeq \frac{r B_T}{R B_P}$$

which is the "0"th order approximation of other configurations.

$$\theta^\star = \theta + \lambda(\psi, \theta, \zeta)$$

is the poloidal angle that makes the magnetic field lines straight. the local rotational number $\frac{\mathbf{B}\cdot\nabla\theta^\star}{\mathbf{B}\cdot\nabla\zeta}$ is a function of ψ alone in the $(\psi, \theta^\star, \zeta)$ coordinate system. and λ is a periodic function of θ and ζ with zero average over a magnetic surface $\int\int d\theta d\zeta \lambda = 0$

In almost all toroidal devices, the rotational transform changes from flux surface to flux surface.

To measure the amount of change, one uses the concept of shear, defined simply as

$$s(\psi) \equiv \frac{d\iota}{d\psi} = i(\psi)$$

In this paper

$$\begin{aligned}
\iota &= \frac{\mathbf{B}\cdot\nabla\theta^\star}{\mathbf{B}\cdot\nabla\zeta} \\
&= \frac{(\chi'_\psi \nabla\zeta\times\nabla\psi + \Phi'_\psi \nabla\psi\times\nabla\theta^\star)\cdot\nabla\theta^\star}{(\chi'_\psi \nabla\zeta\times\nabla\psi + \Phi'_\psi \nabla\psi\times\nabla\theta^\star)\cdot\nabla\zeta} \\
&= \frac{\chi'_\psi \nabla\zeta\times\nabla\psi\cdot\nabla\theta^\star}{\Phi'_\psi \nabla\psi\times\nabla\theta^\star\cdot\nabla\zeta} \\
&= \frac{\chi'_\psi}{\Phi'_\psi} \text{ a function of flux surface } \psi \text{ alone}
\end{aligned}$$

The contravariant basis vectors are

$$\mathbf{e}^i \equiv \nabla\alpha, \alpha = (\psi, \theta, \zeta)$$

The covariant basis vectors are
$\mathbf{e}_i \equiv \frac{\partial \mathbf{x}}{\partial \alpha_i} = \sqrt{g}\mathbf{e}^j \times \mathbf{e}^k, (i, j, k)$ **forms a positive triplet**
$\sqrt{g} = (\nabla\psi \cdot \nabla\theta \times \nabla\zeta)^{-1}$ **is the Jacobian.**

The metric coefficient g_{ij} and g^{ij} forms the matrix of metric tensor

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial \mathbf{R}}{\partial u^i} \cdot \frac{\partial \mathbf{R}}{\partial u^j}, \qquad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j = \nabla u^i \cdot \nabla u^j$$

Obviously they are symmetric:

$$g_{ij} = g_{ji}, \qquad g^{ij} = g^{ji}$$

They are orthogonal coordinate system as well if, and only if:

$$g_{ij} = 0, \qquad g^{ij} = 0 \text{ if} i \neq j$$

The determinant of the tensor is given by

$$g = det[g_{ij}]$$

The contravariant basis vectors are

$$\mathbf{e}^i \equiv \nabla\alpha, \alpha = (\psi, \theta, \zeta)$$

The covariant basis vectors are
$\mathbf{e}_i \equiv \frac{\partial \mathbf{x}}{\partial \alpha_i} = \sqrt{g}\mathbf{e}^j \times \mathbf{e}^k, (i, j, k)$ **forms a positive triplet**
$\sqrt{g} = (\nabla\psi \cdot \nabla\theta \times \nabla\zeta)^{-1}$ **is the Jacobian.**

---

g is related to the transformation Jacobian in the following way:

$$g_{ij} = \frac{\partial \mathbf{R}}{\partial u^i} \cdot \frac{\partial \mathbf{R}}{\partial u^j} = \frac{\partial x}{\partial u^i}\frac{\partial x}{\partial u^j} + \frac{\partial y}{\partial u^i}\frac{\partial y}{\partial u^j} + \frac{\partial z}{\partial u^i}\frac{\partial z}{\partial u^j}$$

$$g = \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^1} \\ \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^2} \\ \frac{\partial z}{\partial u^3} & \frac{\partial z}{\partial u^3} & \frac{\partial z}{\partial u^3} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^2} & \frac{\partial x}{\partial u^3} \\ \frac{\partial y}{\partial u^1} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^3} \\ \frac{\partial z}{\partial u^1} & \frac{\partial z}{\partial u^2} & \frac{\partial z}{\partial u^3} \end{vmatrix} = (\frac{\partial \mathbf{R}}{\partial u^1} \cdot \frac{\partial \mathbf{R}}{\partial u^2} \times \frac{\partial \mathbf{R}}{\partial u^3})^2$$

Thus $\sqrt{g}$ is the Jacobian of coordinate transformation

and from previous slides we know

$$\sqrt{g}^{-1} = \nabla\psi \cdot \nabla\theta \times \nabla\zeta$$

Thus, from eq2, the contravariant components of the magnetic field are $B^j \equiv \mathbf{B} \cdot \mathbf{e}^i$ where

$$B^\theta = \frac{1}{\sqrt{g}}(\chi' - \Phi' \frac{\partial \lambda}{\partial \zeta})$$
$$B^\zeta = \frac{1}{\sqrt{g}}\Phi'(1 + \frac{\partial \lambda}{\partial \theta})$$
$$B^\psi = 0$$

$$\begin{aligned}
B^\theta = \mathbf{B} \cdot \nabla\theta \quad &= \sqrt{g}^{-1}\chi'_\psi + \Phi'_\psi \nabla\psi \times (\lambda'_\zeta \nabla\zeta) \cdot \nabla\theta \\
&= \sqrt{g}^{-1}\chi'_\psi - \Phi'_\psi \lambda'_\zeta \nabla\theta \cdot (\nabla\zeta) \times \nabla\psi \\
&= \sqrt{g}^{-1}(\chi'_\psi - \Phi'_\psi \lambda'_\zeta) \\
B^\zeta = \mathbf{B} \cdot \nabla\zeta \quad &= \Phi'_\psi \nabla\psi \times (1 + \lambda'_\theta \nabla\theta) \cdot \nabla\zeta \\
&= \sqrt{g}^{-1}(\Phi'_\psi (1 + \lambda'_\theta) \\
B^\psi = \mathbf{B} \cdot \nabla\psi \quad &= 0
\end{aligned}$$

The covariant components

$$B_i \equiv \mathbf{B} \cdot \mathbf{e}_i$$

are related to B^i through the metric tensor

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$$

$$B_i = B^\theta g_{\theta i} + B^\zeta g_{\zeta i}$$

as can be verified by taking the scalar product of eq2 with $\mathbf{e}_i$.

Since

$$\mathbf{e}_i \cdot \mathbf{e}^j = \delta_i^j$$

$$B_\psi = \mathbf{B} \cdot \mathbf{e}_\psi = (B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta) \cdot \mathbf{e}_\psi = B^\theta g_{\theta\psi} + B^\zeta g_{\zeta\psi}$$

$$B_\theta = \mathbf{B} \cdot \mathbf{e}_\theta = B^\theta g_{\theta\theta} + B^\zeta g_{\zeta\theta}$$

$$B_\zeta = \mathbf{B} \cdot \mathbf{e}_\zeta = B^\theta g_{\theta\zeta} + B^\zeta g_{\zeta\zeta}$$

Although the function $\lambda(\psi, \theta, \zeta)$ in eq3a can be eliminated by taking $\theta = \theta^\star$ as retention here provides flexibility in specifying the poloidal angle θ. The role of the poloidal angle in the moment exponent of equilibria is to yield rapidly convergent Fourier series for the spatial coordinates $\mathbf{x}(\psi, \theta, \zeta)$.

---

make use of fft to compute Fourier series

$$x = \sum_{m=0} \sum_{n=0} x_{mn}(\psi) e^{i(m\theta - n\zeta)}$$

The complex coefficients x_{mn} can be computed from the coefficients of real Fourier series

$$x = \sum_{m=0} \sum_{n=0} a_{mn}(\psi) \sin(m\theta - n\zeta)$$
$$x = \sum_{m=1} \sum_{n=1} b_{mn}(\psi) \cos(m\theta - n\zeta)$$

FFT

adopt the definition used in the Matlab

It is common to assume that we are given a periodic sequence of numbers $\{f_k\}_{k=1}^N$ of period N.

Then the DFT of the sequence is a sequence F_n for $n = 1, \cdots, N$ defined by

$$F_n = \sum_{k=1}^{N} f_k e^{-2\pi i(n-1)(k-1)/N}.$$

the Inverse Discrete Fourier Transform (IDFT) gives for $k = 1, \cdots, N$

$$f_k = \sum_{n=1}^{N} F_n e^{2\pi i(n-1)(k-1)/N}.$$

Relation to Fourier Transform

the FT of a causal signal $f(t)$, which is effectively zero for $t > T$

1. For window size T and number of sampling points N define the sample spacing $T_s = \frac{T}{N}$
2. Now define the sample points $t_k = kT_s$ for $k = 0, \cdots, (N-1)$, (i.e., $t_k = k\frac{T}{N}$)
3. Then we define $f_k = f(t_k)$.
4. Associated with this we define the frequency sampling points $\omega_n \equiv \frac{2\pi n}{T}$
   where the number $\frac{2\pi}{T}$ is the fundamental frequency.
   Now we consider the problem of approximating the FT of f at the points $\omega_n \equiv \frac{2\pi n}{T}$.
   The exact answer is
   $$F(\omega_n) = \int_{-\infty}^{\infty} e^{-i\omega_n t} f(t)dt, n = 0, \cdots, (N-1)$$
5. For $f \sim 0$ for $|x| > T$,
   $$F(\omega_n) \simeq \int_0^T e^{-i\omega_n t} f(t)dt, n = 0, \cdots, (N-1)$$
6. Let us approximate this integral by a left-endpoint Riemann sum approximation using the points t_k defined above:
   $$F(\omega_n) \simeq T_s \sum_{k=0}^{N-1} e^{-i\omega_n t_k} f(t_k), n = 0, \cdots, (N-1)$$
7. Substituting the definitions of ω_n, t_k and T_s in terms of T and N we have
   $$F(\omega_n) \simeq \frac{T}{N} \sum_{k=0}^{N-1} e^{-2\pi i n/N} f(t_k), n = 0, \cdots, (N-1)$$

8. Thus we see that

$$F(\frac{2\pi n}{T}) \simeq \frac{T}{N} F_n$$

Relation to Fourier Series

if f is a T periodic function defined initially on $[-T/2, T/2]$ then

$$f \sim \sum_{-\infty}^{\infty} c_n e^{2\pi i n/T}$$
$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(x) e^{-2\pi i n x/T} dx$$

Now from what we have written above, for the approximation of the FT of a function which is zero outside the interval $[-T/2, T/2]$, we see that

$$c_n \sim \frac{1}{T}\frac{T}{N}\sum_{-N/2+1}^{N/2} f_k e^{-2\pi i n k/N}, n = -N/2+1, \cdots, N/2$$

Thus by our definition of the DFT we have

$$c_n = \frac{1}{N} F_n$$

Since only truncated series are used, a proper choice for θ is necessary to provide adequate accuracy in the approximate moment solution. In general, this value for θ is incompatible with the requirement that magnetic field lines are straight in $(\theta^\star, \zeta)$ coordinates. The inclusion of λ therefore generates a convenient resummation of the inverse equilibrium Fourier moment expansion. In this context, λ assumes the role of a renormalization parameter. Inserting eq2 into eq1a yields

$$\mathbf{F} = F_\psi \nabla\psi + F_\beta \beta$$

$F_\psi = \sqrt{g}(J^{\boldsymbol{\xi}} B^\theta - J^\theta B^\zeta) + p'$
$F_\beta = J^\beta$
$\beta = \sqrt{g}(B^\zeta \nabla\theta - B^\theta \nabla zeta)$
$J^i \equiv \mathbf{J} \cdot \nabla\alpha_i = \mu_m^{-1} \nabla \cdot \mathbf{B} \times \nabla\alpha_i$

$$\begin{aligned}
\mathbf{F} &= -\mathbf{J} \times \mathbf{B} + \nabla p \\
&= -(J^\psi \mathbf{e}_\psi + J^\theta \mathbf{e}_\theta + J^\zeta \mathbf{e}_\zeta) \times (B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta) + p' \nabla\psi \\
&= -(J^\psi B^\theta \mathbf{e}_\psi \times \mathbf{e}_\theta + J^\psi B^\zeta \mathbf{e}_\psi \times \mathbf{e}_\zeta + J^\theta B^\zeta \mathbf{e}_\theta \times \mathbf{e}_\zeta + J^\zeta B^\theta \mathbf{e}_\zeta \times \mathbf{e}_\theta) + p' \nabla\psi \\
&= -(J^\psi B^\theta \sqrt{g} \nabla\zeta - J^\psi B^\zeta \sqrt{g} \nabla\theta + J^\theta B^\zeta \sqrt{g} \nabla\psi - J^\zeta B^\theta \sqrt{g} \nabla\psi) + p' \nabla\psi \\
&= J^\psi \sqrt{g}(B^\zeta \nabla\theta - B^\theta \nabla\zeta) + \left[\sqrt{g}(J^\zeta B^\theta - J^\theta B^\zeta) + p'\right] \nabla\psi \\
&= F_\beta \beta + F_\psi \nabla\psi
\end{aligned}$$

$$J^\psi = \frac{1}{\mu_0} \nabla \times \mathbf{B} \cdot \nabla\psi = \frac{1}{\mu_0}(\nabla \times \mathbf{B} \cdot \nabla\psi - \mathbf{B} \cdot \nabla \times \nabla\psi) = \frac{1}{\mu_0} \nabla \cdot (\mathbf{B} \times \nabla\psi)$$

There are only 2 independent components of $\mathbf{F}$, since the component

$$\mathbf{B} \cdot \mathbf{F} = p' \cdot \nabla\psi = 0$$

is already incorporated into the representation of B in eq2.

$$\mathbf{B} \cdot \nabla\psi = 0$$

$$\begin{aligned}
\mathbf{B} \cdot \mathbf{F} &= F_\psi \mathbf{B} \cdot \nabla\psi + F_\beta \mathbf{B} \cdot \beta \\
&= F_\beta (B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta) \cdot \sqrt{g}(B^\zeta \nabla\theta - B^\theta \nabla\zeta) \\
&= \sqrt{g} F_\beta (B^\theta B^\zeta \mathbf{e}_\theta \cdot \nabla\theta - B^\zeta B^\theta \mathbf{e}_\zeta \cdot \nabla\zeta) \\
&= \sqrt{g} F_\beta (B^\theta B^\zeta - B^\zeta B^\theta) \\
&= 0
\end{aligned}$$

Writing J^i in terms of the covariant components of B yields expressions for the forces in terms of the flux functions, χ', Φ', p' and the metric:

$$F_\psi = \mu_0^{-1}(B^\theta \frac{\partial B_\theta}{\partial \psi} + B^\zeta \frac{\partial B_\zeta}{\partial \psi} - \mathbf{B} \cdot \nabla B_\psi) + p'$$
$$F_\beta = \frac{1}{\mu_0 \sqrt{g}}(\frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta})$$

As we have already knew

$$\nabla \times \mathbf{D} = \frac{1}{\sqrt{g}} \sum_k (\frac{\partial D_j}{\partial u^i} - \frac{\partial D_i}{\partial u^j} \mathbf{e}_k$$

from Ampere's law, we obtain

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} = \frac{1}{\sqrt{g}} \left[(\frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta} \mathbf{e}_\psi) + (\frac{\partial B_\psi}{\partial \zeta} - \frac{\partial B_\zeta}{\partial \psi} \mathbf{e}_\theta) + (\frac{\partial B_\theta}{\partial \psi} - \frac{\partial B_\psi}{\partial \theta} \mathbf{e}_\zeta) \right]$$

$$J^\zeta = \mathbf{J} \cdot \nabla \zeta = \frac{1}{\mu_0 \sqrt{g}} (\frac{\partial B_\theta}{\partial \psi} - \frac{\partial B_\psi}{\partial \theta} \mathbf{e}_\zeta) \cdot \nabla \zeta$$

$$J^\theta = \mathbf{J} \cdot \nabla \theta = \frac{1}{\mu_0 \sqrt{g}} (\frac{\partial B_\psi}{\partial \zeta} - \frac{\partial B_\zeta}{\partial \psi} \mathbf{e}_\theta) \cdot \nabla \theta$$

$$F_\psi = \frac{1}{\mu_0} (B^\theta \frac{\partial B_\theta}{\partial \psi} - B^\theta \frac{\partial B_\psi}{\partial \theta} - B^\zeta \frac{\partial B_\psi}{\partial \zeta} + B^\zeta \frac{\partial B_\zeta}{\partial \psi}) + p'$$

$$F_\beta \equiv J^\psi = \mathbf{J} \cdot \nabla \psi = \frac{1}{\mu_0 \sqrt{g}} ((\frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta} \mathbf{e}_\psi) \cdot \nabla \psi$$

Here, for any scalar A, the derivative along a magnetic field line is

$$\mathbf{B} \cdot \nabla A = B^\theta \frac{\partial A}{\partial \theta} + B^\zeta \frac{\partial A}{\partial \zeta}$$

Magnetic Differential Equation: $B \cdot \nabla A = S$

S: source term

A: unknown single-valued function to be solved for.

2 solvability conditions on S.

1. $\oint \frac{dl}{B} S = 0$

   coming from $\mathbf{B} \cdot \nabla A = B \frac{\partial A}{\partial l}$, which implies

$$A = A_0 + \int_0^l \frac{dl}{B} S.$$

2. $\langle \mathbf{B} \cdot \nabla A \rangle = \langle S \rangle = 0$

   or $\lim_{\Delta \mathcal{V} \to 0} \int_{\Delta \mathcal{V}} S d^3\mathbf{x} = 0$

As usual in the theory of differential eq, its general solution consists of 2 parts, a homogeneous and a particular solution

$$A = A_h + A_p$$

where A_h

$$\mathbf{B} \cdot \nabla A_h = 0$$

f_p is obtained from Fourier's series approximation.

Equation of a magnetic field line

It is defined as a curve whose tangent at any point is parallel to **B**:

$$\mathbf{B} \propto d\mathbf{R}(\text{ or}\mathbf{B} \times d\mathbf{R} = 0)$$

$d\mathbf{R}$: differential vector tangent to the field line

Using contravariant components of both $\mathbf{B}(B^1, B^2, B^3)$ and $d\mathbf{R}(du^1, du^2, du^3)$ the eq leads to

$$\frac{B^1}{du^1} = \frac{B^2}{du^2} = \frac{B^3}{du^3} = c$$

c: proportionality constant

Recalling $B^i = \mathbf{B} \cdot \nabla u^i$,

$$\frac{\mathbf{B} \cdot \nabla u^1}{du^1} = \frac{\mathbf{B} \cdot \nabla u^2}{du^2} = \frac{\mathbf{B} \cdot \nabla u^3}{du^3} = c$$

How c is determined

If parametrize the filed line curve by its arc length l, the tangent vector $d\mathbf{R}/dl$ is a unit vector, and

$$\frac{\mathbf{B}}{B} = \frac{d\mathbf{R}}{dl}|_{along\mathbf{B}}$$

Thus we have the constant

$$c = \frac{B}{dl}$$

So the equation of a magnetic field line is

$$\frac{B}{dl} = \frac{B^1}{du^1} = \frac{B^2}{du^2} = \frac{B^3}{du^3}$$

or

$$\frac{B}{dl} = \frac{\mathbf{B} \cdot \nabla u^1}{du^1} = \frac{\mathbf{B} \cdot \nabla u^2}{du^2} = \frac{\mathbf{B} \cdot \nabla u^3}{du^3}$$

In the 2D axisymmetric case, $F_\beta = 0$, can be integrated to yield $B_\zeta = F(\psi)$.

---

axisymmetric along angle ζ

$$\frac{\partial}{\partial \zeta} = 0$$
$$\mathbf{e}_\zeta = R\zeta$$
$$\mathbf{e}^\zeta = \frac{1}{R}\zeta$$

$$B_\zeta = \mathbf{B} \cdot e_\zeta = (B^\theta \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta) \cdot \mathbf{e}_\zeta = B^\theta g_{\theta\zeta} + B^\zeta g^{\zeta\zeta} = B^\zeta g^{\zeta\zeta}$$

$$F_\beta = \frac{1}{\mu_0 \sqrt{g}} \left(\frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta} \right) = \frac{1}{\mu_0 \sqrt{g}} \frac{\partial B_\zeta}{\partial \theta}$$

$$\Longrightarrow \frac{\partial B_\zeta}{\partial \theta}$$

together with $\frac{\partial B_\zeta}{\partial \zeta} = 0$, we have B_ζ is a function of ψ.

Noting that

$$g_{\theta\zeta} = g_{\psi\zeta} = 0, \frac{\partial \lambda}{\partial \zeta} = 0$$

due to axisymmetry,

From last slide, we have

$$\frac{\partial \lambda}{\partial \zeta} = 0$$

and

$$B^\zeta = B_\zeta / g_{\zeta\zeta}$$

with

$$g_{\zeta\zeta} = R^2$$

R **is the major radius**

This can be easily verified from cylindrical coordinates

$$\begin{cases} x = R\cos\zeta \\ y = R\sin\zeta \\ z = z \end{cases}$$

$$g_{\zeta\zeta} = \mathbf{e}_\zeta \cdot \mathbf{e}_\zeta = \frac{\partial \mathbf{x}}{\partial \zeta} \cdot \frac{\partial \mathbf{x}}{\partial \zeta} = (-R\sin\zeta, R\cos\zeta, 0) \cdot (-R\sin\zeta, R\cos\zeta, 0) = R^2$$

eq4d becomes the inverse GS eq

$$F_\psi = \frac{\chi'}{\mu_0\sqrt{g}}\left[\frac{\partial}{\partial\psi}\left(\frac{\chi' g_{\theta\theta}}{\sqrt{g}}\right) - \frac{\partial}{\partial\theta}\left(\frac{\chi' g_{\theta\psi}}{\sqrt{g}}\right)\right] + \frac{FF'}{\mu_0 R^2} + p'$$

$$\begin{aligned}
F_\psi &= \frac{1}{\mu_0}(B^\theta\frac{\partial B_\theta}{\partial\psi} + B^\zeta\frac{\partial B_\zeta}{\partial\psi} - B^\theta\frac{\partial B_\psi}{\partial\theta} - B^\zeta\frac{\partial B_\psi}{\partial\zeta}) + p' \\
&= \frac{1}{\mu_0}(B^\theta\frac{\partial}{\partial\psi}(B^\theta g_{\theta\theta} + B^\zeta g_{\zeta\theta}) + B^\zeta\frac{\partial}{\partial\psi}(B^\theta g_{\theta\zeta} + B^\zeta g_{\zeta\zeta}) - B^\theta\frac{\partial}{\partial\theta}(B_\psi) -) + p' \\
&= \frac{1}{\mu_0}(B^\theta\frac{\partial}{\partial\psi}(B^\theta g_{\theta\theta}) + B^\zeta\frac{\partial}{\partial\psi}(B^\zeta g_{\zeta\zeta}) - B^\theta\frac{\partial}{\partial\theta}(B^\theta g_{\theta\psi})) + p' \\
&= \frac{1}{\mu_0}B^\theta(\frac{\partial}{\partial\psi}(\frac{1}{\sqrt{g}}(\chi' - \Phi'\frac{\partial\lambda}{\partial\zeta})g_{\theta\theta}) - \frac{\partial}{\partial\theta}(\frac{1}{\sqrt{g}}(\chi' - \Phi'\frac{\partial\lambda}{\partial\zeta})g_{\theta\psi})) + \frac{1}{\mu_0}B^\zeta\frac{\partial}{\partial\psi}(B^\zeta R^2) + p' \\
&= \frac{1}{\mu_0\sqrt{g}}(\chi' - \Phi'\frac{\partial\lambda}{\partial\zeta})(\frac{\partial}{\partial\psi}(\frac{1}{\sqrt{g}}\chi' g_{\theta\theta}) - \frac{\partial}{\partial\theta}(\frac{1}{\sqrt{g}}\chi' g_{\theta\psi})) + \frac{1}{\mu_0 R^2}(B^\zeta R^2)\frac{\partial}{\partial\psi}(B^\zeta R^2) + p' \\
&= \frac{\chi'}{\mu_0\sqrt{g}}(\frac{\partial}{\partial\psi}(\frac{\chi' g_{\theta\theta}}{\sqrt{g}}) - \frac{\partial}{\partial\theta}(\frac{\chi' g_{\theta\psi}}{\sqrt{g}})) + \frac{1}{\mu_0 R^2}(FF') + p'
\end{aligned}$$

where

$$F \equiv RB^\zeta = \frac{R^2\Phi'}{\sqrt{g}}(1 + \frac{\partial\lambda}{\partial\theta})$$

From eq3c and 3d, note that eq5a cab be written $\frac{R^2\Phi'}{\sqrt{g}}(1+\frac{\partial\lambda}{\partial\theta}) = F(\psi)$ which yields $\Phi'(\psi) = \langle\frac{\sqrt{g}}{R^2}\rangle F(\psi)$ and

$$\frac{\partial\lambda}{\partial\theta} = \frac{\sqrt{g}/R^2}{\langle\sqrt{g}/R^2\rangle} - 1$$

Here the brackets denote a normalization θ average.

From last slide, we have $F \equiv RB^{\zeta} = \frac{R^2\Phi'}{\sqrt{g}}(1+\frac{\partial\lambda}{\partial\theta})$, Thus

$$\Phi'(1+\frac{\partial\lambda}{\partial\theta}) = \frac{\sqrt{g}}{R^2}F$$

Averaging in θ direction

$$\oint \Phi'(1+\frac{\partial\lambda}{\partial\theta})d\theta = \oint \frac{\sqrt{g}}{R^2}Fd\theta$$

$$\Phi' \cdot 2\pi + \Phi'\lambda|_0^{2\pi} = F\oint \frac{\sqrt{g}}{R^2}$$

$$\Phi' = \frac{1}{2\pi}F\oint \frac{\sqrt{g}}{R^2} = F\langle\frac{\sqrt{g}}{R^2}\rangle$$

Hence

$$(1+\frac{\partial\lambda}{\partial\theta}) = \frac{\sqrt{g}}{R^2}F/\Phi' = \frac{\sqrt{g}/R^2}{\langle\sqrt{g}/R^2\rangle}$$

Eq5c shows that in 2D geometry, the straight magnetic field line system for $\lambda = 0$ is one for which $\sqrt{g}/R^2$ is constant on a magnetic surface.

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only constant satisfies

$$\frac{\sqrt{g}}{R^2} = \langle \frac{\sqrt{g}}{R^2} \rangle$$

Consider an equilibrium that is approximated by shifted, elliptical flux surfaces fro which the cylindrical coordinates (R, Z) have the low-order Fourier representation

$$R = R_0(\psi) + R_1(\psi)\cos\theta$$
$$Z = Z_1(\psi)\sin\theta$$

Analysis of this configuration indicates that to leading order in the inverse aspect ration, the condition

$$\frac{\partial}{\partial\theta}(R^2/\sqrt{g}) = 0$$

in the (ψ, θ) coordinate system leads to an unphysical inward shift

$$\Delta = R_0(0) - R_0(\psi) \leq 0$$

which is independent of the plasma pressure.

This type of configuration is for theoretical analysis, practical system

is much more complicate.

the inverse aspect ration,

$$\epsilon = \frac{a}{R}$$

The retention of λ allows for the surface variation of $\sqrt{g}/R^2$ in the (ψ, θ) coordinate system, where the low-order Fourier series representation for (R, Z) is appropriate. It also yields the correct variation of $R_0(\psi)$ with pressure.

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