investigate the small amplitude waves which propagate through a spatially uniform MHD plasma. start by combining the ideal MHD eqs to form a closed set of eqs:

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{V} = 0$$

$$\boldsymbol{\rho\frac{d\mathbf{V}}{dt} + \nabla p - \frac{\nabla\times\mathbf{B}\times\mathbf{B}}{\mu_0} = 0}$$

$$-\frac{\partial\mathbf{B}}{\partial t} + \nabla\times(\mathbf{V}\times\mathbf{B}) = 0$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

Next, we linearized these equations (assuming, for the sake of simplicity, that the equilibrium flow velocity and equilibrium plasma current are both zero) to give

$$\frac{\partial\rho}{\partial t} + \rho_0\nabla\cdot\mathbf{V} = 0$$

$$\boldsymbol{\rho_0\frac{\partial\mathbf{V}}{\partial t} + \nabla p - \frac{\nabla\times\mathbf{B}\times\mathbf{B}_0}{\mu_0} = 0}$$

$$-\frac{\partial\mathbf{B}}{\partial t} + \nabla\times(\mathbf{V}\times\mathbf{B}_0) = 0$$

$$\frac{d}{dt}\left(\frac{p}{p_0} - \frac{\gamma\rho}{\rho_0}\right) = 0$$

the subscript 0 denotes an equilibrium quantity. Perturbed quantities are written without subscripts. Of course, ρ_0, p_0, and $\mathbf{B}_0$ are constants in a spatially uniform plasma.

search for wave-like solutions in which perturbed quantities vary like $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$$-\omega\rho + \rho_0 \mathbf{k}\cdot\mathbf{V} = 0$$

$$-\omega\rho_0\mathbf{V} + \mathbf{k}p - \frac{(\mathbf{k}\times\mathbf{B})\times\mathbf{B}_0}{\mu_0} = 0$$

$$\omega\mathbf{B} + \mathbf{k}\times(\mathbf{V}\times\mathbf{B}_0) = 0$$

$$-\omega(\frac{p}{p_0} - \frac{\gamma\rho}{\rho_0}) = 0$$

Assuming that $\omega \neq 0$, the above equations yield

$$\rho = \rho_0 \frac{\mathbf{k}\cdot\mathbf{V}}{\omega}$$

$$\omega p = \gamma\frac{p_0}{\rho_0}(\omega\rho) = \gamma p_0 \frac{\omega\rho}{\rho_0} = \gamma p_0(\mathbf{k}\cdot\mathbf{V})$$

$$\mathbf{B} = -\frac{\mathbf{k}\times(\mathbf{V}\times\mathbf{B}_0)}{\omega} = \frac{(\mathbf{k}\cdot\mathbf{V})\mathbf{B}_0 - (\mathbf{k}\cdot\mathbf{B}_0)\mathbf{V}}{\omega}$$

Substitution of these expressions into the linearized equation of motion gives

$$
\begin{aligned}
-\omega\rho_0\mathbf{V} &= -\mathbf{k}p - \frac{\mathbf{B}_0\times(\mathbf{k}\times\mathbf{B})}{\mu_0} \\
&= -\gamma p_0\frac{\mathbf{k}\cdot\mathbf{V}}{\omega}\mathbf{k} + \frac{(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}-(\mathbf{B}_0\cdot\mathbf{B})\mathbf{k}}{\mu_0} \\
&= -\gamma p_0\frac{\mathbf{k}\cdot\mathbf{V}}{\omega}\mathbf{k} + \frac{(\mathbf{k}\cdot\mathbf{B}_0)}{\mu_0}\frac{(\mathbf{k}\cdot\mathbf{V})\mathbf{B}_0-(\mathbf{k}\cdot\mathbf{B}_0)\mathbf{V}}{\omega} - \frac{\mathbf{k}}{\mu_0}\mathbf{B}_0\cdot\frac{(\mathbf{k}\cdot\mathbf{V})\mathbf{B}_0-(\mathbf{k}\cdot\mathbf{B}_0)\mathbf{V}}{\omega}
\end{aligned}
$$

$$
-\omega^2\mathbf{V} = -\frac{\gamma p_0}{\rho_0}(\mathbf{k}\cdot\mathbf{V})\mathbf{k} - \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\mu_0\rho_0}\mathbf{V} + \frac{(\mathbf{k}\cdot\mathbf{B}_0)}{\mu_0\rho_0}(\mathbf{k}\cdot\mathbf{V})\mathbf{B}_0 - \frac{(\mathbf{B}_0\cdot\mathbf{B}_0)}{\mu_0\rho_0}(\mathbf{k}\cdot\mathbf{V})\mathbf{k} + \frac{(\mathbf{k}\cdot\mathbf{B}_0)}{\mu_0\rho_0}(\mathbf{B}_0\cdot\mathbf{V})\mathbf{k}
$$

$$
\left[\omega^2 - \frac{(\mathbf{k}\cdot\mathbf{B}_0)^2}{\mu_0\rho_0}\right]\mathbf{V} = \left[\left(\frac{\gamma p_0}{\rho_0} + \frac{\mathbf{B}_0^2}{\mu_0\rho_0}\right)\mathbf{k} - \frac{(\mathbf{k}\cdot\mathbf{B}_0)}{\mu_0\rho_0}\mathbf{B}_0\right](\mathbf{k}\cdot\mathbf{V}) - \frac{(\mathbf{k}\cdot\mathbf{B}_0)(\mathbf{V}\cdot\mathbf{B}_0)}{\mu_0\rho_0}\mathbf{k}
$$

in (x, y, z) coordinate system, assume

$\mathbf{B}_0 = B_0\mathbf{z}$

$\mathbf{k} = k_x\mathbf{x} + k_z\mathbf{z}$

θ be the angle subtended between $\mathbf{B}_0$ and $\mathbf{k}$

So

$$\mathbf{k}_x = k\sin\theta,$$

$$\mathbf{k}_z = k\cos\theta$$

Define

Alfven speed $V_A = \sqrt{\frac{B_0^2}{\mu_0\rho_0}}$

sound speed $V_S = \sqrt{\frac{\gamma p_0}{\rho_0}}$

The equation of motion can be rewritten as

$$\left[\omega^2 - k^2V_A^2\cos\theta^2\right]\mathbf{V} = \left[(V_S^2 + V_A^2)\mathbf{k} - k_zV_A^2\mathbf{z}\right](k_xV_x + k_zV_z) - k_zV_zV_A^2\mathbf{k}$$

its x component is

$$\left[\omega^2 - k^2 V_A^2 \cos^2\theta\right] V_x$$

$$= (V_S^2 + V_A^2) k_x (k_x V_x + k_z V_z) - k_z V_z V_A^2 k_x$$

$$= (V_S^2 + V_A^2)(k_x^2 V_x + k_x k_z V_z) - k_x k_z V_A^2 V_z$$

$$= k^2 V_S^2 \sin^2\theta V_x + k^2 V_S^2 \sin\theta\cos\theta V_z + k^2 V_A^2 \sin^2\theta V_x$$

its y component is

$$\left[\omega^2 - k^2 V_A^2 \cos^2\theta\right] V_y = 0$$

its z component is

$$\left[\omega^2 - k^2 V_A^2 \cos^2\theta\right] V_z$$

$$= (V_S^2 + V_A^2) k_z (k_x V_x + k_z V_z) - k_z V_A^2 (k_x V_x + k_z V_z) - k_z V_z V_A^2 k_z$$

$$= (V_S^2 + V_A^2)(k_x k_z V_x + k_z^2 V_z) - V_A^2 (k_x k_z V_x + k_z^2 V_z) - k_z V_z V_A^2 k_z$$

$$= k_x k_z V_S^2 V_x + k_x k_z V_A^2 V_x + k_z^2 V_S^2 V_z + k_z^2 V_A^2 V_z - k_x k_z V_A^2 V_x - k_z^2 V_A^2 V_z - k_z^2 V_A^2 V_z$$

$$= k^2 V_S^2 \sin\theta\cos\theta V_x + k^2 V_S^2 \cos\theta^2 V_z - k^2 V_A^2 \cos^2\theta V_z - k^2 V_A^2 \cos^2\theta V_z$$

Thus the equation of motion reduces to the eigenvalue problem

$$\begin{bmatrix} \omega^2 - k^2 V_A^2 - k^2 V_S^2 \sin^2\theta & 0 & -k^2 V_S^2 \sin\theta\cos\theta \\ 0 & \omega^2 - k^2 V_A^2 \cos^2\theta & 0 \\ -k^2 V_S^2 \sin\theta\cos\theta & 0 & \omega^2 - k^2 V_S^2 \cos^2\theta \end{bmatrix}$$

The solubility condition is that the determinant of the square matrix is zero.

This yields the dispersion relation

$$(\omega^2 - k^2 V_A^2 \cos^2\theta)\left[\omega^4 - \omega^2 k^2 (V_S^2 + V_A^2) + k^2 V_A^2 V_S^2 \cos^2\theta\right] = 0$$

There are three independent roots of the above dispersion relation corresponding to the three different types of wave that can propagate through an MHD plasma.

The first, and most obvious, root is

$$\omega = kV_A \cos\theta$$

which has the associated eigenvector $(0, V_y, 0)$

This root is characterized by both $\mathbf{k} \cdot \mathbf{V} = 0$ and $\mathbf{v} \cdot \mathbf{B}_0 = 0$.

It immediately follows that there is zero perturbation of the plasma density or pressure associated with this root

In fact, this root can easily be identified as the shear-Alfven wave, which only involves plasma motion perpendicular to the magnetic field.

The remaining two roots of the dispersion relation (721) are written

$$\omega = kV+$$

and

$$\omega = kV-$$

$$V_{\pm} = \left\{ \frac{1}{2} \left[V_A^2 + V_S^2 \pm \sqrt{(V_A^2 + V_S^2)^2 - 4V_A^2 V_S^2 \cos^2\theta} \right] \right\}^{1/2}$$

Note that $V_+ \geq V_-$.

The first root V_+ : termed as the fast magnetosonic wave, or fast wave

the second root V_- : called the slow magnetosonic wave, or slow wave

The eigenvectors for these waves are $(V_x, 0, V_y)$ and thus

$\mathbf{k} \cdot \mathbf{V} \neq 0$, $\mathbf{V} \cdot \mathbf{B}_0 \neq 0$

these waves are associated with non-zero perturbations in the plasma density and pressure, and also involve plasma motion parallel, as well as perpendicular, to the magnetic field.

In order to better understand the nature of the fast and slow waves, let us consider the cold-plasma limit,
which is obtained by letting the sound speed V_S tend to zero.
In this limit, the slow wave ceases to exist (in fact, its phase velocity tends to zero)
whereas the dispersion relation for the fast wave reduces to

$$\omega = kV_A$$

This can be identified as the dispersion relation for the compressional-Alfven wave
Thus, we can identify the fast wave as the compressional-Alfven wave modified by a non-zero plasma pressure.

In the limit $V_A \gg V_S$, which is appropriate to low-β plasmas

$$\beta = \frac{2\mu_0 p}{B^2} = \frac{p}{\rho}\frac{2\mu_0\rho}{B^2} \sim (\frac{V_S}{V_A})^2 \ll 1$$

the dispersion relation for the slow wave reduces to

$$\omega = kV_S \cos\theta$$

This is actually the dispersion relation of a sound wave propagating along magnetic field-lines. Thus, in low-β plasmas the slow wave is a sound wave modified by the presence of the magnetic field.

$$\begin{aligned}
\sqrt{(V_A^2+V_S^2)^2 - 4V_A^2V_S^2\cos^2\theta} &= (V_A^2+V_S^2)\sqrt{(1-\frac{4V_A^2V_S^2\cos^2\theta}{(V_A^2+V_S^2)^2}} \\
&\sim (V_A^2+V_S^2)(1-\frac{1}{2}\frac{4V_A^2V_S^2\cos^2\theta}{(V_A^2+V_S^2)^2}) \\
&= (V_A^2+V_S^2) - \frac{2V_A^2V_S^2\cos^2\theta}{V_A^2+V_S^2} \\
&= (V_A^2+V_S^2) - \frac{2V_S^2\cos^2\theta}{1+\frac{V_S^2}{V_S^2}} \\
&\sim (V_A^2+V_S^2) - 2V_S^2\cos^2\theta
\end{aligned}$$

The distinction between the fast and slow waves can be further understood by comparing the signs of the wave induced fluctuations in the plasma and magnetic pressures:

$$\frac{\mathbf{B}_0 \cdot \mathbf{B}}{\mu_0} = \frac{(\mathbf{k} \cdot \mathbf{V})B_0^2 - (\mathbf{k} \cdot \mathbf{B}_0)(\mathbf{B}_0 \cdot \mathbf{V})}{\mu_0 \omega}$$

From the z component of linearized equation of motion

$$\omega \rho_0 V_z = kp \cos\theta$$

Combining and thus obtaining

$$\begin{aligned}
\frac{\mathbf{B}_0 \cdot \mathbf{B}}{\mu_0} &= \frac{B_0^2}{\mu_0}\frac{p}{\gamma p_0} - \frac{B_0^2}{\mu_0 \rho_0}\frac{k^2}{\omega^2}\cos^2\theta p \\
&= (\frac{B_0^2}{\mu_0 \rho_0}\frac{\rho_0}{\gamma p_0} - \frac{B_0^2}{\mu_0 \rho_0}\frac{k^2}{\omega^2}\cos^2\theta)p \\
&= (\frac{V_A^2}{V_S^2} - V_A^2\frac{k^2}{\omega^2}\cos^2\theta)p \\
&= \frac{V_A^2}{V_S^2}(1 - \frac{k^2 V_S^2 \cos^2\theta}{\omega^2})p
\end{aligned}$$

Hence
p and $\frac{\mathbf{B}_0 \cdot \mathbf{B}}{\mu_0}$ have the same sign if $V > V_S \cos\theta$
and the opposite sign if $V < V_S \cos\theta$

$V = \frac{\omega}{k}$ is the phase velocity.

It is straightforward to show that $V_+ > V_S \cos\theta$ and $V_- < V_S \cos\theta$

$$\omega^2 = \frac{1}{2}\left[k^2(V_A^2 + V_S^2) \pm \sqrt{k^4(V_A^2 + V_S^2)^2 - 4k^4 V_A^2 V_S^2 \cos^2\theta}\right]$$

$$V^2 \equiv \frac{\omega^2}{k^2} = \frac{1}{2}(V_A^2 + V_S^2) \pm \frac{1}{2}\sqrt{(V_A^2 + V_S^2)^2 - 4V_A^2 V_S^2 \cos^2\theta}$$

$$\begin{aligned} V_+^2 &= \tfrac{1}{2}(V_A^2 + V_S^2) + \tfrac{1}{2}\sqrt{(V_A^2 + V_S^2)^2 - 4V_A^2 V_S^2 \cos^2\theta} \\ &\geq \tfrac{1}{2}(V_A^2 + V_S^2) + \tfrac{1}{2}\sqrt{(V_A^2 + V_S^2)^2 - 4V_A^2 V_S^2} \\ &= \tfrac{1}{2}(V_A^2 + V_S^2) + \tfrac{1}{2}(V_A^2 - V_S^2) \\ &= V_A^2 \end{aligned}$$

or

$$\begin{aligned} &= \tfrac{1}{2}(V_A^2 + V_S^2) + \tfrac{1}{2}(V_S^2 - V_A^2) \\ &= V_S^2 \end{aligned}$$

for $0 \leq \theta \leq \pi$, $\cos\theta > 0$, so

$$V_+ > V_S \cos\theta$$

Similarly

$$
\begin{aligned}
V_-^2 &= \tfrac{1}{2}(V_A^2 + V_S^2) - \tfrac{1}{2}\sqrt{(V_A^2 + V_S^2)^2 - 4V_A^2 V_S^2 \cos^2\theta} \\
&\leq \tfrac{1}{2}(V_A^2 + V_S^2) - \tfrac{1}{2}\sqrt{(V_A^2 + V_S^2)^2 - 4V_A^2 V_S^2} \\
&= \tfrac{1}{2}(V_A^2 + V_S^2) - \tfrac{1}{2}(V_A^2 - V_S^2) \\
&= V_A^2
\end{aligned}
$$

or

$$
\begin{aligned}
&= \tfrac{1}{2}(V_A^2 + V_S^2) - \tfrac{1}{2}(V_S^2 - V_A^2) \\
&= V_S^2
\end{aligned}
$$

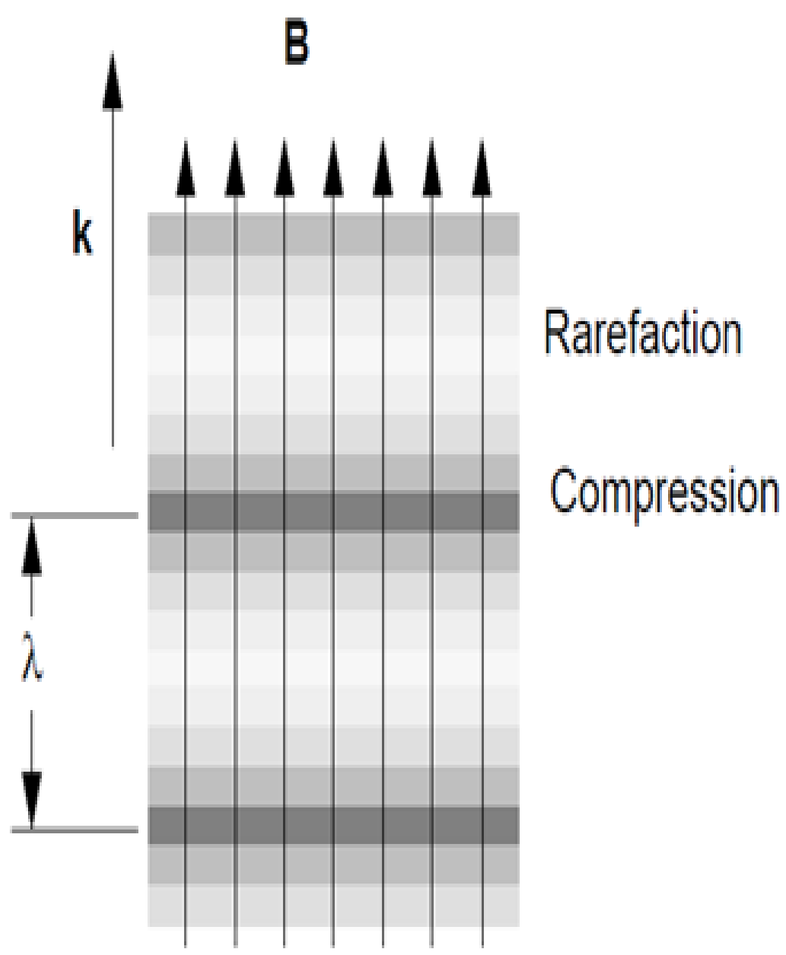
and

$$V_- < V_S \cos\theta$$

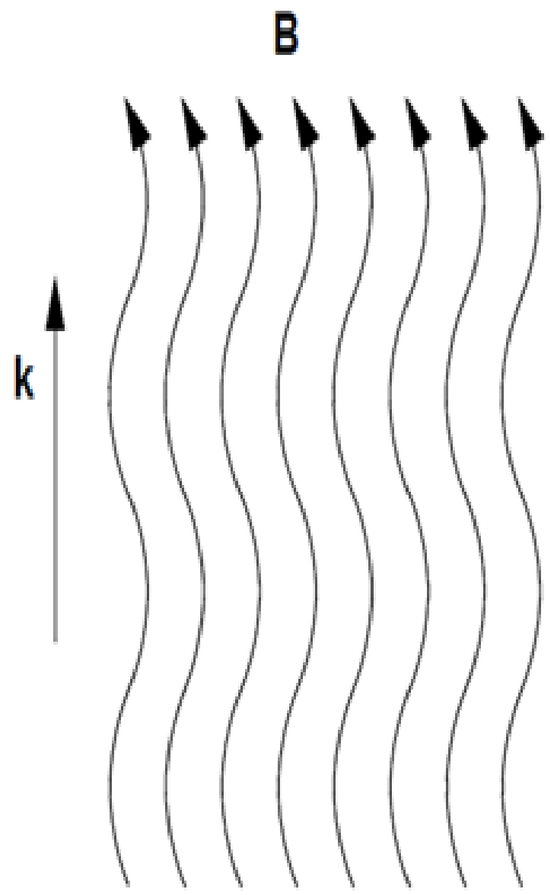
Thus, we conclude that in the fast magnetosonic wave the pressure and magnetic energy fluctuations reinforce one another,

whereas the fluctuations oppose one another in the slow magnetosonic wave.

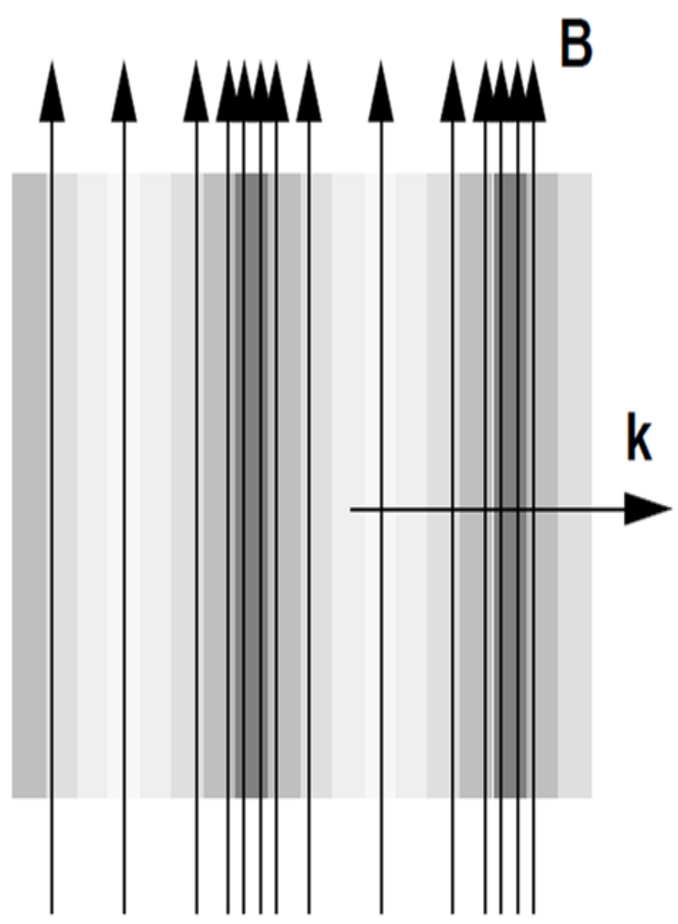
The slow wave always has a smaller phase velocity than the shear-Alfven wave
which, in turn, always has a smaller phase velocity than the fast wave.



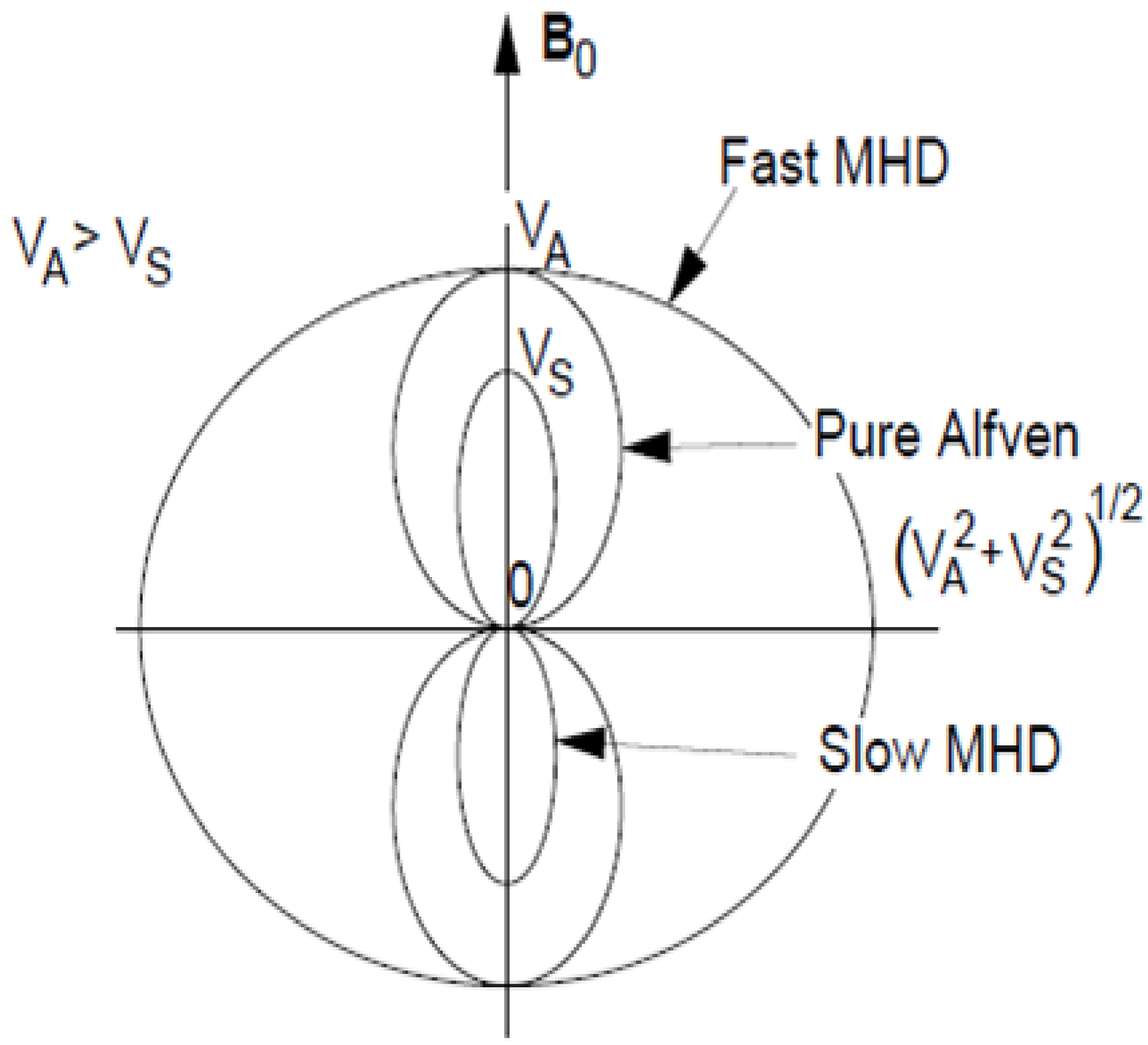
slow wave ($\parallel \mathbf{B}_0$)

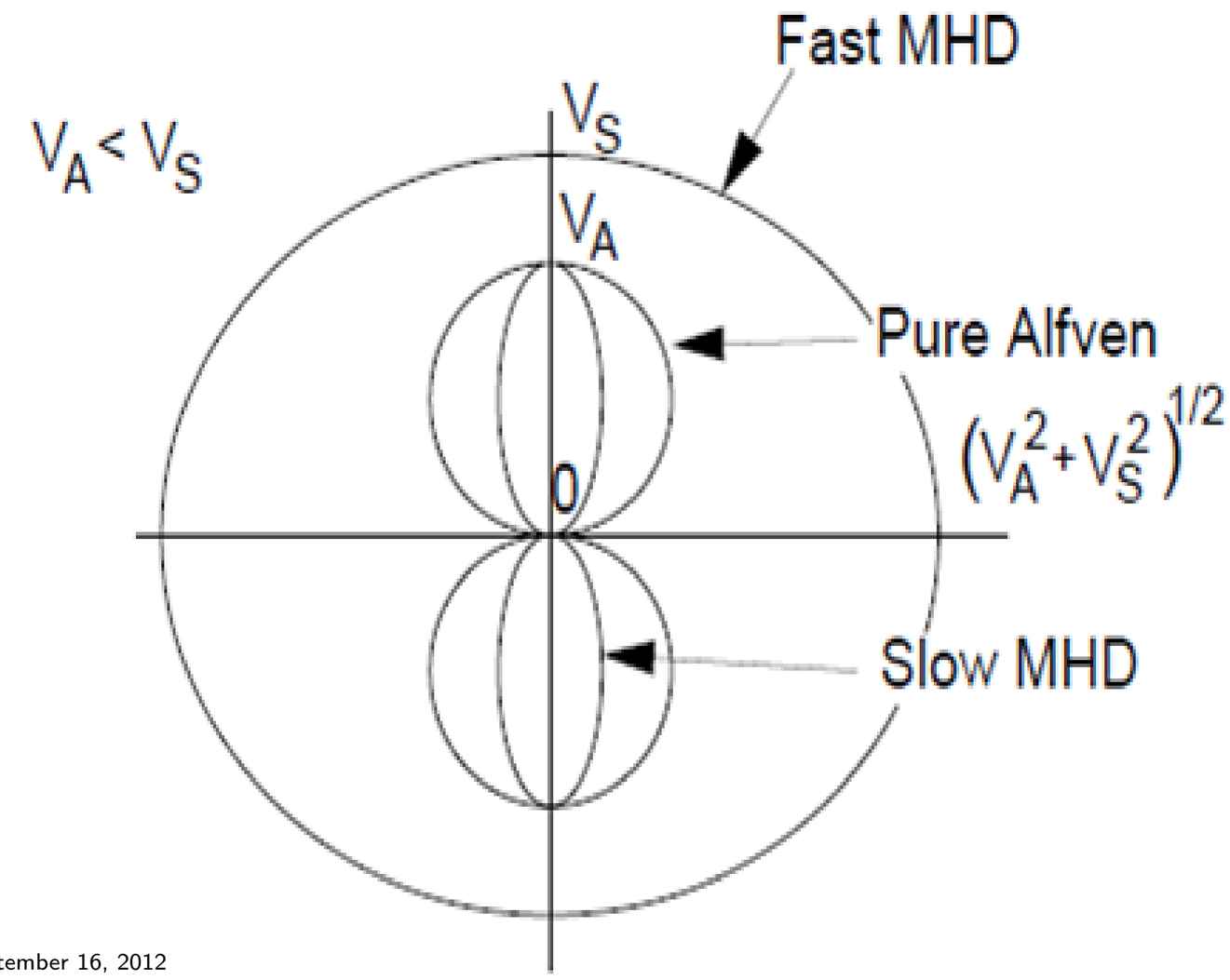


Alfven wave ($\parallel \mathbf{B}_0$)



fast wave ($\perp \mathbf{B}_0$)


B0
Fast MHD
VA > VS
VA
VS
Pure Alfven
(VA2+VS2)1/2
0
Slow MHD


Fast MHD
VA < VS
VS
VA
Pure Alfven
(VA2+VS2)1/2
0
Slow MHD

The Second Variation

The linearized ideal MHD eqs are found by looking for solution

which deviate only infinitesimally from the equilibrium,

and by keeping terms only to $1st$ order in this small deviation.

We take as equilibrium quantities

$\mathbf{J}_0(\mathbf{x})$ the current density
$\mathbf{B}_0(\mathbf{x})$ the magnetic field
$p_0(\mathbf{x})$ the pressure
$\mathbf{V}_0(\mathbf{x})$ the fluid velocity

They satisfy

$$\mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0$$
$$\nabla \times \mathbf{B}_0 = \mu_0 \mathbf{J}_0$$
$$\nabla \cdot \mathbf{B}_0 = 0$$
$$\mathbf{V}_0 = 0$$

The Second Variation

The perturbed variables $\mathbf{B}_1$, $\mathbf{J}_1$, p_1, $\mathbf{V}_1$ are added into $\mathbf{B}$, $\mathbf{J}$, p, $\mathbf{V}$

$$\mathbf{B}(\mathbf{x},t) = \mathbf{B}_0(\mathbf{x}) + \mathbf{B}_1(\mathbf{x},t)$$
$$\mathbf{J}(\mathbf{x},t) = \mathbf{J}_0(\mathbf{x}) + \mathbf{J}_1(\mathbf{x},t)$$
$$p(\mathbf{x},t) = p_0(\mathbf{x}) + p_1(\mathbf{x},t)$$
$$\mathbf{V}(\mathbf{x},t) = \mathbf{V}_1(\mathbf{x},t)$$

Assume that they are much smaller than their equilibrium parts.

From here on, we drop the subscript "0" from the equilibrium quantities.

The Second Variation

Introducing a displacement field $\boldsymbol{\xi}(\mathbf{x}, t)$,

which is just the displacement of a fluid element originally located at position $\mathbf{x}$.

Taking as initial conditions that the perturbed quantities vanish at $t = 0$

$$\boldsymbol{\xi}(\mathbf{x}, 0) = \mathbf{B}_1(\mathbf{x}, 0) = p_1(\mathbf{x}, 0) = 0$$

the linearized ideal MHD eqs take the form

$$p_1 = -\boldsymbol{\xi} \cdot \nabla p - \gamma p \nabla \cdot \boldsymbol{\xi}$$

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$$

$$\mathbf{J}_1 = \frac{1}{\mu_0} \nabla \times \mathbf{B}_1$$

The linearized momentum eq then becomes

$$\rho \frac{\partial^2}{\partial t^2} \boldsymbol{\xi} = \mathbf{J} \times \mathbf{B}_1 + \mathbf{J}_1 \times \mathbf{B} - \nabla p \equiv \mathbf{F}(\boldsymbol{\xi})$$

$$\mathbf{F}(\boldsymbol{\xi}) = \frac{1}{\mu_0} [\nabla \times \mathbf{B} \times \mathbf{B}_1 + \nabla \times \mathbf{B}_1 \times \mathbf{B}] + \nabla(\boldsymbol{\xi} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{\xi})$$

It is known as the linear eq of motion for an ideal MHD plasma.

The Second Variation

We look for separable eigenmode solution. If we assume time dependence

$$\boldsymbol{\xi}(\mathbf{x}, t) = \boldsymbol{\xi}(\mathbf{x}) e^{-i\omega t}$$

Then it takes the time independent form

$$-\rho_0 \omega^2 \boldsymbol{\xi}(\mathbf{x}) = \mathbf{F}(\boldsymbol{\xi})$$

This is an eigenvalue eq for the frequency ω^2.

A variational statement is possible and offer many advantages over a direct solution since $\mathbf{F}$ is self-adjoint

$$\int d^3\mathbf{x} \boldsymbol{\eta} \cdot \mathbf{F}(\boldsymbol{\xi}) = \int d^3\mathbf{x} \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\eta})$$

for any two vector fields $\boldsymbol{\eta}(\mathbf{x})$ and $\boldsymbol{\xi}(\mathbf{x})$.

It follows immediately that

1. The eigenvalues of $\mathbf{F}$ are real.

   Subtracting the inner product of $\boldsymbol{\xi}^\star(\mathbf{x})$, the complex conjugate of $\boldsymbol{\xi}(\mathbf{x})$, with $\mathbf{F}(\boldsymbol{\xi})$ from the inner product of $\boldsymbol{\xi}(\mathbf{x})$ with $\mathbf{F}(\boldsymbol{\xi}^\star)$,

$$(\omega^2 - \omega^{\star 2})\rho_0 \int d^3\mathbf{x}|\boldsymbol{\xi}|^2 = -\int d^3\mathbf{x}[\boldsymbol{\xi}^\star \cdot \mathbf{F}(\boldsymbol{\xi}) - \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}^\star)] = 0$$

   Thus

$$\omega^2 = \omega^{\star 2}$$

   or, in words, ω^2 is purely real.

2. The eigenvectors of $\mathbf{F}$ are orthogonal.

   Suppose $\boldsymbol{\xi}_n$ is the eigenvectors of $\mathbf{F}$ with eigenvalues ω_n and $\boldsymbol{\xi}_m$ is the eigenvectors of $\mathbf{F}$ with eigenvalues ω_m

   We subtract

$$\int d^3\mathbf{x}\boldsymbol{\xi}_m \cdot \left[-\rho_0\omega_n^2\boldsymbol{\xi}_n = \mathbf{F}(\boldsymbol{\xi}_n)\right]$$

   from

$$\int d^3\mathbf{x}\boldsymbol{\xi}_n \cdot \left[-\rho_0\omega_m^2\boldsymbol{\xi}_m = \mathbf{F}(\boldsymbol{\xi}_m)\right]$$

   to obtain

$$\rho_0(\omega_n^2-\omega_m^2)\int d^3\mathbf{x}\boldsymbol{\xi}_m^\star\cdot\boldsymbol{\xi}_n = \int d^3\mathbf{x}(\boldsymbol{\xi}_m^\star\cdot\mathbf{F}(\boldsymbol{\xi}_n)-\boldsymbol{\xi}_n\cdot\mathbf{F}(\boldsymbol{\xi}_m^\star) = 0$$

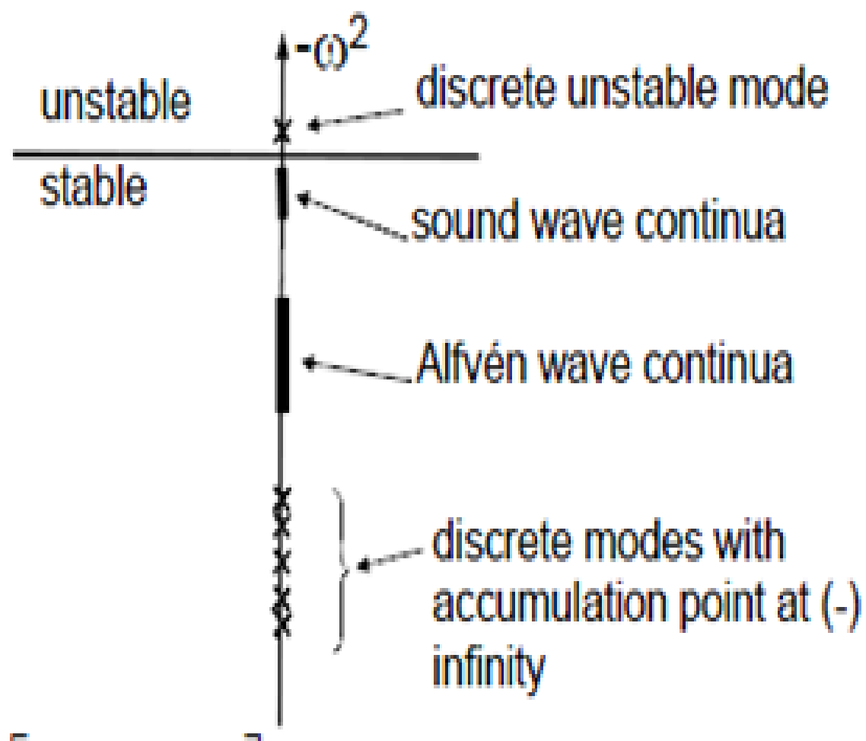
   This implies that the inner product of $\boldsymbol{\xi}_n$ and $\boldsymbol{\xi}_m$ is zero, or that the eigenvectors are orthogonal if $n \neq m$.

3. $\boldsymbol{\xi}_n$ is a complete set eigenfunctions of $\mathbf{F}$.

   Any square integrable displacement field $\boldsymbol{\xi}$ can be represented as linear combination of the $\boldsymbol{\xi}_n$

$$\boldsymbol{\xi} = \sum_{n=0}^{\infty} a_n \boldsymbol{\xi}_n$$

4. The spectrum of $\mathbf{F}$



$$\left[-\frac{1}{\rho_0}F-\omega^2\right]\vec{\xi}(\vec{x})=0$$

FIGURE 8.4: A typical ideal MHD spectrum.

The Second Variation

Let us define the quantity δw which is the change in potential energy due to a perturbation

$$\delta w(\boldsymbol{\xi}^\star, \boldsymbol{\xi}) = -\frac{1}{2}\int d^3\mathbf{x}\boldsymbol{\xi}^\star \cdot F(\boldsymbol{\xi})$$

quantify K such that $\omega^2 K$ is the kinetic energy

$$K(\boldsymbol{\xi}^\star, \boldsymbol{\xi}) = \frac{1}{2}\int d^3\mathbf{x}\rho|\boldsymbol{\xi}|^2$$

Consider now the functional Ω^2, known as the Rayleigh quotient

$$\Omega^2(\boldsymbol{\xi}^\star, \boldsymbol{\xi}) = \frac{\delta w(\boldsymbol{\xi}^\star, \boldsymbol{\xi})}{K(\boldsymbol{\xi}^\star, \boldsymbol{\xi})}$$

Any allowable function $\boldsymbol{\xi}$ (and $\boldsymbol{\xi}^\star$) for which Ω^2 becomes stationary is an eigenfunction of the ideal MHD normal mode eqs with eigenvalue

$$\omega^2 = \Omega^2(\boldsymbol{\xi}^\star, \boldsymbol{\xi})$$

The Second Variation

To prove this, consider the variation

$$\begin{aligned}\delta(\Omega^2) &= \frac{\delta[\delta w]}{K} - \frac{\delta w \delta[K]}{K^2} \\ &= \frac{1}{K}\left[\delta[\delta w] - \Omega^2 \delta[K]\right] \\ &= -\frac{1}{2K}\int d^3\mathbf{x}\left\{\delta\boldsymbol{\xi}^\star \cdot [F(\boldsymbol{\xi}) + \rho\Omega^2\boldsymbol{\xi}] + \delta\boldsymbol{\xi}\cdot[F(\boldsymbol{\xi}^\star) + \rho\Omega^2\boldsymbol{\xi}^\star]\right\} \\ &= 0\end{aligned}$$

Since $\delta\boldsymbol{\xi}^\star$ and $\delta\boldsymbol{\xi}$ are arbitrary variations, for the variation of Ω^2 to vanish, we must have

$$\mathbf{F}(\boldsymbol{\xi}) = -\rho\omega^2\boldsymbol{\xi}$$

The energy principle states that there is an instability if, and only if, there exists a vector field $\boldsymbol{\eta}(\mathbf{x})$ that satisfies the boundary condition and such that

$$\delta w(\boldsymbol{\eta}^\star, \boldsymbol{\eta}) < 0$$

One can thus investigate the stability of a system by looking at the sign of δw for various trial functions $\boldsymbol{\eta}$ without ever solving for the eigenfunctions $\boldsymbol{\xi}(\mathbf{x})$.

Since K is positive definite, an actual eigenvalue ω^2 must exist such that

$$\omega^2 \leq \frac{\delta w(\boldsymbol{\eta}^\star, \boldsymbol{\eta})}{K(\boldsymbol{\eta}^\star, \boldsymbol{\eta})}$$

so that the actual system will always be more unstable than that found with trial functions.

Suppose we have a truncated set of basis functions,

$$\boldsymbol{\xi}_n, n = 0, 1, 2, \cdots, N$$

We can form trial functions $\boldsymbol{\xi}$ by taking linear combinations of these

$$\boldsymbol{\xi} = \sum a_n \boldsymbol{\xi}_n$$

We can then evaluate δw as a quadratic form

$$\delta w = \sum_{m=0}^{N} \sum_{n=0}^{N} \delta w(\boldsymbol{\xi}_m, \boldsymbol{\xi}_n) a_m a_n$$

Minimize it, subject to any convenient normalization, wrt the amplitude a_n, leads to a linear algebraic eq for the set of a_n.