Plasma Instabilities

An important field of plasma physics is the stability of the plasma.

It usually only makes sense to analyze the stability of a plasma

once it has been established that the plasma is in equilibrium.

"Equilibrium" asks whether there are net forces that will accelerate any part of the plasma.

If there are not, then "stability" asks whether a small perturbation will grow, oscillate, or be damped out.

There are

- Ideal MHD Instabilities
  - Linear Instabilities
  - Nonlinear Instabilities
- Resistive MHD Instabilities

  The addition of dissipation to the ideal MHD eqs produce new instabilities: resistive instabilities

  Removing constraints from the ideal eqs makes states of lower potential energy accessible to the plasma.

  The relaxation of constraints can also make existing instabilities grow faster.
  - Linear Instabilities
  - Nonlinear Instabilities

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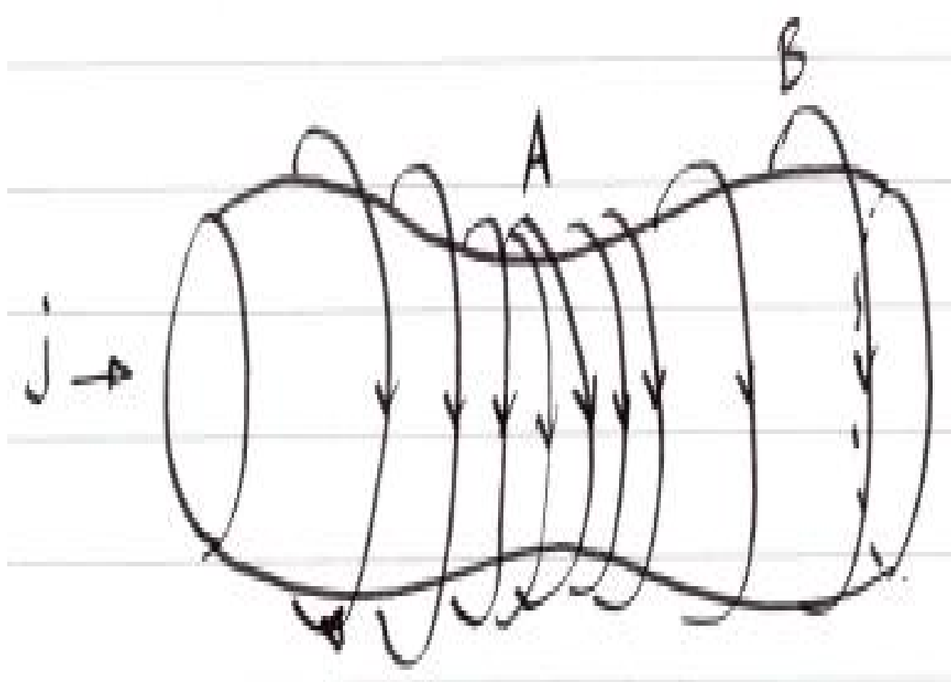
Plasma Instabilities

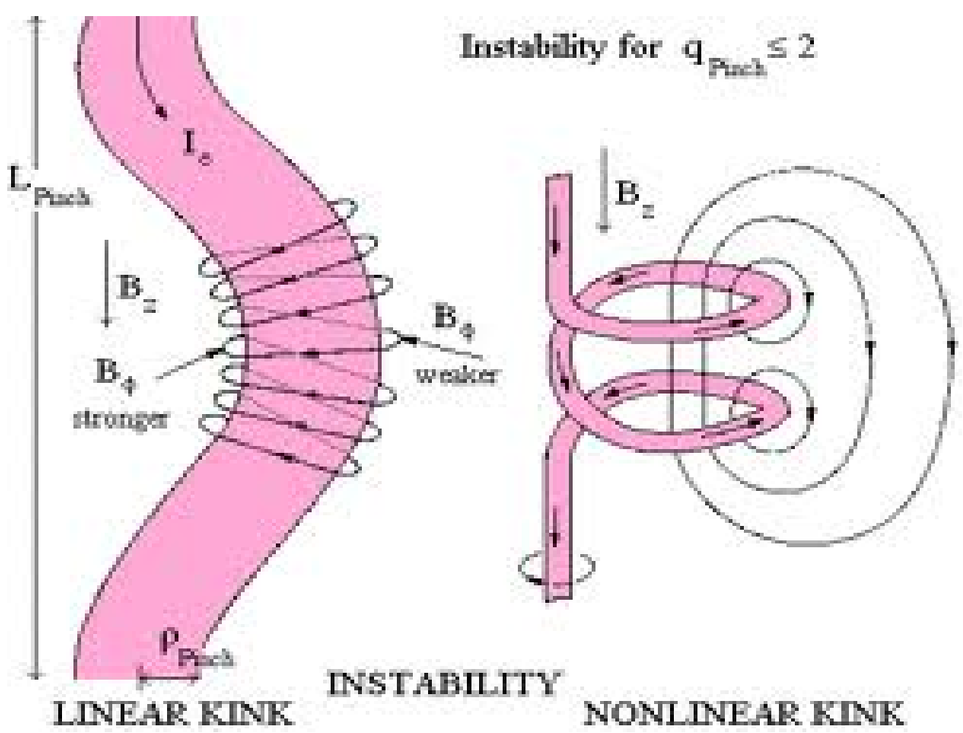
Most common observed instabilities in experiments

Ideal	$m = 0$	$m = 1$	$m \geq 2$	high-β
	sausage	kink	kink	interchange & ballooning
Resistive		$m = 1$	$m \geq 2$	resistive $\cdots$
		tearing mode	tearing mode	interchange mode
Nonlinear	$\cdots$	$\cdots$	$\cdots$	$\cdots$

$m = 0$ sausage instability

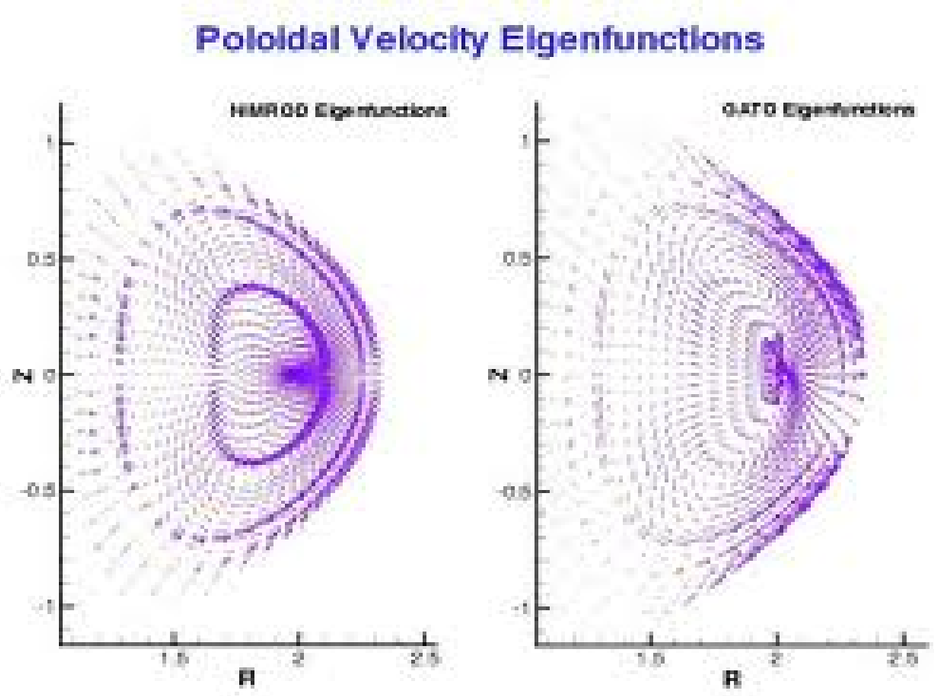


$m = 1$ sausage instability

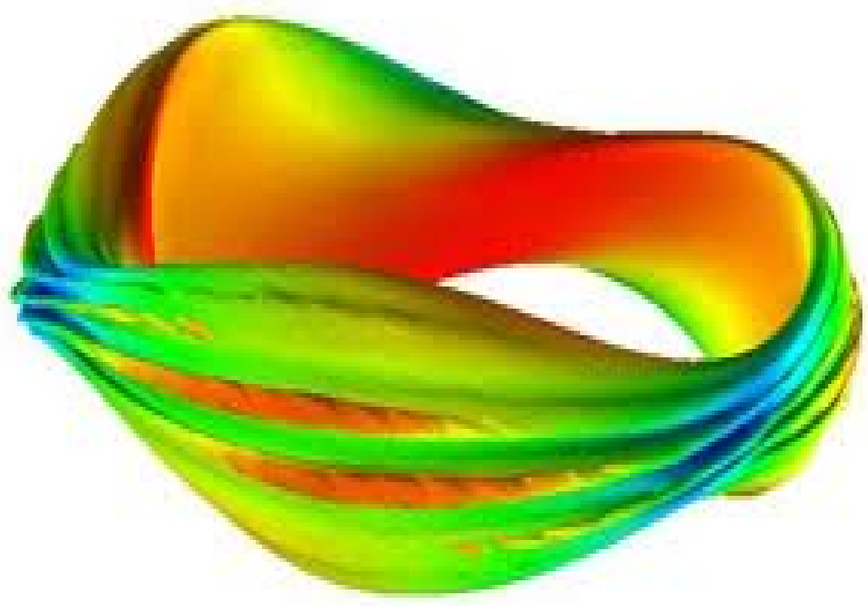


Plasma Instabilities

interchange mode



ballooning mode



Plasma Instabilities

Now it is becoming clear that completely stable plasma are not necessarily the best plasma for controlled thermonuclear fusion.

It is evidently more important to learn to live with instabilities than to avoid them completely.

Develop nonlinear theory to predict the consequences of instabilities when they do occur.

In any event,

- What is the nonlinear development of the instabilities?
- Do they saturate at a finite amplitude?
- Does their structure change dramatically from their appearance during the linear phase?
- Do they break up into fine scale structure?
- Do they carry the plasma to the wall?

Magnetic Reynolds and Lundquist Number

For motion with a characteristic scalelength L

the ratio of the convection term to the (resistive) diffusion term

is a dimensionless quantity

$$R_M = \mu_0 u L / \boldsymbol{\eta}$$

which is sometimes called the magnetic Reynolds number

If the magnetic Reynolds number is sufficiently large, the infinite-conductivity assumption is valid.

For fully developed magnetohydrodynamic motion,

the characteristic velocities are very large,

approaching the Alfven speed,

the magnetic Reynolds is then called the Lundquist number S.

Lundquist numbers in low-resistivity plasmas can

range up to 10^8, or higher.

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Linear Instabilities Analysis: 2nd variation,

The linearized ideal MHD equations are found by looking for solutions which deviate only infinitesimally from the equilibrium, and by keeping terms only to first order in this small deviation. We take as equilibrium quantities the current density $\mathbf{J}^0$, the magnetic field $\mathbf{B}^0$, the pressure p^0, and the fluid velocity $\mathbf{V}^0$. They satisfy MHD equilibrium.

The quantities $\mathbf{B}$,$\mathbf{J}$,p, and $\mathbf{V}$ are decomposed into equilibrium and perturbed parts,

$$\begin{aligned}
\mathbf{B} &= \mathbf{B}^0 + \mathbf{B}^1 \\
\mathbf{J} &= \mathbf{J}^0 + \mathbf{J}^1 \\
p &= p^0 + p^1 \\
\mathbf{V} &= \mathbf{V}^1
\end{aligned}$$

It is customary to introduce a displacement field $\boldsymbol{\xi}(\mathbf{x}, t)$, which is just the displacement of a fluid element originally located at position $\mathbf{x}$.

If we take as initial conditions that the perturbed quantities vanish at time zero,

$$\boldsymbol{\xi}(x\,0) = \mathbf{B}(\mathbf{x}, 0) = p(\mathbf{x}, 0) = 0,$$

then the linearized momentum equation becomes

$$\rho^0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}(\boldsymbol{\xi})$$

and then the fluid potential energy δW_f can be defined as

$$\delta W_f = \frac{1}{2}\int d^3\mathbf{x} = \frac{1}{2}\int d^3\mathbf{x}\boldsymbol{\xi}^\star \cdot \mathbf{F}(\boldsymbol{\xi})$$

$$\left[\begin{array}{ll}
\frac{1}{\mu_0}\mathbf{Q}_\perp^2 & \text{shear Alfven wave} \\
+\frac{1}{\mu_0}B^2[\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}]^2 & \text{fast magnetoacoustic wave} \\
+\gamma p|\nabla\cdot\boldsymbol{\xi}|^2 & \text{slow magnetoacoustic wave} \\
-\mathbf{2}(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\nabla}\boldsymbol{p})(\boldsymbol{\kappa}\cdot\boldsymbol{\xi}_\perp) & \text{pressure driven instabilities} \\
-\sigma\boldsymbol{\xi}_\perp\times\mathbf{B}\cdot\mathbf{Q}_\perp & \text{kink instabilities}
\end{array}\right]$$

Each one of the terms in this equation has a physical interpretation.

Linear Instabilities Analysis: 2nd variation,

(1) Shear Alfven wave term $\frac{1}{\mu_0}\mathbf{Q}_\perp^2$

This is the energy necessary to bend the magnetic field lines.

It is always stabilizing,

and is the dominant potential energy contribution to

the shear Alfven waves.

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Linear Instabilities Analysis: 2nd variation,

(2) Fast Magnetoacoustic wave term

$\frac{1}{\mu_0}B^2[\nabla\cdot\boldsymbol{\xi}_\perp + 2\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}]^2$

This is the energy necessary to compress the magnetic field lines.

It is always stabilizing,

and is the dominant potential energy

for compressional fast magnetosonic waves.

If pure toroidal field were present,

$$\boldsymbol{\kappa} = -\frac{\mathbf{R}}{R}$$

in cylindrical coordinate,

this would be equivalent to

$$B^2\left[R^2\nabla\cdot(\frac{\boldsymbol{\xi}}{R^2})\right]^2$$

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Linear Instabilities Analysis: 2nd variation,

(3) Sound (slow magnetoacoustic) wave term

$\gamma p|\nabla \cdot \boldsymbol{\xi}|^2$

This is the energy required to compress the plasma

(not the magnetic field).

It is always stabilizing

and is the potential energy for the slow magnetosonic (sound) waves.

It is the only term that contains the parallel displacement $\boldsymbol{\xi}_{||}$.

This is consistent with the discussion of the three wave types.

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Linear Instabilities Analysis: 2nd variation,

(4) Pressure driven instabilities term

$-2(\boldsymbol{\xi}_\perp \cdot \boldsymbol{\nabla} p)(\boldsymbol{\kappa} \cdot \boldsymbol{\xi}_\perp)$

This term is not positive definite

and therefore contributes to MHD instabilities.

When this term is dominant, we have an interchange instability.

When ∇p and $\boldsymbol{\kappa}$ are parallel, this term is destabilizing.

We call this bad curvature.

Anytime that ∇p and $\boldsymbol{\kappa}$ are not parallel,

it is possible to find a displacement $\boldsymbol{\xi}$

that makes this term destabilizing.

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Linear Instabilities Analysis: 2nd variation,

(5) Kink instability term

$-\sigma \boldsymbol{\xi}_{\perp} \times \mathbf{B} \cdot \mathbf{Q}_{\perp}$

This term is not positive definite

and is proportional to the parallel current.

When This term is dominant, we have a kink instability.

We see from the definition of $\mathbf{Q} = \boldsymbol{\nabla} \times \boldsymbol{\xi} \times \mathbf{b}$ that this term is proportional to the scalar product of a vector field $\boldsymbol{\xi}_{\perp} \times \mathbf{B}$,

and its curl.

It is a measure of the twist of the vector field.

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Linear Instabilities Analysis: 2nd variation,

Considering the wavenumber limit $|\boldsymbol{k}_\perp| \to \infty$

and expanding $\boldsymbol{\xi}_\perp$ as

$$\boldsymbol{\xi}_\perp = \boldsymbol{\xi}_\perp^0 + \boldsymbol{\xi}_\perp^1 + \cdots,$$
$$\frac{|\boldsymbol{\xi}_\perp^0|}{|\boldsymbol{\xi}_\perp^1|} \sim \frac{1}{k_\perp a}$$

we obtain the energy principle of the incompressible high-mode-number ballooning mode

$$\delta W_F = \frac{1}{2}\int dr \left[|\boldsymbol{k}_\perp|^2 |\boldsymbol{b}\cdot\nabla\Phi|^2 - \frac{2}{B^2}(\boldsymbol{b}\times\boldsymbol{k}_\perp\cdot\nabla p)(\boldsymbol{b}\times\boldsymbol{k}_\perp\cdot\boldsymbol{\kappa})|\Phi|^2\right]$$

where $\boldsymbol{\xi}_\perp^0 = Y\boldsymbol{b}\times\boldsymbol{k}_\perp$

and $\Phi \equiv YB$ exhibits the competition

between the stabilizing effect of the field line bending,

representing by $|\boldsymbol{k}_\perp|^2$,

and the destabilizing effect of the pressure gradient

when the magnetic curvature is unfavorable.

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Linear Instabilities Analysis: 2nd variation,

Considering only the perturbation perpendicular to **B**,

the kinetic energy can be obtained in the same way.

$$K = \frac{1}{2}\int \rho|\boldsymbol{\xi}_\perp|^2 d\boldsymbol{r} = \frac{1}{2}\int \rho|\boldsymbol{\xi}|^2 d\boldsymbol{r} = \frac{1}{2}\int \rho|Y|^2|\boldsymbol{b}\times\boldsymbol{k}_\perp|^2 d\boldsymbol{r}$$

$$= \frac{1}{2}\int \rho\frac{|\boldsymbol{k}_\perp|^2}{B^2}|\Phi|^2 d\boldsymbol{r}.$$

Then the Euler equation can be obtained through the variational principle

$$\omega^2 = \frac{\delta W}{K}.$$

Letting $\Phi \to \Phi + \delta\Phi$ and $\omega^2 \to \omega^2 + \delta\omega^2$,

and setting $\delta\omega^2 \to 0$

the corresponding vector form is an ordinary differential equation in the parallel direction

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\Big[\frac{|\boldsymbol{k}_\perp|^2}{B^2}\boldsymbol{B}\cdot\boldsymbol{\nabla}\Phi\Big] + \frac{\rho}{B^2}\omega^2|\boldsymbol{k}_\perp|^2\Phi$$
$$+\frac{2}{B^2}(\boldsymbol{b}\times\boldsymbol{k}_\perp\cdot\boldsymbol{\nabla}p)(\boldsymbol{b}\times\boldsymbol{k}_\perp\cdot\boldsymbol{\kappa})\Phi = 0$$

which is the well-known ballooning equation.

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Linear Instabilities Analysis: 2nd variation,

This is an eigenvalue problem.

The stability of the eigenfunctions is determined by the Mercier criterion,

which can be given as a function of normalized toroidal flux Φ_T

$$
\begin{aligned}
D_M = \; & \frac{(\iota')^2}{4} - p'\mathcal{V}'[-\langle\frac{B^2}{|\nabla\Phi_T|^2}\rangle\mathcal{V}'' + \iota'\langle\frac{(\boldsymbol{J}\cdot\boldsymbol{B})_{PSN}}{|\nabla\Phi_T|^2}\rangle] \\
& -(p'\mathcal{V}')^2[\langle\frac{(\boldsymbol{J}\cdot\boldsymbol{B})^2_{PSN}}{|\nabla\Phi_T|^2 B^2}\rangle\langle\frac{B^2}{|\nabla\Phi_T|^2}\rangle \\
& -\langle\frac{(\boldsymbol{J}\cdot\boldsymbol{B})_{PSN}}{|\nabla\Phi_T|^2}\rangle^2 + \langle\frac{B^2}{|\nabla\Phi_T|^2}\rangle\langle\frac{1}{B^2}\rangle].
\end{aligned}
$$

$(\boldsymbol{J}\cdot\boldsymbol{B})_{PSN}$: the Pfirsch-Schluter current divided by p' and is given by

$$
\begin{aligned}
(\boldsymbol{J}\cdot\boldsymbol{B})_{PSN} &= -\boldsymbol{B}\times\boldsymbol{\nabla}\Phi_T\cdot\nabla\tilde{\beta}_N \\
\boldsymbol{B}\cdot\boldsymbol{\nabla}\tilde{\beta}_N &= 1 - \frac{B^2}{\langle B^2\rangle}.
\end{aligned}
$$

Mercier stability corresponds to satisfying the condition $D_M > 0$.

The first term on the right-hand side represents the stabilizing effect of global magnetic shear.

The last term is destabilizing, due to the Pfirsch-Schluter current (geodesic curvature) and the diamagnetic current.

only the second term can change sign, depending on the average magnetic well index $\mathcal{V}''$ and the global magnetic shear ι'.

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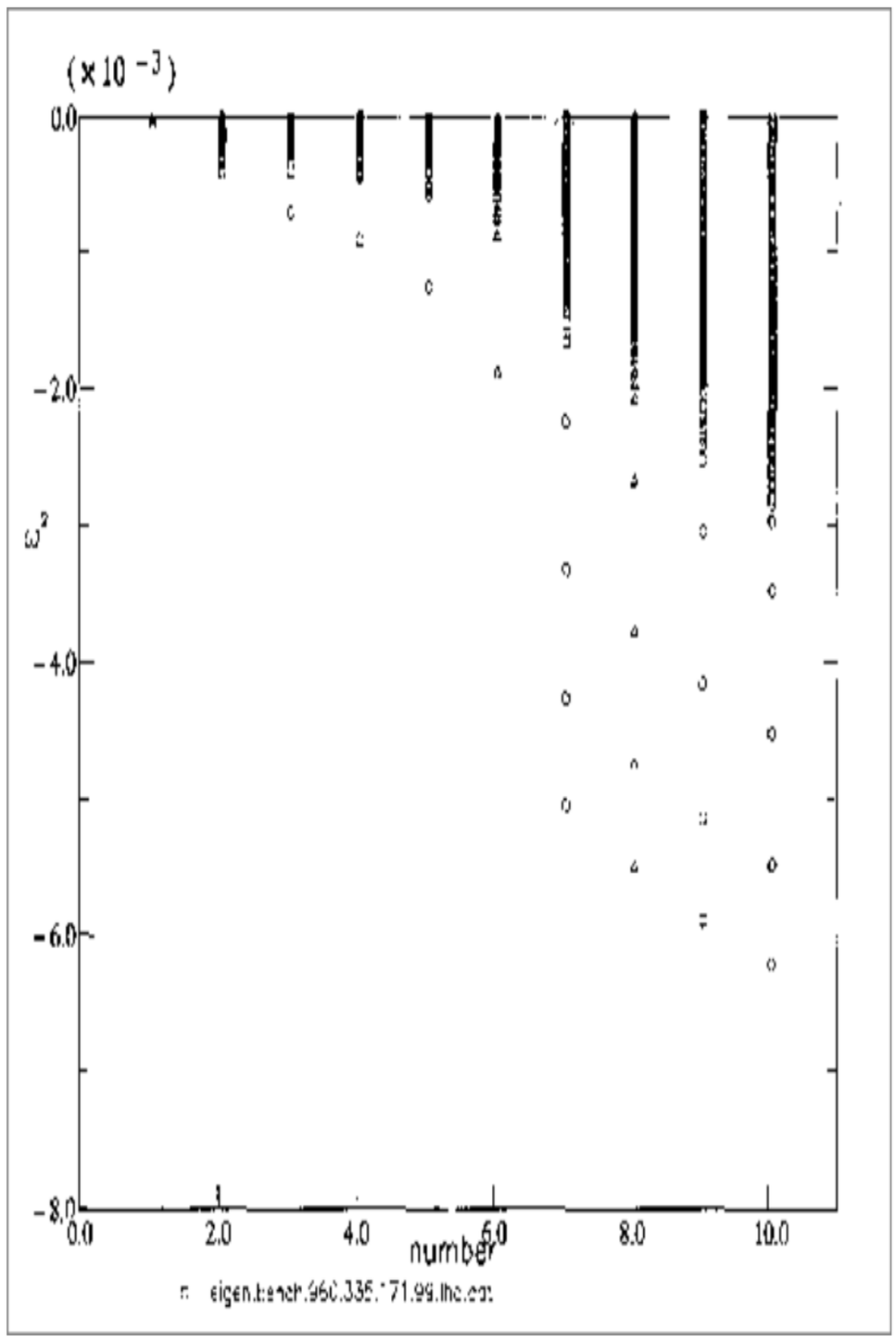
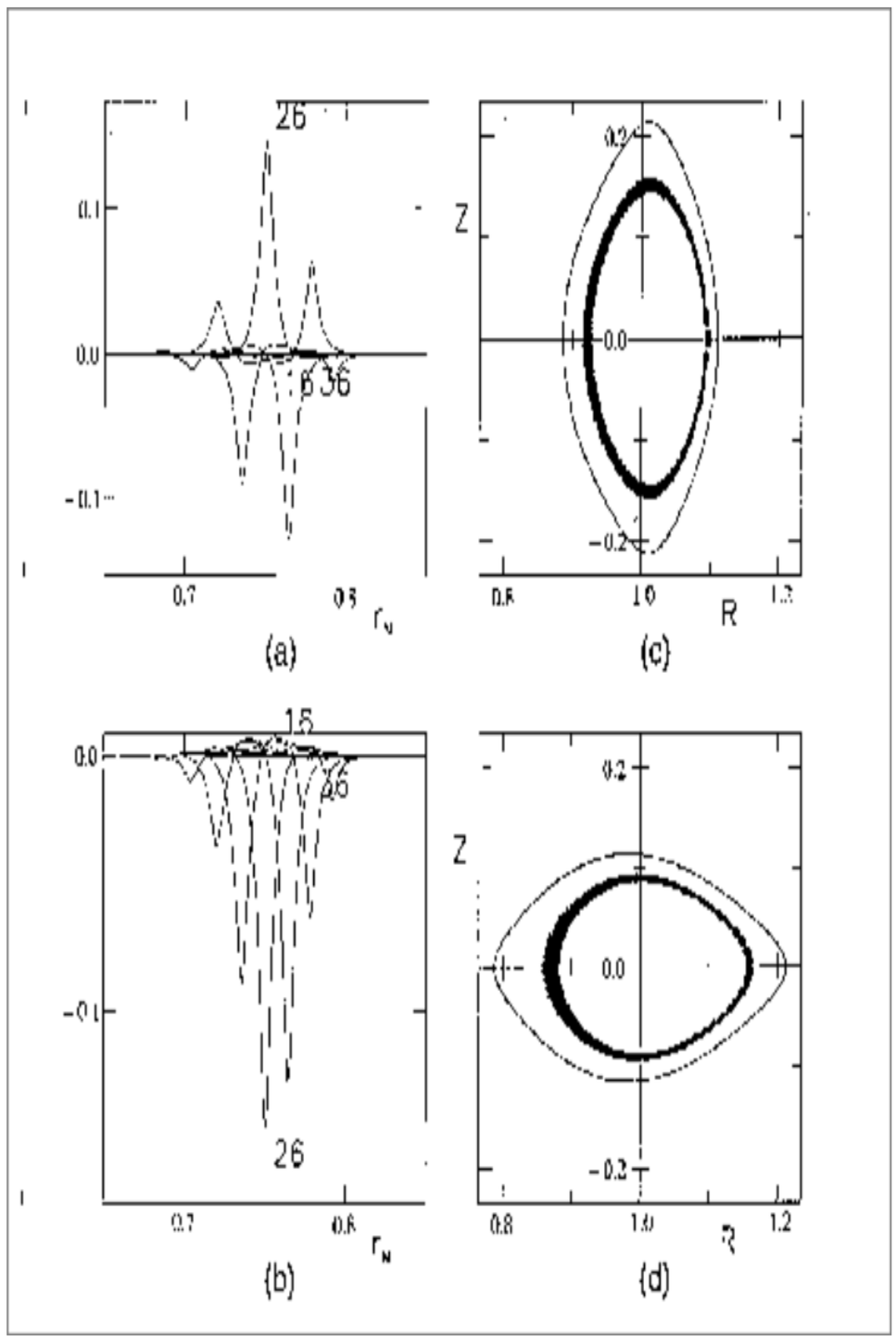
Linear Instabilities Analysis: 2nd variation,

Eigenspectrun



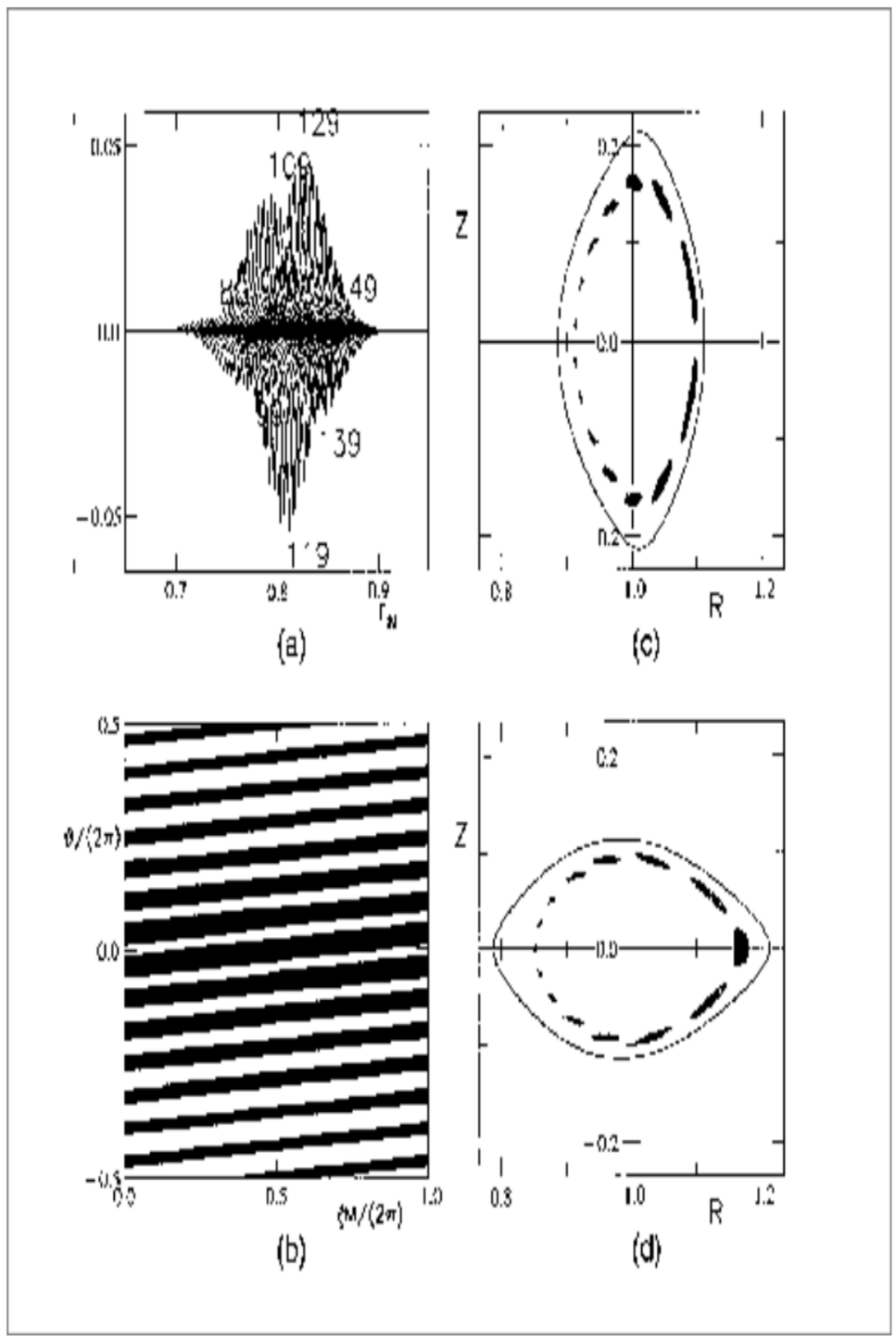
Linear Instabilities Analysis: 2nd variation,

Pressure-driven interchange mode



Linear Instabilities Analysis: 2nd variation,

Pressure-driven ballooning mode



Nonlinear Instability Analysis

More than 5 overlapping techniques used to study nonlinear MHD instabilities:

- computer simulation
- bifurcation theory
- singular perturbation analysis
- the study of convection cells
- the study of the fully developed turbulence

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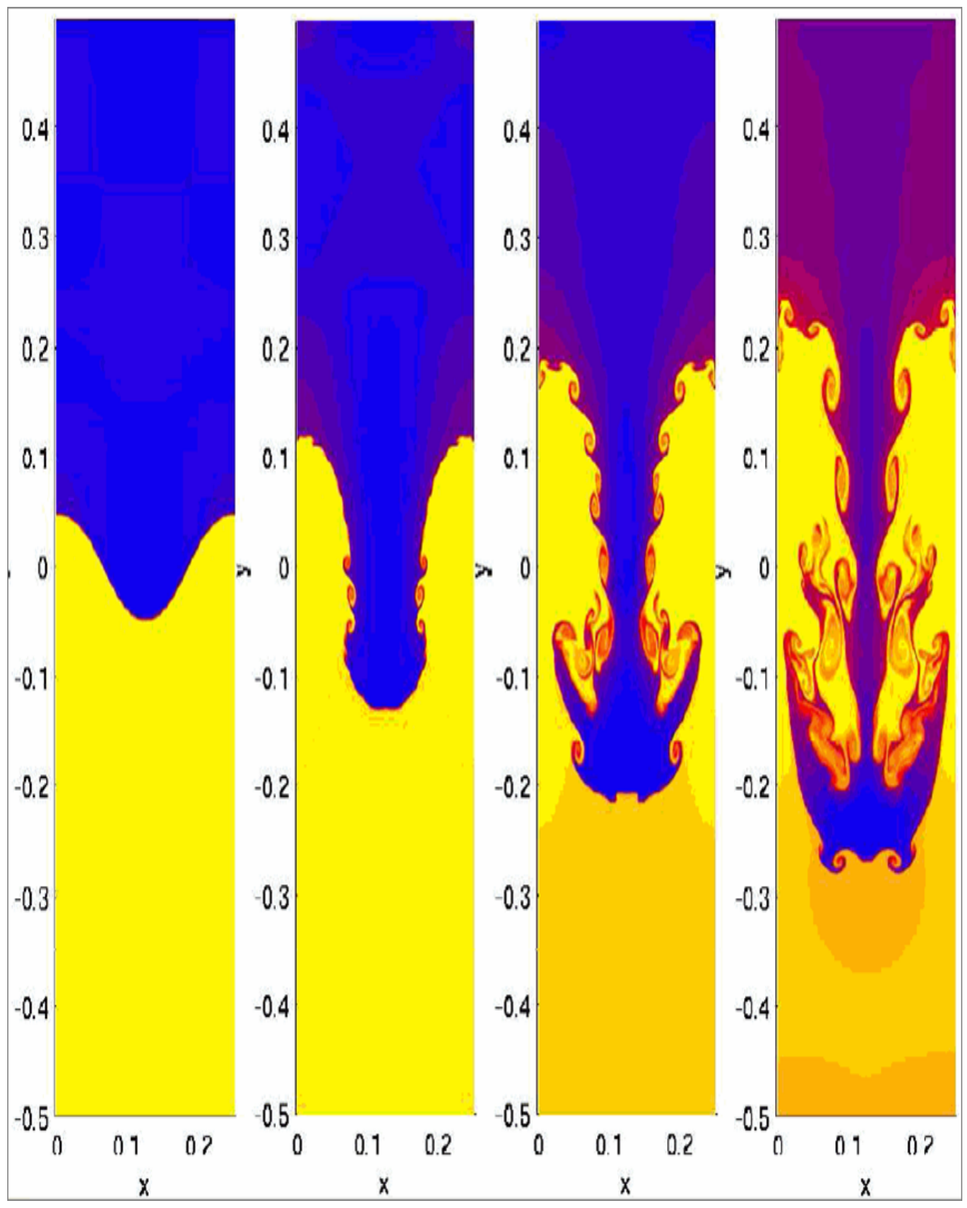
I. The Rayleigh-Taylor, flute instabilities, resistive interchange mode

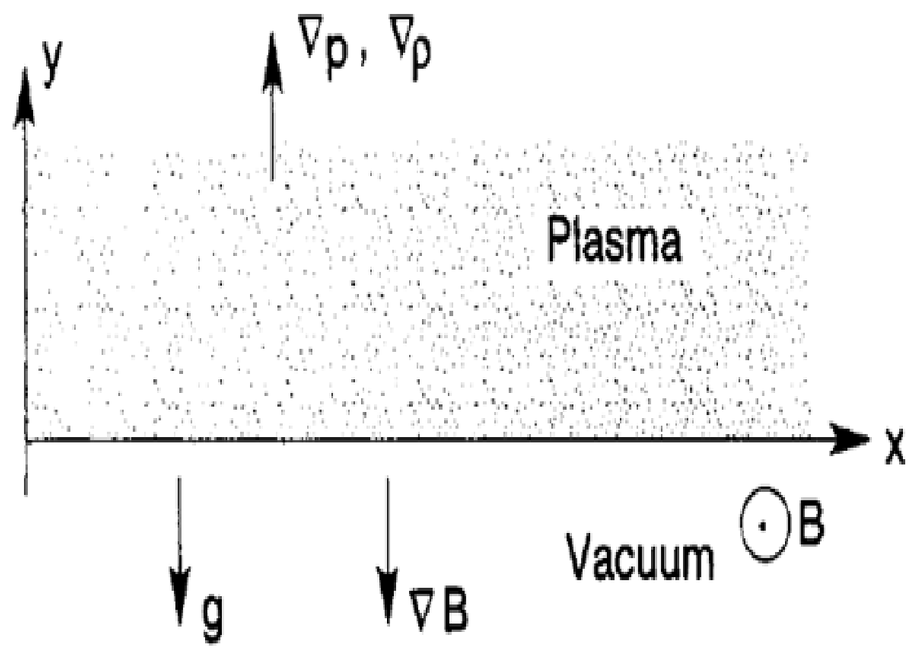
Instability occurs when a dense plasma is supported against gravity by the pressure of a magnetic field.



In curved magnetic fields, the centrifugal force on the plasma due to particle motion along the curved field lines acts like a 'gravitational force'.

This field curvature driven instability is the most virulent type of MHD instability in non-uniform plasmas.

I. The Rayleigh-Taylor, flute instabilities, resistive interchange mode



Assuming equilibrium quantities

$$p^0(y), B^0(y), \rho^0(y), \mathbf{u}^0 = 0$$

the equilibrium pressure balance eq $-\nabla p + \mathbf{J} \times \mathbf{B} - \rho g = 0$ yields

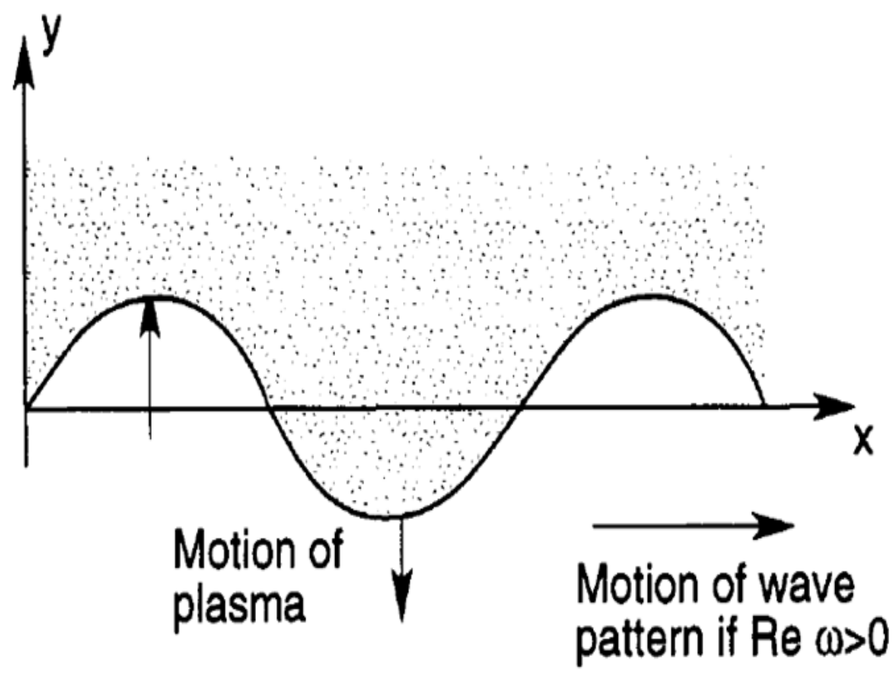
$$\frac{\partial}{\partial y}\left(p^0 + \frac{B^{0,2}}{2\mu_0}\right) + \rho^0 g = 0$$

Adding wave-like perturbations in the form of

$$\psi \propto \hat{\psi}(y) e^{ikx - i\omega t}$$

at plasma-vacuum interface,

I. The Rayleigh-Taylor, flute instabilities, resistive interchange mode



the linearized eq of motion is simply

$$\rho^0 \frac{\partial \mathbf{V}^1}{\partial t} = \rho^1 \mathbf{g} - \nabla \left(p^1 + \frac{B^0 B_z^1}{\mu_0} \right)$$

The $\mathbf{z} \cdot \nabla \times$ operation yields

$$-i\omega \left(ik\rho^0 V_y - \frac{\partial}{\partial y}(\rho^0 V_x) \right) = -ik\rho^1 g$$

Supposing that the plasma motion is incompressible, i.e.

$$0 = \nabla \cdot \mathbf{V}^1 = ikV_x + \frac{\partial V_y}{\partial y}$$
$$V_x = \frac{i}{k}\frac{\partial V_y}{\partial y}$$

the density perturbation can be obtained from the continuity eq

$$\frac{\partial \rho^1}{\partial t} + \mathbf{V}^1 \cdot \nabla \rho^0 = 0$$

I. The Rayleigh-Taylor, flute instabilities, resistive interchange mode

gives

$$-i\omega\rho^1 = -V_y\frac{\partial\rho^0}{\partial y} = -\frac{\rho^0 V_y}{s},$$
$$\rho^1 = \frac{\rho^0 V_y}{i\omega s}$$

where $\rho^0(y) \propto e^{y/s}$ is assumed. s denotes density-gradient 'scale length'.

With the expressions for ρ^1 and V_x, the eq of motion becomes

$$\frac{1}{\rho^0}\frac{\partial}{\partial y}\left(\rho^0\frac{\partial V_y}{\partial y}\right) - k^2\left(1+\frac{g}{s\omega^2}\right)V_y = 0$$

Prescribing the conducting wall boundary conditions

$$V_y = 0 \text{ at } y = 0, h$$

the discrete solutions can be found of the form

$$V_y(y) = \sin\left(\frac{n\pi y}{h}\right)e^{-\frac{y}{2s}}$$

and the eigenvalue ω is determined by the dispersion relation

$$k^2\left(1+\frac{g}{s\omega^2}\right) = -\frac{1}{4s^2} - \frac{n^2\pi^2}{h^2}$$

Here s and g are both positive.

There are no solutions unless ω^2 is negative, corresponding to ω being pure imaginary. Solving for ω, we obtain

$$\omega = \pm i\left(\frac{g}{s}\frac{h^2k^2}{n^2\pi^2 + h^2k^2 + h^2/4s^2}\right)^{1/2}$$

ω positive imaginary part:	exponentially growing perturbation, i.e. instability
ω negative imaginary part:	decaying perturbation

I. The Rayleigh-Taylor, flute instabilities, resistive interchange mode

$n = 1$: the lowest mode to satisfy the bc

longest wavelength mode in the y direction

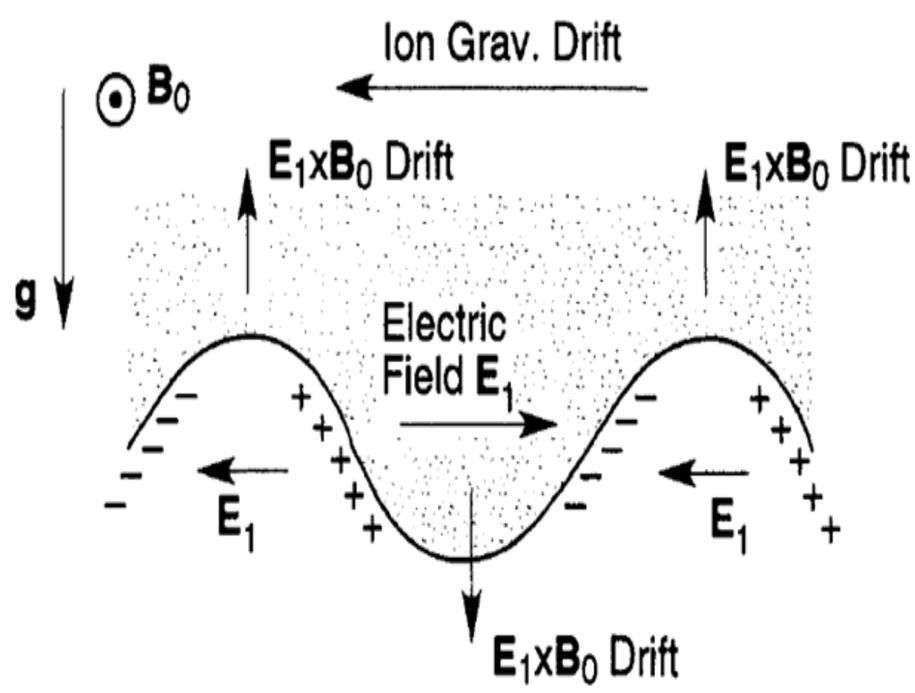
more rapidly growing than modes with $n > 1$

The largest growth rate is

$$\gamma = \sqrt{\frac{g}{s}}$$

corresponding shortest wavelength in x direction, i.e. large k values.

Physical Mechanisms



An external force $\mathbf{F}$ perpendicular to a magnetic field $\mathbf{B}$ causes a charged particle to drift with a velocity

$$\mathbf{v}_d = \frac{\mathbf{F} \times \mathbf{B}}{eB^2} = \frac{M\mathbf{g} \times \mathbf{B}}{eB^2}$$

This gravitational drift is in the negative-x direction.

Suppose a small wave-like ripple should develop on a plasma-vacuum interface. Thus, the drift of ion causes positive charge to build up on the one side of the ripple; the depletion of ions causes a negative charge to build up on the other side of the ripple.

Due to this separation of charges, a small electric field $\mathbf{E}^1$ develops, and this electric field changes sign going from crest to trough of the

I. The Rayleigh-Taylor, flute instabilities, resistive interchange mode

perturbation.

It is apparent that the resulting $\mathbf{E}^1 \times \mathbf{B}^0$ drift is always upward in those regions where the interface has already moved upward, and downward in those regions where the interface has already moved downward.

Thus the initial ripples grows larger, as a result of $\mathbf{E} \times \mathbf{B}$ drifts that are phased so as to amplify the initial perturbation.

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I+. Flute Instability

Because the close analog between gravitational drifts and the ∇B and curvature drifts that arise in non-uniform magnetic fields

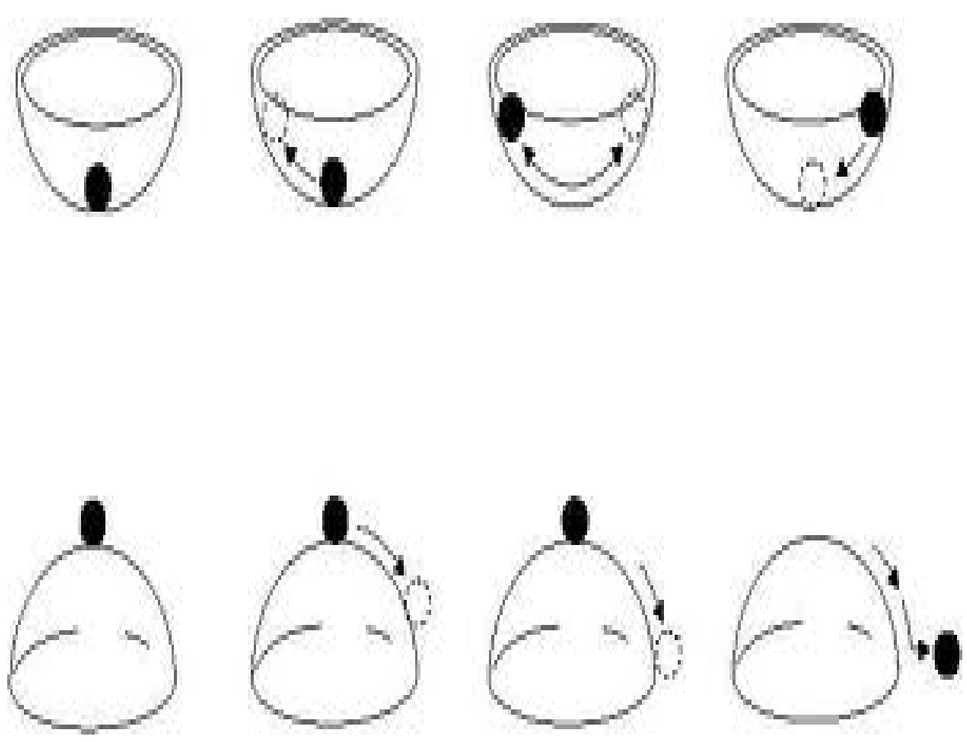
$$\mathbf{V}_d = \frac{M}{e}\left(\frac{V_\perp^2}{2} + V_\parallel^2\right)\frac{\boldsymbol{\kappa}\times\mathbf{B}}{B^2}$$

$\boldsymbol{\kappa} = -\mathbf{b}\times\boldsymbol{\nabla}\times\mathbf{b} = \mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{b}$

a plasma in a curved magnetic field can be viewed as having analogous particle drifts to a plasma in a gravitational field – and therefore a potential for charge build-up and unstable growth of perturbations.

Since the Rayleigh-Taylor instability arises whenever the gravitational force is directed away from the region of maximum density, the corresponding instability of a plasma in a curved field arises whenever the curvature is directed away from the region of maximum plasma pressure, i.e. whenever the plasma is confined by a magnetic field that is concave towards the plasma:

$\boldsymbol{\kappa}$ and ∇p are oppositely directed.



The growth rate γ of the instability can be estimated by replacing g by $\frac{2p\kappa}{\rho}$ in the expression for γ and by equating the scale-length s to

I+. Flute Instability

the pressure-gradient scale-length, i.e.

$$s^{-1} = \frac{|\nabla p|}{p}$$

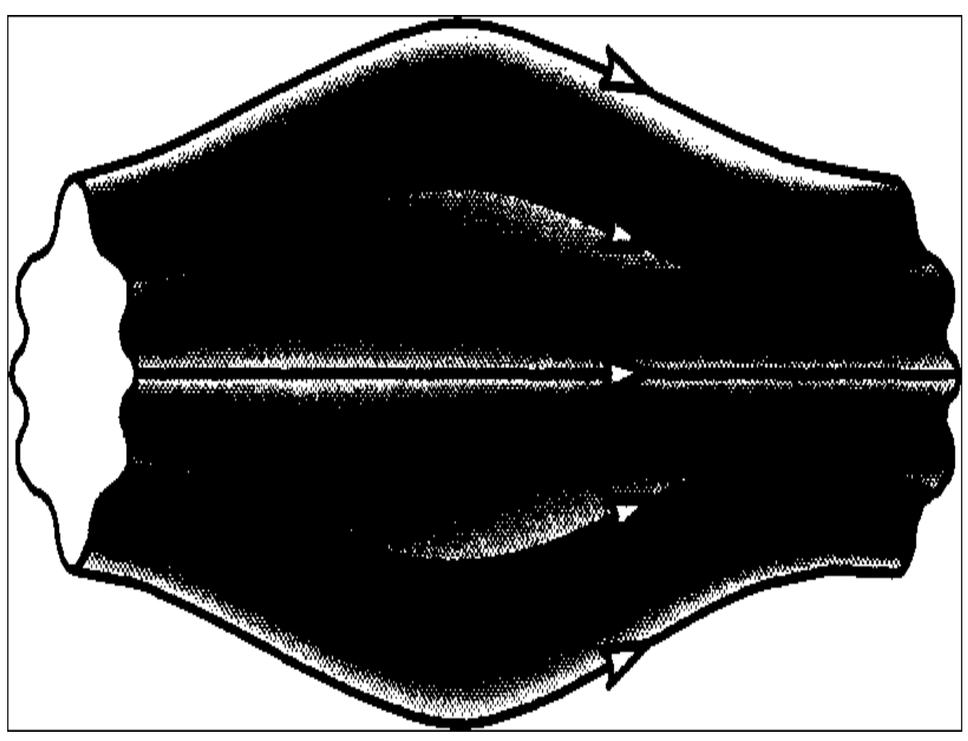
the growth rate becomes

$$\gamma \sim \sqrt{2|\nabla p|\kappa/\rho}$$

This instability is rapidly growing and the growth rate can be re-written as

$$\gamma \sim C_S \left(\frac{\kappa}{s}\right)^{1/2}$$

C_S is the sound speed. Thus, the characteristic growth time is the time it takes a sound wave to transverse a distance that is the geometric mean of the pressure-gradient scale-length and the curvature.



I++. Resistive Interchange Mode

This instability is driven by an inverted pressure gradient in a curved magnetic field.

Consider a plane slab with an inverted density gradient with a sheared magnetic field.

the eq of motion $\rho^0 \frac{\partial}{\partial t} V_x^1 = -\rho^1 g - J_z^1 B_\star$

the eq of density $\frac{\partial}{\partial t}\rho^1 = -V_x^1 \frac{\partial}{\partial x}\rho^0(x)$

for the incompressible plasma motion.

Since V_y^1 is much larger than V_x^1, the best estimate of the inertial effects is made by balancing the rate of change of kinetic energy

$$\frac{1}{2}\rho^0 \frac{\partial}{\partial t}\left[(V_x^1)^2 + (V_y^1)^2\right] \simeq \rho^2 \gamma (V_x^1)^2/(k\epsilon)^2$$

against the work done by the gravitational field

$$\mathbf{F}\cdot V_x^1 = -\rho^1 g V_x^1$$

and the work done by the $\mathbf{J}\times\mathbf{B}$ force

$$\mathbf{F}_{\mathbf{J}\times\mathbf{B}}\cdot V_x^1 = -J_z^1 B_\star V_x^1, \;\; J_z^1 \simeq V_x^1 B_\star/\eta$$

within the boundary layer. Balancing these terms and using

$$B_\star \simeq B_\star' \epsilon$$

yields an estimate for the growth rate

$$\gamma \simeq \left(\frac{\eta}{\rho}\right)^{1/3}\left(\frac{\rho' g k}{B_\star'}\right)^{2/3} \simeq \tau_R^{-1/3}\tau_A^{2/3}(kg)^{2/3}\left(\frac{\rho' q}{\rho q'}\right)^{2/3}$$

and the boundary layer thickness

$$\frac{\epsilon}{a} \simeq \tau_R^{-1/3}\tau_A^{2/3}\left(\frac{\rho' g}{\rho}\right)^{1/6}(ka)^{-1/3}\left(\frac{q}{q' a}\right)^{2/3}$$

II. The resistive tearing instability

Consider a plasma current slab with current directed parallel to the surface of the slab

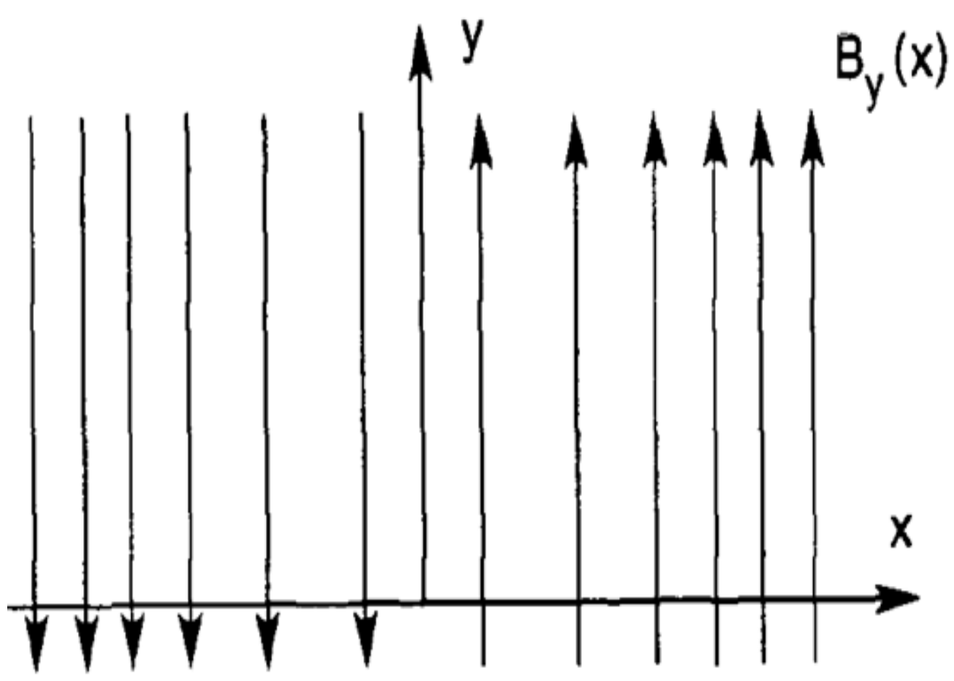
$$J_z = \begin{cases} J_z^0 & |x| \leq a \\ 0 & |x| > a \end{cases}$$

The plasma is uniform in the y and z directions. Solving Ampere's law,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \Longrightarrow \frac{dB_y}{dx} = \mu_0 J_z(x)$$

we obtain

$$B_y(x) = \begin{cases} B_y'(0)x & |x| \leq a \\ -B_y'(0)a = -\mu_0 J_z^0 a & x < -a \\ B_y'(0)a & x > a \end{cases}$$



II. The resistive tearing instability

The inclusion of plasma resistivity will allow the negative B_y field on the left of $x = 0$ to diffuse into the region of positive B_y filed on the right of $x = 0$.

This annihilation of the magnetic field will clearly occur most effectively in the vicinity of $x = 0$, which is where we will find the largest plasma flows in the resistive tearing instability.

Since the magnetic flux through any plasma surface element in the (x, z) plane must remain fixed, this Annihilation of magnetic field is energetically favored.

These plane slab equilibria are stationary in time and uniform in the y and z directions, and the linearized perturbations of the equilibria may be Fourier analyzed into normal mode of the form

$$\psi^1(x,t) = \hat{\psi}^1(x) e^{iky - i\omega t}$$

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II. The resistive tearing instability

First consider ideal MHD case,

the linearization of Faraday's law and Ohm's law gives

$$\begin{aligned}\frac{\partial \mathbf{B}^1}{\partial t} &= -\nabla \times \mathbf{E}^1 = \nabla \times (\mathbf{V}^1 \times \mathbf{B}^0) \\ &= (\mathbf{B}^0 \cdot \nabla)\mathbf{V}^1 - (\mathbf{V}^1 \cdot \nabla)\mathbf{B}^0 - \mathbf{B}^0(\nabla \cdot \mathbf{V}^1)\end{aligned}$$

Its x component is just

$$\frac{\partial B_x^1}{\partial t} = ikB_y^0 V_x^1$$

For the normal mode varying as $e^{-i\omega t}$, this eq becomes

$$\omega B_x^1 = -kB_y^0 V_x^1$$

Writing $(B^2)^1 = 2\mathbf{B}^0 \cdot \mathbf{B}^1$, the linearized eq of motion is

$$\begin{aligned}\rho\frac{\partial \mathbf{V}^1}{\partial t} &= -\nabla p^1 + (\mathbf{J} \times \mathbf{B})^1 \\ &= -\nabla p^1 + \frac{1}{\mu_0}((\nabla \times \mathbf{B}) \times \mathbf{B})^1 \\ &= -\boldsymbol{\nabla}\left(\boldsymbol{p^1} + \frac{\mathbf{B^0}\cdot\mathbf{B^1}}{\boldsymbol{\mu_0}}\right) + \frac{\mathbf{1}}{\boldsymbol{\mu_0}}\left[(\mathbf{B^0}\cdot\boldsymbol{\nabla})\mathbf{B^1} + (\mathbf{B^1}\cdot\boldsymbol{\nabla})\mathbf{B^0}\right]\end{aligned}$$

Taking curl operation, we will see that the magnetic energy is eliminated. But we will show later that it is available for instability through resistivity.

Now the x and y components are

$$-i\omega\rho^0 V_x = -\frac{\partial}{\partial x}\left(p^1 + \frac{\mathbf{B}^0\cdot\mathbf{B}^1}{\mu_0}\right) + \frac{1}{\mu_0}ikB_y^0 B_x$$

$$-i\omega\rho^0 V_y = -ik\left(p^1 + \frac{\mathbf{B}^0\cdot\mathbf{B}^1}{\mu_0}\right) - \frac{1}{\mu_0}\left(B_y^0\frac{\partial B_x}{\partial x} - B_x\frac{\partial B_y^0}{\partial x}\right)$$

II. The resistive tearing instability

The z component of curl operation gives

$$\begin{aligned}
&-i\omega\left(\frac{\partial}{\partial x}(\rho^0 V_y) - ik\rho^0 V_y\right) \\
&= \frac{1}{\mu_0}\left[\frac{\partial}{\partial x}\left(B_x\frac{\partial B_y^0}{\partial x} - B_y^0\frac{\partial B_x}{\partial x}\right) + k^2 B_y^0 B_x\right] \\
&= -\frac{1}{\mu_0}\left\{\frac{\partial}{\partial x}\left[B_y^{0,2}\frac{\partial}{\partial x}\left(\frac{B_x}{B_y^0}\right)\right] - k^2 B_y^0 B_x\right\}
\end{aligned}$$

Assuming the plasma flow in the (x, y) plane is incompressible

$$0 = \nabla\cdot\mathbf{V}_\perp = \frac{\partial V_x}{\partial x} + ikV_y$$

which is used to eliminate V_y from the above eq, we get

$$-\frac{\omega\mu_0}{k}\left[\frac{\partial}{\partial x}\left(\rho^0\frac{\partial V_x}{\partial x}\right) - k^2\rho^0 V_x\right] = \frac{\partial}{\partial x}\left[B_y^{0,2}\frac{\partial}{\partial x}\left(\frac{B_x}{B_y^0}\right)\right] - k^2 B_y^0 B_x$$

Multiplying through by $-\omega k$, with the expression of $\frac{B_x}{B_y^0} = -\frac{k}{\omega}V_x$ we obtain

$$\frac{\partial}{\partial x}\left((\rho^0\mu_0\omega^2 - k^2 B_y^{0,2})\frac{\partial V_x}{\partial x}\right) - k^2(\rho^0\mu_0\omega^2 - k^2 B_y^{0,2})V_x = 0$$

This the the ideal MHD Shear Alfven waves eq.

Examining its quadratic expression by multiplying it with the complex conjugate $V_x^\star$ and integrating over all x

$$\int_{-\infty}^{\infty}(\rho^0\mu_0\omega^2 - k^2 B_y^{0,2})\left(\left|\frac{\partial V_x}{\partial x}\right|^2 + k^2\left|V_x\right|^2\right)dx = 0$$

we found that the plasma is completely stable to the perturbation.

II. The resistive tearing instability

Including Resistivity

With the introduction of resistivity into Ohm's law

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$$

the linearized Faraday's law is now

$$\begin{aligned}\frac{\partial \mathbf{B}^1}{\partial t} &= -\nabla \times \mathbf{E}^1 = \nabla \times (\mathbf{V}^1 \times \mathbf{B}^0) - \eta \nabla \times \mathbf{J}^1 \\ &= (\mathbf{B}^0 \cdot \nabla)\mathbf{V}^1 - (\mathbf{V}^1 \cdot \nabla)\mathbf{B}^0 - \mathbf{B}^0(\nabla \cdot \mathbf{V}^1) + \frac{\eta}{\mu_0}\nabla^2 \mathbf{B}^1\end{aligned}$$

and its x component becomes

$$\frac{\partial B_x^1}{\partial t} = ikB_y^0 V_x^1 + \frac{\eta}{\mu_0}\frac{\partial^2 B_x}{\partial x^2}$$

i.e.

$$\underbrace{\omega B_x^1 = -kB_y^0 V_x^1}_{\text{ideal part}} + \frac{i\eta}{\mu_0}\frac{\partial^2 B_x}{\partial x^2}$$

Investigation shows that the resistive term is likely to be most important in a narrow region around the point where $B_y^0 = 0$, i.e. around $x = 0$ i.e.

the "resistive layer".

Since $\mathbf{k} \cdot \mathbf{B} = 0$ in this layer, the perturbation can be considered to be "resonant" in it, such that the unperturbed $\mathbf{B}$ lies parallel to wave-fronts on this surface.

The non-zero η in the resistive layer then allows the $\mathbf{B}$ lines to connect across the resonance, via a finite value of B_x.

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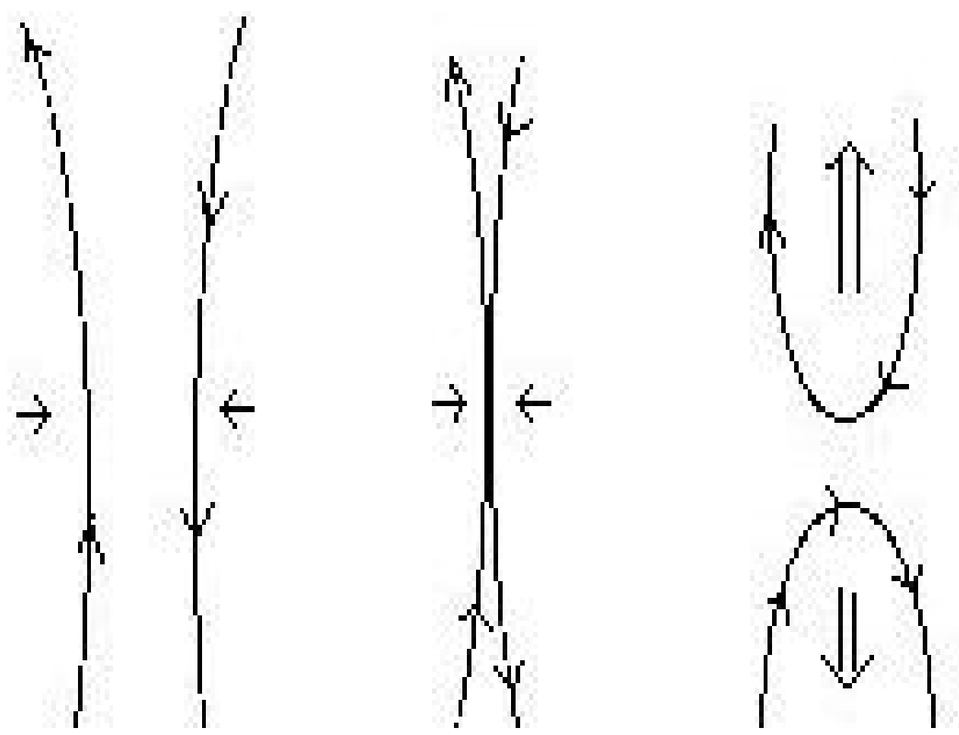
II. The resistive tearing instability



In the ideal case, $B_x = 0$ at the point $B_y^0 = 0$.

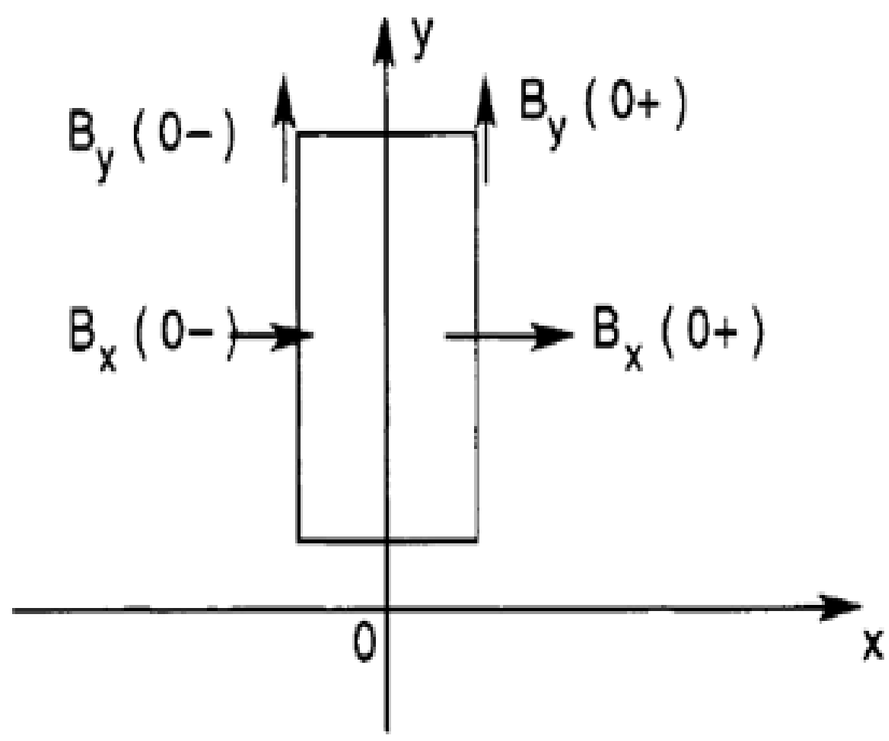
But when resistivity is introduced, B_x can approach some non-zero const as $x \to 0$.

It is just this non-vanishing of B_x at the point where $B_y^0 = 0$ that characterizes the "resistive tearing" instability.

Thinking of the region around $x = 0$ as forming a 'boundary layer' between the two ideal MHD regions to the left and right of it.

II. The resistive tearing instability

Set up a box :
infinitesimal width in x, but wider than resistive layer
finite height in y, but less than the characteristic wavelength of the perturbation
arbitrary extent in z



Integrating $\nabla \cdot \mathbf{B}^1 = 0$ over the volume of the box and applying Gauss' theorem gives that B_x must be continuous across the boundary

$$B_x(x \to 0+) = B_x(x \to 0-)$$

From this we deduce that the value of B_x at each y-value may be taken to be constant throughout the resistive layer around $x = 0$.

Similarly, integrating the Ampere's law over the surface of the box in the (x, y)-plane and applying Stokes' theorem for the surface integral of a curl, we find that any discontinuity in B_y^1 must be associated with a 1st order 'surface current' J_z^1 flowing in the boundary layer, i.e.

$$B_y(x \to 0+) - B_y(x \to 0-) = \mu_0 J_z^1$$

which indicates that the y component of the field perturbation can be discontinuous across the boundary layer.

II. The resistive tearing instability

From $\nabla \cdot \mathbf{B}^1 = 0$, i.e.

$$\frac{\partial B_x}{\partial x} + ikB_y^1 = 0$$

the discontinuity in B_y^1 implies a discontinuity in $\frac{\partial B_x}{\partial x}$.

Thus, although B_x itself is continuous across the boundary layer, its gradient in x is not. The quantity

$$\Delta' = \frac{1}{B_x}\left(\frac{\partial B_x}{\partial x}|_{x=0+} - \frac{\partial B_x}{\partial x}|_{x=0-}\right)$$

measures this discontinuous jump across the boundary layer at $x = 0$.

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II. The resistive tearing instability

The growth rate of the resistive tearing mode

Within the layer, we may certainly take

$$B_y^0 = B_y'(0)x$$

and make use of the perturbed field component $B_x \simeq \bar{B}_x(const)$

Then Faraday's law becomes

$$\omega \bar{B}_x^1 + kB_y'(0)xV_x^1 = \frac{i\eta}{\mu_0}\frac{\partial^2 B_x}{\partial x^2}$$

In the full form of eq of motion, the x derivative tends to dominant over the y derivative in the thin resistive layer. Thus it can be approximated by

$$\begin{aligned} -\omega\rho^0\mu_0\frac{\partial^2 V_x}{\partial x^2} &= kB_y'(0)\frac{\partial}{\partial x}\left[x^2\frac{\partial}{\partial x}\left(\frac{B_x}{x}\right)\right] \\ &= kB_y'(0)\frac{\partial}{\partial x}\left(x\frac{\partial B_x}{\partial x} - B_y^0 B_x\right) \\ &= kB_y'(0)\frac{\partial^2 B_x}{\partial x^2} \end{aligned}$$

Substituting for $\frac{\partial^2 B_x}{\partial x^2}$ from the previous eq, this becomes

$$\gamma\eta\rho^0\frac{\partial^2 V_x}{\partial x^2} = kB_y'(0)x\left(i\gamma\bar{B}_x + kB_y'(0)xV_x\right)$$

The characteristic width of the resistive layer can be determined simply by balancing the term on the lhs against the 2nd term on the rhs

$$x \sim \delta = (\gamma\eta\rho^0)^{1/4}/(kB_y'(0))^{1/2}$$

The resistive layer becomes thinner as the resistivity η decreases.

We can also show that the resistive tearing instabilities grow on time-scales that are intermediate between the very short MHD time-scale, τ_A, and the very long resistive time-scale τ_R.

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II. The resistive tearing instability

Magnetic Island

The resistive tearing instability produces a change in the topology of the magnetic field.

before onset of the instability
The filed lines are straight and lie in the flat surfaces parallel to the (y, z) plane, and the direction of B_y component reverses across $x = 0$.

After onset of the instability, the magnetic configuration is deformed, and the field lines now lie on modified surfaces.

The eq of field lines is given by

$$\frac{dx}{dy} = \frac{B_x}{B_y}$$

For small amplitude perturbations, the B_y component can be approximated by its equilibrium value

$$B_y \sim B_y'(0)x$$

A particular chosen phase gives the B_x component at some particular time

$$B_x = \bar{B}_x e^{\gamma t} \sin(ky)$$

$\bar{B}_x$ takes an approximate const within the resistive layer around $x = 0$.

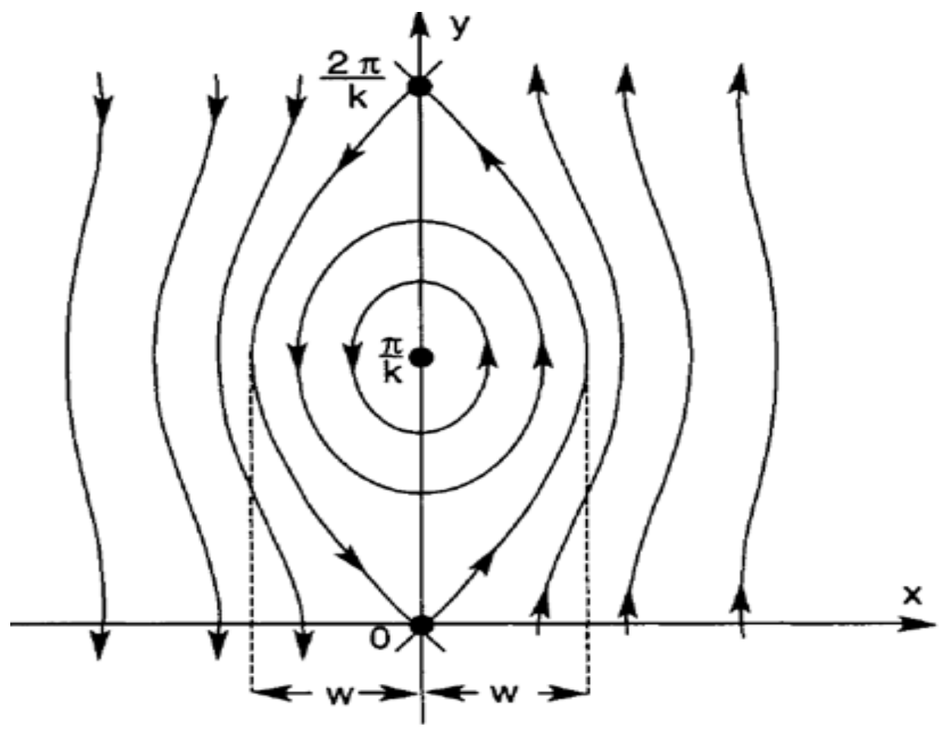
Then the eq of field line can be integrated to give

$$\frac{1}{2} B_y'(0) x^2 + \frac{\bar{B}_x}{k} e^{\gamma t} \cos(ky) = const$$

Where different values of the const give different field lines in the (x, y) plane.

The solutions can easily be plotted.

II. The resistive tearing instability



At relatively large values of $|x|$, corresponding to large values of the const, the field lines are only slightly distorted from the unperturbed configuration.

The distortion increases for smaller values of $|x|$, corresponding to smaller values of the const, the field lines eventually become 'closed on themselves'.

The 'closed' field lines arise from values of the const less than

$$const < \frac{\bar{B}x}{k} e^{\gamma t}$$

The closed field line regions are called 'magnetic island'.

The surface that separate the closed field line surfaces from the open field line surfaces is usually called the 'magnetic separatrix'.

The separatrix corresponds to a value of the const exactly equal to

$$const = \frac{\bar{B}x}{k} e^{\gamma t}$$

The half-width w of the magnetic island is simply the value of x for this value of const and at $ky = \pi$:

$$w = 2\sqrt{\frac{\bar{B}x}{kB_y'(0)}} e^{\gamma t/2}$$

Obviously it is proportional to the square-root of the field perturbation $\bar{B}_x$, so it increases exponentially in time.

If we attempted a numerical treatment of the effects of resistive tearing instabilities, we would expect to find not only a primary island chain, but also secondary chains of small islands.

When the mode amplitude grows so large that secondary islands begin to overlap with primary island or primary island chains begin to overlap with each other, then the magnetic structure becomes 'stochastic'.

III. Draft Wave Instabilities

Consider a uniform and finite extent in the y and z directions plasma slab, Any perturbation can be written as

$$\psi^1(\mathbf{x},t) = \hat{\psi}^1(x)e^{-i\omega t + ik_y y + ik_z z}$$

The perturbed eq of motion is written as

$$\begin{aligned}\rho^0 \frac{\partial \mathbf{V}^1}{\partial t} &= -\nabla p + (\mathbf{J}\times\mathbf{B})^1 \\ &= -\boldsymbol{\nabla}\left(\boldsymbol{p^1} + \frac{\mathbf{B^0}\cdot\mathbf{B^1}}{\boldsymbol{\mu_0}}\right) + \frac{\mathbf{1}}{\boldsymbol{\mu_0}}\left[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}\right]^{\mathbf{1}}\end{aligned}$$

Since the equilibrium field is entirely in the z direction, the x and y components are

$$\begin{aligned}-i\omega\rho^0 V_x &= -\frac{\partial}{\partial x}\left(p^1 + \frac{B_z^0 B_z^1}{\mu_0}\right) + \frac{ik_z}{\mu_0}B_z^0 B_x \\ -i\omega\rho^0 V_y &= -ik_y\left(p^1 + \frac{B_z^0 B_z^1}{\mu_0}\right) + \frac{ik_z}{\mu_0}B_z^0 B_y\end{aligned}$$

Assuming $k_z \ll k_y$, the divergence free condition of the perturbed magnetic field $0 = \nabla\cdot\mathbf{B}^1 = \frac{\partial B_x}{\partial x} + ik_y B_y + ik_z B_z = 0$ gives

$$\frac{\partial B_x}{\partial x} + ik_y B_y = 0$$

Taking the curl of eq of motion and considering the z component we have

$$\begin{aligned}-i\omega\left(\frac{\partial}{\partial x}(\rho^0 V_y) - ik_y\rho^0 V_x\right) &= \frac{ik_z B_z^0}{\mu_0}\left(\frac{\partial B_y}{\partial x} - ik_y B_x\right) \\ &= -\frac{k_z B_z^0}{\mu_0 k_y}\left(\frac{\partial^2 B_x}{\partial x^2} - k_y^2 B_x\right)\end{aligned}$$

With perp incompressible condition $0 = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = \frac{\partial V_x}{\partial x} + ik_y V_y$, we have

$$\frac{\omega\rho^0}{k_y}\left(\frac{\partial^2 V_x}{\partial x^2} - k_y^2 V_x\right) = -\frac{k_z B_z^0}{\mu_0 k_y}\left(\frac{\partial^2 B_x}{\partial x^2} - k_y^2 B_x\right)$$

III. Draft Wave Instabilities

Using WKB approximation, the derivative of a slowly varying function can be approximated by

$$\frac{\partial}{\partial x} = ik_x$$

Thus we have

$$-\frac{\omega\rho^0}{k_y}k_\perp^2 V_x = \frac{k_z B_z^0}{\mu_0 k_y}k_\perp^2 B_x, \;\; (k_\perp^2 = k_x^2 + k_y^2)$$

which is simplified to

$$\omega V_x = -k_z V_A^2 \frac{B_x}{B_z^0}$$

which is as much information as we can obtain from the perp components of the perturbed eq of motion. V_x and B_x will be related to each other through Ohm's law.

The perturbed Ohm's law is given by

$$\mathbf{E}^1 + \mathbf{V}^1 \times \mathbf{B}^0 = \eta \mathbf{J}^1 + \frac{1}{ne}(\mathbf{J} \times \mathbf{B} - \nabla p_e)^1$$

coupled with perturbed Faraday's law

$$\begin{aligned}\frac{\partial \mathbf{B}^1}{\partial t} &= -\nabla \times \mathbf{E}^1 \\ &= (\mathbf{B^0} \cdot \boldsymbol{\nabla})\mathbf{V^1} - (\mathbf{V^1} \cdot \boldsymbol{\nabla})\mathbf{B^0} - \mathbf{B^0}(\boldsymbol{\nabla} \cdot \mathbf{V^1}) - \boldsymbol{\nabla} \times (\mathbf{J} \times \mathbf{B} - \boldsymbol{\nabla} p_e)^1\end{aligned}$$

In the meantime, assuming ideal Ohm's law is a satisfactory approximation in the perp direction

$$\mathbf{E}_\perp = \mathbf{V} \times \mathbf{B}$$

the x component of Faraday's law can be written as

$$-i\omega B_x = ik_z B_z^0 V_x - ik_y \left[\eta J_z - \frac{1}{ne}\left(ik_z p_e^1 + \frac{B_x}{B_z^0}\frac{dp_e^0}{dx}\right)\right]$$

where J_z can be expressed in terms of B_x with Ampere's law

III. Draft Wave Instabilities

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

$$\begin{aligned} J_z &= \frac{1}{\mu_0}\left(\frac{\partial B_y}{\partial x} - ik_y B_x\right) \\ &= \frac{i}{\mu_0 k_y}\left(\frac{\partial^2 B_x}{\partial x^2} - k_y^2 B_x\right) \\ &\simeq -\frac{ik_\perp^2}{\mu_0 k_y} B_x \quad (k_\perp^2 = k_x^2 + k_y^2) \end{aligned}$$

Thus it becomes

$$\omega B_x + k_x B_z^0 V_x = -\frac{i\eta}{\mu_0} k_\perp^2 B_x - \frac{k_x}{ne}\left(ik_z p_e^1 + \frac{B_x}{B_z^0}\frac{dp_e^0}{dx}\right)$$

In order to eliminate p_e^1, we assume that the electrons are isothermal along the magnetic field line

$$\mathbf{B} \cdot \nabla Te = 0$$

Its perturbed form is

$$ik_z T_e^1 + \frac{B_x}{B_z^0}\frac{dT_e^0}{dx} = 0$$

Then

$$\begin{aligned} ik_z p_e^1 + \frac{B_x}{B_z^0}\frac{dp_e^0}{dx} &= ik_z T_e^0 n_e^1 + \frac{B_x}{B_z^0}\left(\frac{dp_e^0}{dx} - n_e^0 \frac{dT_e^0}{dx}\right) \\ &= T_e^0\left(ik_z n_e^1 + \frac{B_x}{B_z^0}\frac{dn_e^0}{dx}\right) \end{aligned}$$

Now we have a new variable n_e^1 needed to be eliminated. Using the perturbed continuity eq

$$\frac{\partial n_e^1}{\partial t} + \mathbf{V}_\perp \cdot \nabla n_e^0 + \nabla_\parallel (n_e^0 V_\parallel) = 0$$

i.e.

$$-i\omega n_e^1 + V_x \frac{dn_e^0}{dx} + ik_z n_e^0 V_z = 0$$

where $\nabla \cdot \mathbf{V}_\perp = 0$ is used.

The new variable V_z can be obtained from the parallel component of eq of motion

$$\rho^0 \frac{\partial V_\parallel}{\partial t} = -\nabla_\parallel p_e$$

Its perturbed form is given by

$$\begin{aligned} -i\omega\rho^0 V_z &= -ik_z p_e^1 - \frac{B_x}{B_z^0}\frac{dp_e^0}{dx} \\ &= -T_e^0\left(ik_z n_e^1 + \frac{B_x}{B_z^0}\frac{dn_e^0}{dx}\right) \end{aligned}$$

Combination of these expressions yields

$$-i\left(\omega - \frac{k_z^2 T_e^0}{\omega M}\right) n_e^1 + V_x \frac{dn_e^0}{dx} + \frac{k_z T_e^0}{\omega M}\frac{B_x}{B_z^0}\frac{dn_e^0}{dx} = 0$$

$\Longrightarrow$

$$(\omega B_x + k_z B_z^0 V_x)\left(1 - \frac{k_y V_{de}}{\omega - k_z^2 C_s^2/\omega}\right) = -\frac{i\eta}{\mu_0} k_\perp^2 B_x$$

Here

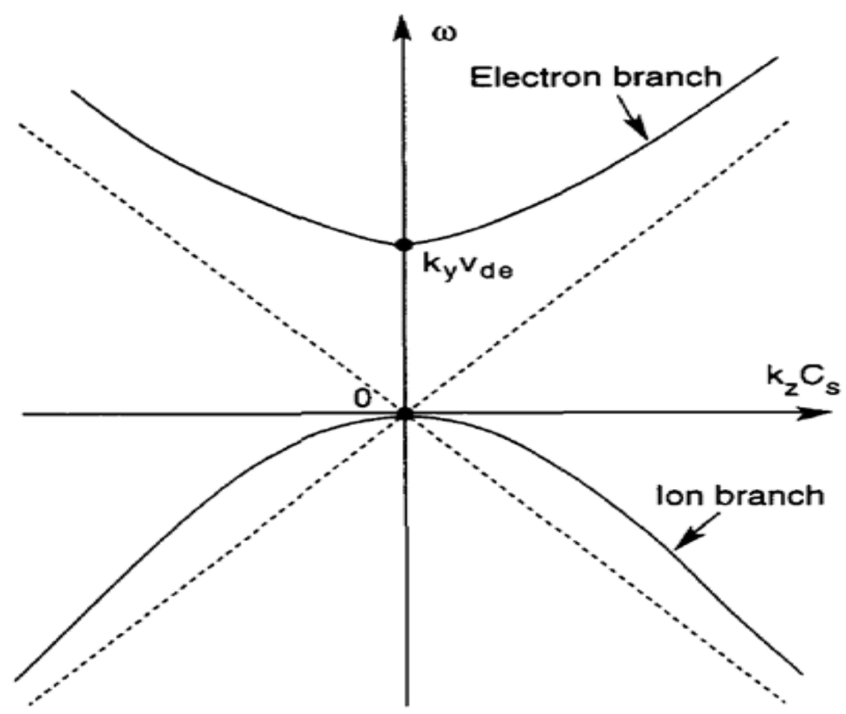
$$V_{de} = -\frac{T_e^0}{n_e^0 e B_z^0}\frac{dn_e^0}{dx}$$

is similar in form to the electron diamagnetic drift velocity.

Combining these two expressions for V_x and B_x yields the following dispersion relation

$$\left(\omega - \frac{k_z^2 V_A^2}{\omega}\right)\left(1 - \frac{k_y V_{de}}{\omega - k_z^2 C_s^2/\omega}\right) = -\frac{i\eta}{\mu_0} k_\perp^2$$

III. Draft Wave Instabilities



In the limit of zero resistivity, we see that there are 2 distinct branches of the dispersion relation. One branch has

$$\omega = k_z V_A$$

and clearly corresponds to the shear Alfven wave.

The 2nd branch has a dispersion relation

$$\omega - k_y V_{de} - \frac{k_z^2 C_s^2}{\omega} = 0$$

This is the drift-wave branch.

The effect of non-zero resistivity are to couple these two branches of the spectrum and to add an imaginary part, either a growth rate or a damping decrement, to their frequencies.

So when $\eta \neq 0$, the higher-frequency branch, i.e. the shear Alfven wave, is

$$\omega = \frac{k_z^2 V_A^2}{\omega} = -\frac{i\eta}{\mu_0} k_\perp^2$$

and the lower-frequency branch , i.e. the drift wave, is

$$\omega - k_y V_{de} - \frac{k_z^2 C_s^2}{\omega} = \frac{i\eta k_\perp^2}{\mu_0} \frac{\omega^2 - k_z^2 C_s^2}{k_z^2 V_A^2}$$

III. Draft Wave Instabilities

We examine the effect of resistivity first on the shear Alfven waves.

Treating the imaginary term on rhs as a small correction and allowing ω to acquire a correspondingly small imaginary part

$$\omega \rightarrow \omega + i\gamma$$

the imaginary part is simply

$$\gamma + (k_z^2 V_A^2/\omega^2) \simeq 2\gamma$$

which yields

$$\gamma \simeq -\eta k_\perp^2/2\mu_0$$

indicating that the shear Alfven waves are damped by resistivity (negative η).

Carrying out a similar analysis for the drift waves, letting

$$\omega \rightarrow \omega + i\gamma$$

and equating the imaginary part of order γ

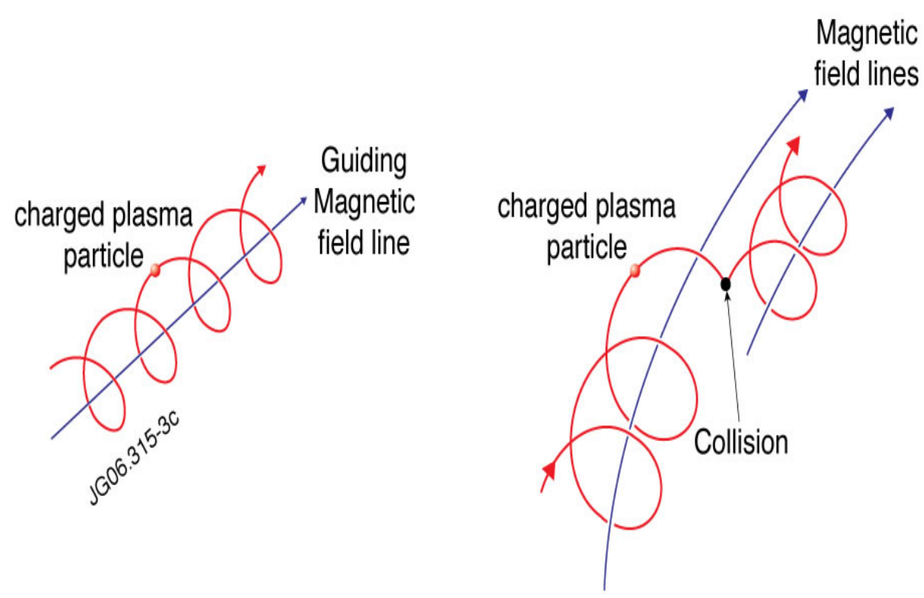
$$\gamma + (k_z^2 C_s^2/\omega^2)\gamma$$

on lhs to the imaginary expression on the rhs, we obtain

$$\gamma = \frac{\eta k_\perp^2}{\mu_0} \frac{\omega^2(\omega^2 - k_z^2 C_s^2)}{k_z^2 V_A^2(\omega^2 + k_z^2 C_s^2)}$$

which shows that the electron drift wave is always unstable (positive γ); whereas the ion branch is always damped.

IV. Edge Localized Modes (ELMs)



The plasma is composed of electrically charged particles, electrons and ions. Such electrically charged particles have the natural property of following magnetic field lines.

Some of these magnetic field lines intersect the solid materials of the vessel at some location. Charged plasma particles that happen to be on such field lines are therefore guided into collisions with the first wall and deposit their energy onto the plasma-facing material.

In an idealized scenario, plasma particles are safe from collisions with the first wall as long as they are guided along these closed field lines. However, there are processes that force plasma particles to leak out from the confined volume, particles diffuse across the magnetic field.

Particles may leave the confined volume simply due to the fact that their orbit around each field line has a finite radius.

They can jump from one guiding field line to another due to collisions with other plasma particles or due to fluctuating electric fields causing so called turbulent cross-field transport.

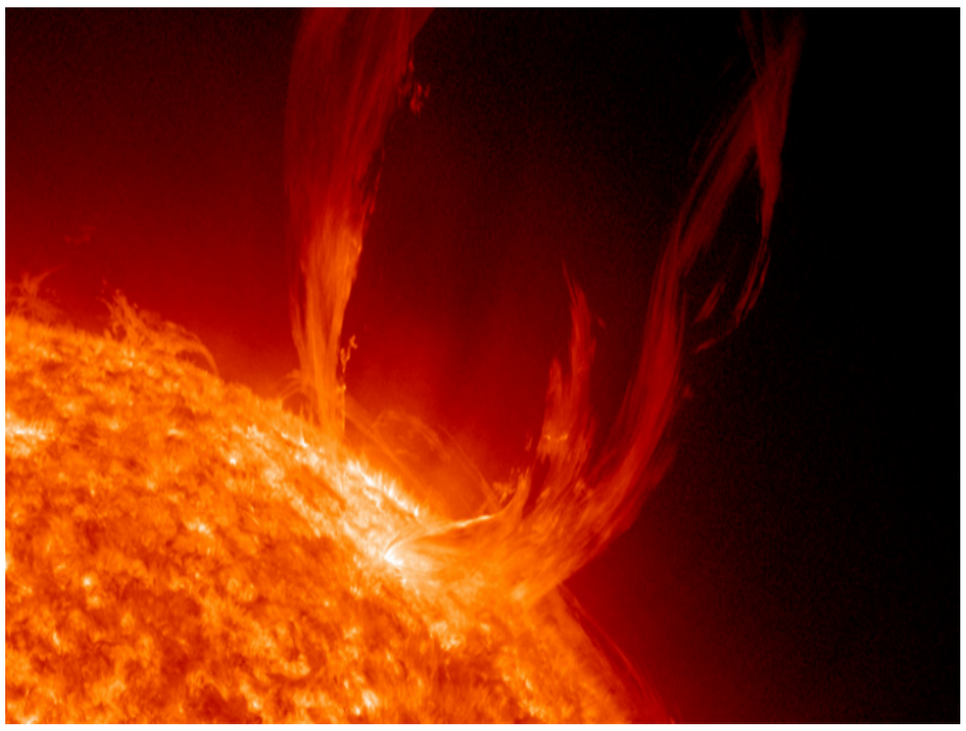
Because of complex instabilities, plasma particles can be ejected out of the region of closed field lines in big quantities during bursts, commonly this process is called Edge Localized Modes (ELMs).

Example: Magnetic Reconnection in solar system

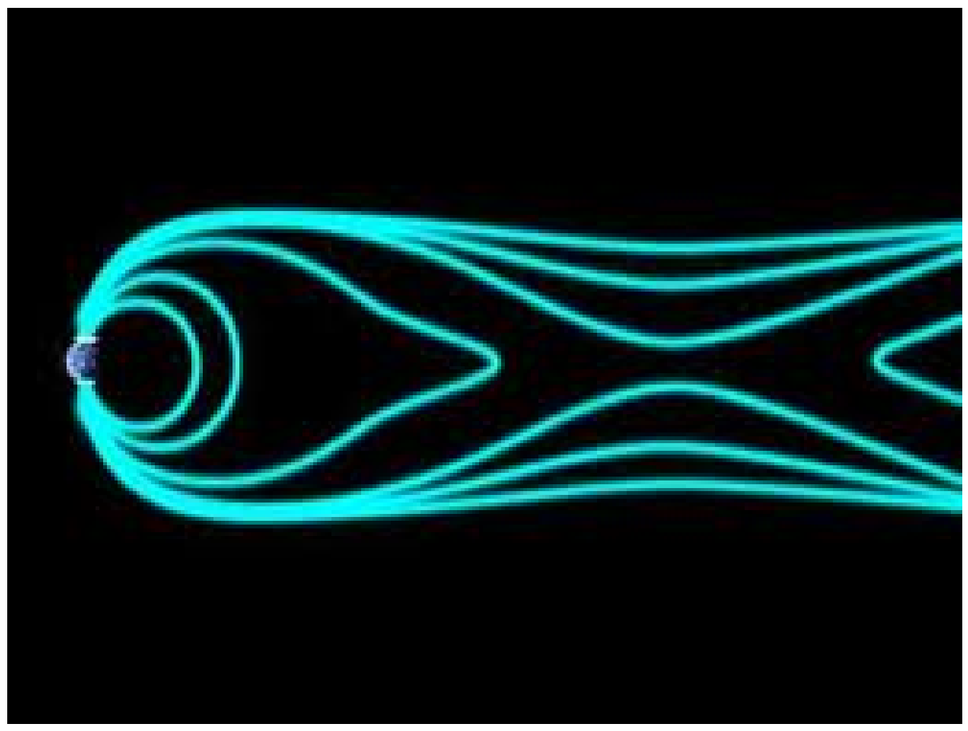
rapid release to the plasma of energy stored in the large scale structure of the coronal magnetic field

heat the corona, drive the outflow of the solar wind

eject high energy charged particles into space.



Example: Magnetic Reconnection on earth's Aurora



When the particles reach earth, they can disrupt power grids and communications systems and threaten space craft and satellites.

A related phenomenon is the aurora seen near the polar regions of earth.

The Earth's own magnetic field is constantly perturbed by the impinging solar wind.

During strong bursts reconnection can be induced in the near-earth magnetotail (a narrow magnetic field structure located on the night side many earth-radii away).

The tenuous plasma in that region is then accelerated down magnetic field lines into the polar regions, striking Earth's atmosphere and exciting nitrogen and oxygen atoms as well as other atoms present in our atmosphere.

The immediate de-excitation of these atoms then emit the wonderful and often intricate display of light we know as the aurora or northern (and southern) lights.

Magnetic Reconnection in Confined Device

The evolution of the magnetic field in a resistive-MHD plasma is governed by

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + (\eta/\mu_0)\boldsymbol{\nabla}^2\mathbf{B}$$

describing the convection of the magnetic field by the plasma flow and the resistive diffusion of the field through the plasma.

The relative magnitude of the two terms is conventionally measured as Lundquist number:

$$S = \frac{\mu_0 V L}{\eta} \sim \frac{|\nabla \times (\mathbf{V} \times \mathbf{B})|}{|(\eta/\mu_0)\nabla^2\mathbf{B}|}$$

V: the characteristic flow speed
L: the characteristic length-scale of the plasma.

$S > 1$: convection dominates, and the frozen flux constraint prevails
$S < 1$: diffusion dominates, and the coupling between the plasma flow and the magnetic field is relatively weak.

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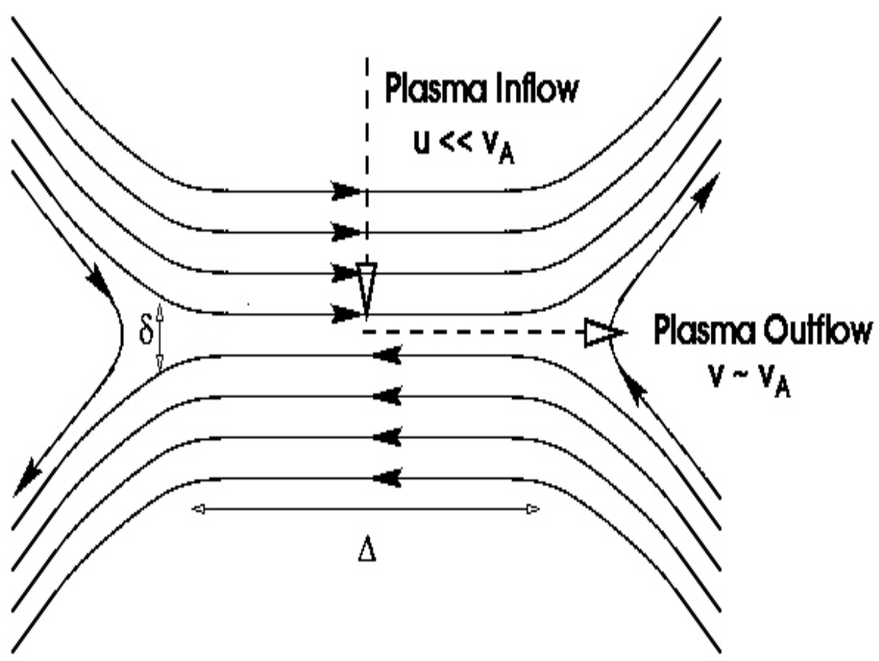
Magnetic Reconnection in Confined Device

when two initially separate plasma regions come into contact with one another Assuming that each plasma is frozen to its own magnetic field, and that cross-field diffusion is absent, we conclude that the two plasmas will not mix, but, instead, that a thin boundary layer will form between them, separating the two plasmas and their respective magnetic fields.

In equilibrium, the location of the boundary layer will be determined by pressure balance. This layer is called current sheet

The effects of diffusion can not be neglected in these boundary layers and leads to the localized breakdown of the flux freezing constraint in the boundary regions, due to diffusion

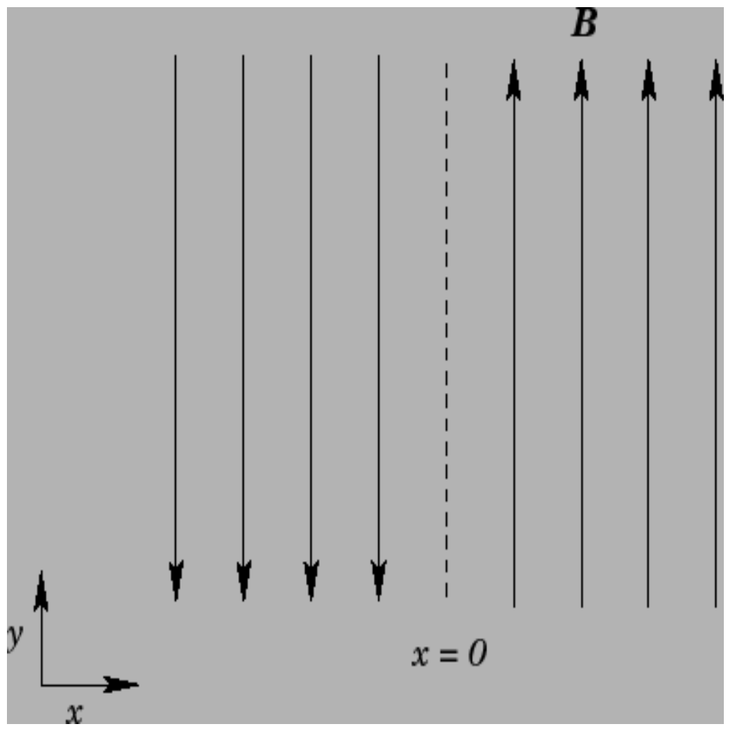


Linear Tearing Mode

We are particularly interested in plasma instabilities which are stable in the absence of resistivity, and only grow when the resistivity is non-zero.

Such instabilities are conventionally termed tearing modes, which is the tendency for a plasma to break up into magnetic islands in order to reduce the magnetic energy in the regions away from the islands.

Consider the interface between two plasmas containing magnetic fields of different orientations.



With the neglect of plasma resistivity, the field configuration represents a stable equilibrium state.

take resistivity into account $\boldsymbol{\eta} > 0$:

an instability is developed to relax the configuration to one possessing lower magnetic energy. and entails the breaking and reconnection of magnetic field lines, i.e., magnetic reconnection

The magnetic energy released during the reconnection process eventually appears as plasma thermal energy.

Thus, magnetic reconnection also involves plasma heating.

Linear Tearing Mode: theory

The equilibrium magnetic field is written

$$\mathbf{B}^0 = B_{0y}(x)\mathbf{y}, \qquad B_{0y}(-x) = -B_{0y}(x)$$

Considering no equilibrium plasma flow, and incompressible plasma fluid, the linearized equations of resistive-MHD,

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{B}) + (\eta/\mu_0)\boldsymbol{\nabla}^2\mathbf{B} \\
\rho^0 \frac{\partial \mathbf{V}}{\partial t} &= -\boldsymbol{\nabla} p + \frac{(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}^0}{(\boldsymbol{\nabla}\times\mathbf{B}^0)\times\mathbf{B}} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \cdot \mathbf{V} &= 0
\end{aligned}$$

ρ^0 : the equilibrium plasma density
B: the perturbed magnetic field,
V : the perturbed plasma velocity,
p: the perturbed plasma pressure

Suppose that all perturbed quantities vary like

$$A(x, y, z, t) = A(x)e^{iky+\gamma t}$$

γ : the instability growth-rate
k : the wavenumber

Linear Tearing Mode: theory

The x component and the z-component of the curl of it reduce to

$$\begin{aligned} \gamma B_x &= ikB_{0y}V_x + \frac{\boldsymbol{\eta}}{\mu_0}(\frac{d^2}{dx^2} - k^2)B_x \\ \gamma\rho^0(\frac{d^2}{dx^2} - k^2)V_x &= \frac{ikB_{0y}}{\mu_0}(\frac{d^2}{dx^2} - k^2 - \frac{B_y''(0)}{B_y^0}B_x \end{aligned}$$

defining

the Alfven time-scale : $\tau_A = \frac{a}{v_A}$ with Alfven velocity $v_A = \frac{B^0}{\sqrt{\mu_0\rho^0}}$

resistive diffusion time-scale : $\tau_R = \frac{\mu_0 a^2}{\boldsymbol{\eta}}$

The ratio of these two time-scales is the Lundquist number: $S = \frac{\tau_R}{\tau_A}$

and normalizing

$\psi = B_x/B^0$

$\phi = ikV_y/\gamma$

$\bar{x} = x/a$

$\bar{y} = y/a$

$F = B_{0y}/B^0$

$F' = dF/d\bar{x}$

$\bar{\gamma} = \gamma\tau_A$

$\bar{k} = ka$

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Linear Tearing Mode: theory

We have

$$\begin{aligned}\bar{\gamma}(\psi - F\phi) &= S^{-1}(\frac{d^2}{d\bar{x}^2} - \bar{k}^2)\psi \\ \bar{\gamma}^2(\frac{d^2}{d\bar{x}^2} - \bar{k}^2)\phi_x &= -\bar{k}^2(\frac{d^2}{d\bar{x}^2} - \bar{k}^2 - \frac{F''}{F})\psi\end{aligned}$$

left term: representing plasma inertia

right term: representing plasma resistivity

Solving these to get

The linear tearing instability grows on a hybrid time-scale

$$\tau_A \gg t_{\text{tearing mode}} \gg \tau_R$$

in boundary layer width.

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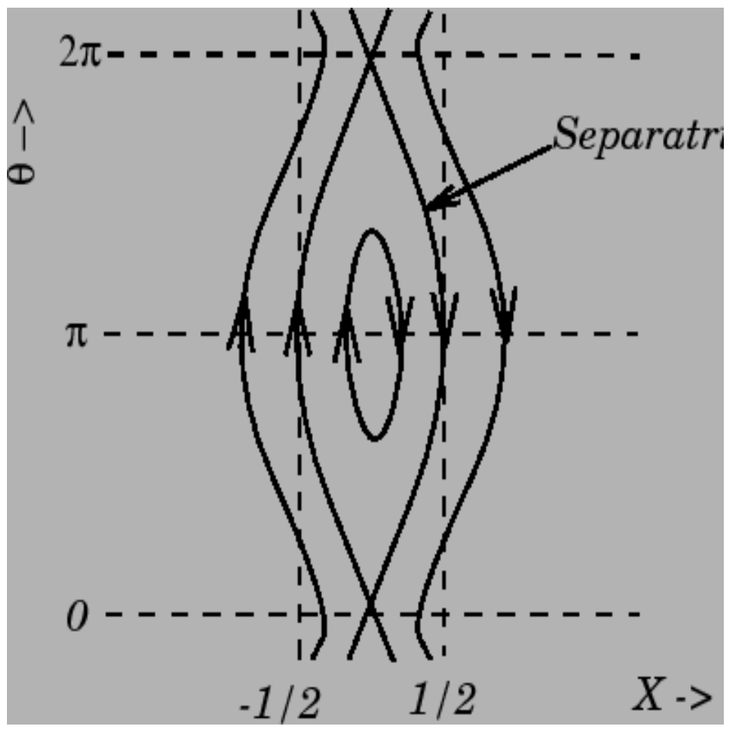
Nonlinear Tearing Mode



The tearing mode gives rise to the formation of a magnetic island centered on the interface.

Magnetic field lines situated outside the separatrix are displayed by the tearing mode, but still retain their original topology.

Magnetic field lines inside the separatrix have been broken and reconnected and now posses quite different topology. The maximum width of the reconnected region is given by the island width.

Once the tearing mode enters the nonlinear regime i.e., the normalized island width exceeds the normalized linear layer width,

the growth rate of the instability slows down considerably, until the mode eventually ends up growing on the extremely slow resistive time-scale τ_R.

It stops growing when it attained a saturated island width.

Nonlinear Tearing Mode

A magnetic island is a filament of plasma with its own set of nested flux surfaces surrounding its own magnetic axis.

Each island twists helically around the torus following the closed field line that forms its magnetic axis.

The whole structure of each island closes upon itself after going the long way around the torus one ore more times, depending upon the rational order of its magnetic axis.

Magnetic island can be traced out by following a field line within the island many times around the torus until the surface covered by that field line becomes clear.

The last closed surface forms a separatrix between the surfaces that close around the islands and the open surfaces.

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$m \geq 2$ resistive tearing mode

consider a cylindrical or toroidal magnetic field configuration with simply nested flux surfaces and finite shear

$m = 1$ ideal kink and resistive tearing mode

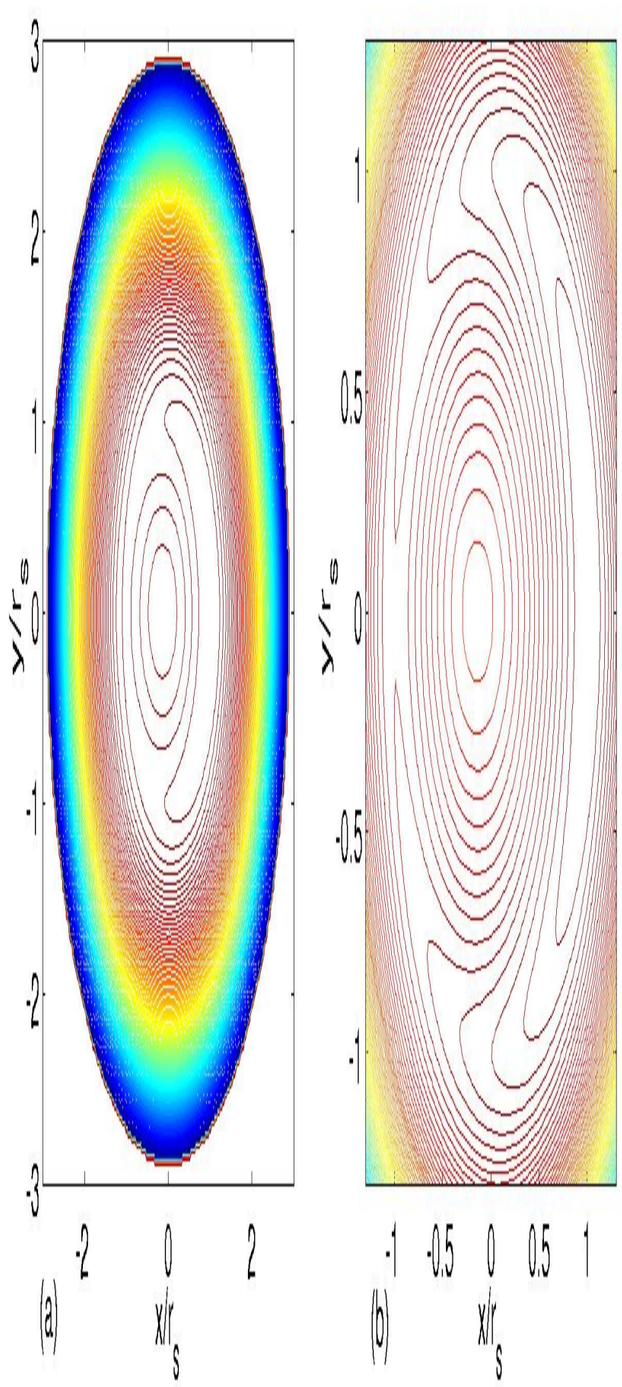
When any part of plasma column is bent, the poloidal field on the inner edge of the bend becomes stronger than the field on the outer edge. The resulting magnetic pressure then pushes the column to bend further and rapidly drives the plasma toward the wall.

the $m = 1$ resistive kink mode is a special case of a tearing mode

When there is sufficient resistivity in the boundary layer, an $m = 1$ tearing mode with one magnetic island is produced.

ideal mode	resistive mode
greater growth rate	smaller growth rate
growth rate decreases with shear	grow rate increases with shear
nonlinearly saturated at a small amplitude	continues to grow until the center of the plasma column is completely mixed
completely stabilized by finite Larmor radius effect	growth rate is only reduced without completely stabilization, nonlinearly develops into sawtooth oscillations

$m = 1$ ideal kink and resistive tearing mode



m=1 magnetic island

Magnetic Reconnection

However, when plasmas carrying oppositely directed magnetic field lines are brought together, a strong current sheet is established, in the presence of which even a vanishingly small amount of resistivity in a small volume can become important, allowing plasma diffusion and, thus, magnetic reconnection to occur.

Magnetic reconnection is an irreversible process in which magnetic fields with different direction merge together and dissipate in the diffusive region breaking the magnetic frozen-in condition and quickly converting magnetic energy into kinetic and thermal energy.

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