Chapter 5

Plasma Descriptions I: Kinetic, Two-Fluid

Descriptions of plasmas are obtained from extensions of the kinetic theory of gases and the hydrodynamics of neutral fluids (see Sections A.4 and A.6). They are much more complex than descriptions of charge-neutral fluids because of the complicating effects of electric and magnetic fields on the motion of charged particles in the plasma, and because the electric and magnetic fields in the plasma must be calculated self-consistently with the plasma responses to them. Additionally, magnetized plasmas respond very anisotropically to perturbations — because charged particles in them flow almost freely along magnetic field lines, gyrate about the magnetic field, and drift slowly perpendicular to the magnetic field.

The electric and magnetic fields in a plasma are governed by the Maxwell equations (see Section A.2). Most calculations in plasma physics assume that the constituent charged particles are moving in a vacuum; thus, the microscopic, "free space" Maxwell equations given in (**??**) are appropriate. For some applications the electric and magnetic susceptibilities (and hence dielectric and magnetization responses) of plasmas are derived (see for example Sections 1.3, 1.4 and 1.6); then, the macroscopic Maxwell equations are used. Plasma effects enter the Maxwell equations through the charge density and current "sources" produced by the response of a plasma to electric and magnetic fields:

$$\rho_q = \sum_s n_s q_s, \quad \mathbf{J} = \sum_s n_s q_s \mathbf{V}_s, \quad \text{plasma charge, current densities.} \tag{5.1}$$

Here, the subscript s indicates the charged particle species ($s = e, i$ for electrons, ions), n_s is the density (#/m^3) of species s, q_s the charge (Coulombs) on the species s particles, and $\mathbf{V}_s$ the species flow velocity (m/s). For situations where the currents in the plasma are small (e.g., for low plasma pressure) and the magnetic field, if present, is static, an electrostatic model ($\mathbf{E} = -\boldsymbol{\nabla}\phi$, $\boldsymbol{\nabla}\cdot\mathbf{E} = \rho_q/\epsilon_0 \Longrightarrow -\nabla^2\phi = \rho_q/\epsilon_0$) is often appropriate; then, only the charge density

ρ_q is needed. The role of a plasma description is to provide a procedure for calculating the charge density ρ_q and current density $\mathbf{J}$ for given electric and magnetic fields $\mathbf{E}, \mathbf{B}$.

Thermodynamic or statistical mehanics descriptions (see Sections A.3 and A.5) of plasmas are possible for some applications where plasmas are close to a Coulomb collisional equilibrium. However, in general such descriptions are not possible for plasmas — because plasmas are usually far from a thermodynamic or statistical mechanics equilibrium, and because we are often interested in short-time-scale plasma responses before Coulomb collisional relaxation processes become operative (on the $1/\nu$ time scale for fluid properties). Also, since the lowest order velocity distribution of particles is not necessarily an equilibrium Maxwellian distribution, we frequently need a kinetic decsription to determine the velocity as well as the spatial distribution of charged particles in a plasma.

The pedagogical approach we employ in this Chapter begins from a rigorous *microscopic* description based on the sum of the motions of all the charged particles in a plasma and then takes successive averages to obtain kinetic, fluid moment and (in the next chapter) magnetohydrodynamic (MHD) descriptions of plasmas. The first section, 5.1, averages the microscopic equation to develop a plasma kinetic equation. This fundamental plasma equation and its properties are explored in Section 5.2. [While, as indicated in (5.1), only the densities and flows are needed for the charge and current sources in the Maxwell equations, often we need to solve the appropriate kinetic equation and then take velocity-space averages of it to obtain the needed density and flow velocity of a particle species.] Then, we take averages over velocity space and use various approximations to develop macroscopic, fluid moment descriptions for each species of charged particles within a plasma (Sections 5.3*, 5.4*). The properties of a two-fluid (electrons, ions) description of a magnetized plasma [e.g., adiabatic, fluid (inertial) responses, and electrical resistivity and diffusion] are developed in the next section, 5.5. Then in Section 5.6*, we discuss the flow responses in a magnetized two-fluid plasma — parallel, cross ($\mathbf{E}\times\mathbf{B}$ and diamagnetic) and perpendicular (transport) to the magnetic field. Finally, Section 5.7 discusses the relevant time and length scales on which the kinetic and two-fluid models of plasmas are applicable, and hence useful for describing various unmagnetized plasma phenomena. This chapter thus presents the procedures and approximations used to progress from a rigorous (but extremely complicated) microscopic plasma description to succesively more approximate (but progressively easier to use) kinetic, two-fluid and MHD macroscopic (in the next chapter) descriptions, and discusses the key properties of each of these types of plasma models.

5.1 Plasma Kinetics

The word kinetic means "of or relating to motion." Thus, a kinetic description includes the effects of motion of charged particles in a plasma. We will begin from an exact (albeit enormously complicated), microscopic kinetic description that is based on and encompasses the motions of all the individual charged particles in the plasma. Then, since we are usually interested in average rather than individual particle properties in plasmas, we will take an appropriate average to obtain a general plasma kinetic equation. Here, we only indicate an outline of the derivation of the plasma kinetic equation and some of its important properties; more complete, formal derivations and discussions are presented in Chapter 13.

The microsopic description of a plasma will be developed by adding up the behavior and effects of all the individual particles in a plasma. We can consider charged particles in a plasma to be point particles — because quantum mechanical effects are mostly negligible in plasmas. Hence, the spatial distribution of a single particle moving along a trajectory $\mathbf{x}(t)$ can be represented by the delta function $\delta[\mathbf{x}-\mathbf{x}(t)] = \delta[x-x(t)]\,\delta[y-y(t)]\,\delta[z-z(t)]$ — see B.2 for a discussion of the "spikey" (Dirac) delta functions and their properties. Similarly, the particle's velocity space distribution while moving along the trajectory $\mathbf{v}(t)$ is $\delta[\mathbf{v}-\mathbf{v}(t)]$. Here, $\mathbf{x},\mathbf{v}$ are Eulerian (fixed) coordinates of a six-dimensional phase space (x, y, z, v_x, v_y, v_z), whereas $\mathbf{x}(t), \mathbf{v}(t)$ are the Lagrangian coordinates that move with the particle.

Adding up the products of these spatial and velocity-space delta function distributions for each of the $i = 1$ to N (typically $\sim 10^{16}$–10^{24}) charged particles of a given species in a plasma yields the "spikey" microscopic (superscript m) distribution for that species of particles in a plasma:

$$f^m(\mathbf{x},\mathbf{v},t) = \sum_{i=1}^{N} \delta[\mathbf{x}-\mathbf{x}_i(t)]\,\delta[\mathbf{v}-\mathbf{v}_i(t)], \quad \text{microscopic distribution function.} \tag{5.2}$$

The units of a distribution function are the reciprocal of the volume in the six-dimensional phase space $\mathbf{x},\mathbf{v}$ or # $/(\mathrm{m}^6\mathrm{s}^{-3})$ — recall that the units of a delta function are one over the units of its argument (see B.2). Thus, $d^3x\,d^3v\,f$ is the number of particles in the six-dimensional phase space differential volume between $\mathbf{x},\mathbf{v}$ and $\mathbf{x}+d\mathbf{x},\mathbf{v}+d\mathbf{v}$. The distribution function in (5.2) is normalized such that its integral over velocity space yields the particle density:

$$n^m(\mathbf{x},t) \equiv \int d^3v\,f^m(\mathbf{x},\mathbf{v},t) = \sum_{i=1}^{N} \delta[\mathbf{x}-\mathbf{x}_i(t)], \quad \text{particle density } (\#/\mathrm{m}^3). \tag{5.3}$$

Like the distribution f^m, this microscopic density distribution is very singular or spikey — it is infinite at the instantaneous particle positions $\mathbf{x} = \mathbf{x}_i(t)$ and zero elsewhere. Integrating the density over the volume V of the plasma yields the total number of this species of particles in the plasma: $\int_V d^3x\,n(\mathbf{x},t) = N$.

Particle trajectories $\mathbf{x}_i(t), \mathbf{v}_i(t)$ for each of the particles are obtained from their equations of motion in the microscopic electric and magnetic fields $\mathbf{E}^m, \mathbf{B}^m$:

$$m\,d\mathbf{v}_i/dt = q\,[\mathbf{E}^m(\mathbf{x}_i,t) + \mathbf{v}_i\times\mathbf{B}^m(\mathbf{x}_i,t)], \quad d\mathbf{x}_i/dt = \mathbf{v}_i, \quad i = 1,2,\ldots,N. \tag{5.4}$$

(The portion of the $\mathbf{E}^m, \mathbf{B}^m$ fields produced by the i^{th} particle is of course omitted from the force on the i^{th} particle.) In Eqs. (5.2)–(5.4), we have suppressed the species index s ($s = e, i$ for electrons, ions) on the distribution function f^m, the particle mass m and the particle charge q; it will be reinserted when needed, particularly when summing over species.

The microscopic electric and magnetic fields $\mathbf{E}^m, \mathbf{B}^m$ are obtained from the free space Maxwell equations:

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E}^m &= \rho_q^m/\epsilon_0, \quad & \boldsymbol{\nabla}\times\mathbf{E}^m &= -\partial\mathbf{B}^m/\partial t, \\ \boldsymbol{\nabla}\cdot\mathbf{B}^m &= 0, & \boldsymbol{\nabla}\times\mathbf{B}^m &= \mu_0(\mathbf{J}^m + \epsilon_0\partial\mathbf{E}^m/\partial t). \end{aligned} \tag{5.5}$$

The required microscopic charge and current sources are obtained by integrating the distribution function over velocity space and summing over species:

$$\begin{aligned} \rho_q^m(\mathbf{x},t) &\equiv \sum_s q_s \int d^3v\, f_s^m(\mathbf{x},\mathbf{v},t) = \sum_s q_s \sum_{i=1}^N \delta[\mathbf{x}-\mathbf{x}_i(t)], \\ \mathbf{J}^m(\mathbf{x},t) &\equiv \sum_s q_s \int d^3v\, \mathbf{v} f_s^m(\mathbf{x},\mathbf{v},t) = \sum_s q_s \sum_{i=1}^N \mathbf{v}_i(t)\, \delta[\mathbf{x}-\mathbf{x}_i(t)]. \end{aligned} \tag{5.6}$$

Equations (5.2)–(5.6) together with initial conditions for all the N particles provide a complete and exact microscopic description of a plasma. That is, they describe the exact motion of all the charged particles in a plasma, their consequent charge and current densities, the electric and magnetic fields they generate, and the effects of these microscopic fields on the particle motion — all of which must be calculated simultaneously and self-consistently. In principle, one can just integrate the N particle equations of motion (5.4) over time and obtain a complete description of the evolving plasma. However, since typical plasmas have 10^{16}–10^{24} particles, this procedure involves far too many equations to ever be carried out in practice[1] — see Problem 5.1. Also, since this description yields the detailed motion of all the particles in the plasma, it yields far more detailed information than we need for practical purposes (or could cope with). Thus, we need to develop an averaging scheme to reduce this microscopic description to a tractable set of equations whose solutions we can use to obtain physically measurable, average properties (e.g., density, temperature) of a plasma.

To develop an averaging procedure, it would be convenient to have a single evolution equation for the entire microscopic distribution f^m rather than having

[1]However, "particle-pushing" computer codes carry out this procedure for up to millions of scaled "macro" particles. The challenge for such codes is to have enough particles in each relevant phase space coordinate so that the noise level in the simulation is small enough to not mask the essential physics of the process being studied. High fidelity simulations are often possible for reduced dimensionality applications. Some relevant references for this fundamental computational approach are: J.M. Dawson, Rev. Mod. Phys. **55**, 403 (1983); C.K. Birdsall and A.B. Langdon, *Plasma Physics Via Computer Simulation* (McGraw-Hill, New York, 1985); R.W. Hockney and J.W. Eastwood, *Computer Simulation Using Particles* (IOP Publishing, Bristol, 1988).

to deal with a very large number (N) of particle equations of motion. Such an equation can be obtained by calculating the total time derivative of (5.2):

$$\begin{aligned}
\frac{df^m}{dt} &\equiv \left[\frac{\partial}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{d\mathbf{v}}{dt}\cdot\frac{\partial}{\partial \mathbf{v}}\right]\sum_{i=1}^{N}\delta[\mathbf{x}-\mathbf{x}_i(t)]\,\delta[\mathbf{v}-\mathbf{v}_i(t)] \\
&= \sum_{i=1}^{N}\left[\frac{\partial}{\partial t} + \frac{d\mathbf{x}_i}{dt}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{d\mathbf{v}_i}{dt}\cdot\frac{\partial}{\partial \mathbf{v}}\right]\delta[\mathbf{x}-\mathbf{x}_i(t)]\,\delta[\mathbf{v}-\mathbf{v}_i(t)] \\
&= \sum_{i=1}^{N}\left[-\frac{d\mathbf{x}_i}{dt}\cdot\frac{\partial}{\partial \mathbf{x}} - \frac{d\mathbf{v}_i}{dt}\cdot\frac{\partial}{\partial \mathbf{v}}\right. \\
&\qquad \left. + \frac{d\mathbf{x}_i}{dt}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{d\mathbf{v}_i}{dt}\cdot\frac{\partial}{\partial \mathbf{v}}\right]\delta[\mathbf{x}-\mathbf{x}_i(t)]\,\delta[\mathbf{v}-\mathbf{v}_i(t)] \\
&= 0.
\end{aligned} \tag{5.7}$$

Here in successive lines we have used three-dimensional forms of the properties of delta functions given in (**??**), and (**??**): $\mathbf{x}\,\delta(\mathbf{x}-\mathbf{x}_i) = \mathbf{x}_i\,\delta(\mathbf{x}-\mathbf{x}_i)$ and $\mathbf{v}\,\delta(\mathbf{v}-\mathbf{v}_i) = \mathbf{v}_i\,\delta(\mathbf{v}-\mathbf{v}_i)$, and $(\partial/\partial t)\,\delta[\mathbf{x}-\mathbf{x}_i(t)] = -d\mathbf{x}_i/dt\cdot(\partial/\partial\mathbf{x})\,\delta[\mathbf{x}-\mathbf{x}_i(t)]$ and $(\partial/\partial t)\,\delta[\mathbf{v}-\mathbf{v}_i(t)] = -d\mathbf{v}_i/dt\cdot(\partial/\partial\mathbf{v})\,\delta[\mathbf{v}-\mathbf{v}_i(t)]$. Substituting the equations of motion given in (5.4) into the second line of (5.7) and using the delta functions to change the functional dependences of the partial derivatives from $\mathbf{x}_i, \mathbf{v}_i$ to $\mathbf{x}, \mathbf{v}$, we find that the result $df^m/dt = 0$ can be written in the equivalent forms

$$\begin{aligned}
\frac{df^m}{dt} &\equiv \frac{\partial f^m}{\partial t} + \frac{d\mathbf{x}}{dt}\cdot\frac{\partial f^m}{\partial \mathbf{x}} + \frac{d\mathbf{v}}{dt}\cdot\frac{\partial f^m}{\partial \mathbf{v}} \\
&= \frac{\partial f^m}{\partial t} + \mathbf{v}\cdot\frac{\partial f^m}{\partial \mathbf{x}} + \frac{q}{m}[\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)]\cdot\frac{\partial f^m}{\partial \mathbf{v}} = 0.
\end{aligned} \tag{5.8}$$

This is called the Klimontovich equation.[2] Mathematically, it incorporates all N of the particle equations of motion into one equation because the mathematical characteristics of this first order partial differential equation in the seven independent, continuous variables $\mathbf{x}, \mathbf{v}, t$ are $d\mathbf{x}/dt = \mathbf{v}$, $d\mathbf{v}/dt = (q/m)[\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)]$, which reduce to (5.4) at the particle positions: $\mathbf{x} \to \mathbf{x}_i$, $\mathbf{v} \to \mathbf{v}_i$ for $i = 1, 2, \ldots, N$. That is, the first order partial differential equation (5.8) advances positions in the six-dimensional phase space $\mathbf{x}, \mathbf{v}$ along trajectories (mathematical characteristics) governed by the single particle equations of motion, independent of whether there is a particle at the particular phase point $\mathbf{x}, \mathbf{v}$; if say the $i^{\rm th}$ particle is at this point (i.e., $\mathbf{x} = \mathbf{x}_i,\ \mathbf{v} = \mathbf{v}_i$), then the trajectory (mathematical characteristic) is that of the $i^{\rm th}$ particle.

Equations (5.2), (5.5), (5.6) and (5.8) provide a complete, exact description of our microscopic plasma system that is entirely equivalent to the one given by (5.2)–(5.6); this Klimintovich form of the equations is what we will average below to obtain our kinetic plasma description. These and other properties of the Klimontovich equation are discussed in greater detail in Chapter 13.

[2] Yu. L. Klimontovich, *The Statistical Theory of Non-equilibrium Processes in a Plasma* (M.I.T. Press, Cambridge, MA, 1967); T.H. Dupree, Phys. Fluids **6**, 1714 (1963).

The usual formal procedure for averaging a microscopic equation is to take its ensemble average.[3] We will use a simpler, more physical procedure. We begin by defining the number of particles N_{6D} in a small box in the six-dimensional ($6D$) phase space of spatial volume $\Delta V \equiv \Delta x\,\Delta y\,\Delta z$ and velocity-space volume $\Delta V_v \equiv \Delta v_x\,\Delta v_y\,\Delta v_z$: $N_{6D} \equiv \int_{\Delta V} d^3x \int_{\Delta V_v} d^3v\; f^m$. We need to consider box sizes that are large compared to the mean spacing of particles in the plasma [i.e., $\Delta x >> n^{-1/3}$ in physical space and $\Delta v_x >> v_T/(n\lambda_D^3)^{1/3}$ in velocity space] so there are many particles in the box and hence the statistical fluctuations in the number of particles in the box will be small ($\delta N_{6D}/N_{6D} \sim 1/\sqrt{N_{6D}} << 1$). However, it should not be so large that macroscopic properties of the plasma (e.g., the average density) vary significantly within the box. For plasma applications the box size should generally be smaller than, or of order the Debye length λ_D for which $N_{6D} \sim (n\lambda_D^3)^2 >>>> 1$ — so collective plasma responses on the Debye length scale can be included in the analysis. Thus, the box size should be large compared to the average interparticle spacing but small compared to the Debye length, a criterion which will be indicated in its one-dimensional spatial form by $n^{-1/3} < \Delta x < \lambda_D$. Since $n\lambda_D^3 >> 1$ is required for the plasma state, a large range of Δx's fit within this inequality range.

The average distribution function $\langle f^m \rangle$ will be defined as the number of particles in such a small six-dimensional phase space box divided by the volume of the box:

$$\langle f^m(\mathbf{x},\mathbf{v},t)\rangle \equiv \lim_{n^{-1/3}<\Delta x<\lambda_D} \frac{N_{6D}}{\Delta V\,\Delta V_v} = \lim_{n^{-1/3}<\Delta x<\lambda_D} \frac{\int_{\Delta V} d^3x \int_{\Delta V_v} d^3v\, f^m}{\int_{\Delta V} d^3x \int_{\Delta V_v} d^3v}, \quad \text{average distribution function.} \tag{5.9}$$

From this form it is clear that the units of the average distribution function are the number of particles per unit volume in the six-dimensional phase space, i.e., $\#/(\mathrm{m}^6\,\mathrm{s}^{-3})$. In the next section we will identify the average distribution $\langle f^m \rangle$ as the fundamental plasma distribution function f.

The deviation of the complete microscopic distribution f^m from its average, which by definition must have zero average, will be written as δf^m:

$$\delta f^m \equiv f^m - \langle f^m \rangle, \quad \langle \delta f^m \rangle = 0, \quad \text{discrete particle distribution function.} \tag{5.10}$$

The average distribution function $\langle f^m \rangle$ represnts the smoothed properties of the plasma species for $\Delta x \gtrsim \lambda_D$; the microscopic distribution δf^m represents the "discrete particle" effects of individual charged particles for $n^{-1/3} \lesssim \Delta x < \lambda_D$.

This averaging procedure is illustrated graphically for a one-dimensional system in Fig. 5.1. As indicated, the microscopic distribution f^m is spikey — because it represents the point particles by delta functions. The average distribution function $\langle f^m \rangle$ indicates the average number of particles over length

[3] In an ensemble average one obtains "expectation values" by averaging over an infinite number of similar plasmas ("realizations") that have the same number of particles and macroscopic parameters (e.g., density n, temperature T) but whose particle positions vary randomly (in the six-dimensional phase space) from one realization to the next.

Figure 5.1: One-dimensional illustration of the microscopic distribution function f^m, its average $\langle f^m \rangle$ and its particle discreteness component δf^m.

scales that are large compared to the mean interparticle spacing. Finally, the discrete particle distribution function δf^m is spikey as well, but has a baseline of $-\langle f^m(x) \rangle$, so that its average vanishes.

In addition to splitting the distribution function into its smoothed and discrete particle contributions, we need to split the electric and magnetic fields, and charge and current densities into their smoothed and discrete particle parts components:

$$
\begin{aligned}
\mathbf{E}^m &= \langle \mathbf{E}^m \rangle + \delta \mathbf{E}^m, \quad & \mathbf{B}^m &= \langle \mathbf{B}^m \rangle + \delta \mathbf{B}^m, \\
\rho_q^m &= \langle \rho_q^m \rangle + \delta \rho_q^m, \quad & \mathbf{J}^m &= \langle \mathbf{J}^m \rangle + \delta \mathbf{J}^m.
\end{aligned} \tag{5.11}
$$

Substituting these forms into the Klimontovich equation (5.8) and averaging the resultant equation using the averaging definition in (5.9), we obtain our fundamental plasma kinetic equation:

$$
\begin{aligned}
\frac{\partial \langle f^m \rangle}{\partial t} + \mathbf{v} \cdot \frac{\partial \langle f^m \rangle}{\partial \mathbf{x}} + \frac{q}{m} [\langle \mathbf{E}^m \rangle + \mathbf{v} \times \langle \mathbf{B}^m \rangle] \cdot \frac{\partial \langle f^m \rangle}{\partial \mathbf{v}} &= \\
-\frac{q}{m} \left\langle [\delta \mathbf{E}^m + \mathbf{v} \times \delta \mathbf{B}^m] \cdot \frac{\partial \delta f^m}{\partial \mathbf{v}} \right\rangle .&
\end{aligned} \tag{5.12}
$$

The terms on the left describe the evolution of the smoothed, average distribution function in response to the smoothed, average electric and magnetic fields in the plasma. The term on the right represents the two-particle correlations between discrete charged particles within about a Debye length of each other. In fact, as can be anticipated from physical considerations and as will be shown in detail in Chapter 13, the term on the right represents the "small" Coulomb collision effects on the average distribution function $\langle f^m \rangle$, whose basic effects were calculated in Chapter 2. Similarly averaging the microscopic Maxwell equations (5.5) and charge and current density sources in (5.6), we obtain smoothed, average equations that have no extra correlation terms like the right side of (5.12).

5.2 Plasma Kinetic Equations

We now identify the smoothed, average [defined in (5.9)] of the microscopic distribution function $\langle f^m \rangle$ as the fundamental distribution function $f(\mathbf{x}, \mathbf{v}, t)$ for a species of charged particles in a plasma. Similarly, the smoothed, average of the microscopic electric and magnetic fields, and charge and current densities will be written in their usual unadorned forms: $\langle \mathbf{E}^m \rangle \to \mathbf{E}$, $\langle \mathbf{B}^m \rangle \to \mathbf{B}$, $\langle \rho_q^m \rangle \to \rho_q$, and $\langle \mathbf{J}^m \rangle \to \mathbf{J}$. Also, we write the right side of (5.12) as $\mathcal{C}(f)$ — a Coulomb collision operator on the average distribution function f which will be derived and discussed in Chapter 11. With these specifications, (5.12) can be written as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \cdot \frac{\partial f}{\partial \mathbf{v}} \quad = \quad \mathcal{C}(f), \quad f = f(\mathbf{x}, \mathbf{v}, t),$$

PLASMA KINETIC EQUATION. (5.13)

This is the fundamental plasma kinetic equation[4] we will use thoughout the remainder of this book to provide a kinetic description of a plasma. To complete the kinetic description of a plasma, we also need the average Maxwell equations, and charge and current densities:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho_q}{\epsilon_0}, \quad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \quad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \quad (5.14)$$

$$\rho_q(\mathbf{x}, t) \equiv \sum_s q_s \int d^3v \, f_s(\mathbf{x}, \mathbf{v}, t), \quad \mathbf{J}(\mathbf{x}, t) \equiv \sum_s q_s \int d^3v \, \mathbf{v} f_s(\mathbf{x}, \mathbf{v}, t). \quad (5.15)$$

Equations (5.13)–(5.15) are the fundamental set of equations that provide a complete kinetic description of a plasma. Note that all of the quantities in them are smoothed, average quantities that have been averaged according to the prescription in (5.9). The particle discreteness effects (correlations of particles due to their Coulomb interactions within a Debye sphere) in a plasma are manifested in the Coulomb collsion operator on the right of the plasma kinetic equation (5.13). In the averaging procedure we implicitly assume that the particle discreteness effects do not extend to distances beyond the Debye length λ_D. Chapter 13 discusses two cases (two-dimensional magnetized plasmas and convectively unstable plasmas) where this assumption breaks down. Thus, while we will hereafter use the average plasma kinetic equation (5.13) as our fundamental kinetic equation, we should keep in mind that there can be cases where the particle discreteness effects in a plasma are not completely represented by the Coulomb collision operator.

For low pressure plasmas where the plasma currents are negligible and the magnetic field (if present) is constant in time, we can use an electrostatic approximation for the electric field ($\mathbf{E} = -\boldsymbol{\nabla}\phi$). Then, (5.13)–(5.15) reduce to

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}[-\boldsymbol{\nabla}\phi + \mathbf{v} \times \mathbf{B}] \cdot \frac{\partial f}{\partial \mathbf{v}} = \mathcal{C}(f), \quad (5.16)$$

[4] Many plasma physics books and articles refer to this equation as the Boltzmann equation, thereby implicity indicating that the appropriate collision operator is the Boltzmann collision operator in (**??**). However, the Coulomb collision operator is a special case (small momentum transfer limit — see Chapter 11) of the Boltzmann collision operator $\mathcal{C}_B$, and importantly involves the cumulative effects (the $\ln \Lambda$ factor) of multiple small-angle, elastic Coulomb collsions within a Debye sphere that lead to diffusion in velocity-space. Also, the Boltzmann equation usually does not include the electric and magnetic field effects on the charged particle trajectories during collisions or on the evolution of the distribution function. Thus, this author thinks it is not appropriate to call this the Boltzmann equation.

$$-\nabla^2\phi = \frac{\rho_q}{\epsilon_0}, \quad \rho_q = \sum_s q_s \int d^3v f(\mathbf{x},\mathbf{v},t), \tag{5.17}$$

which provides a complete electrostatic, kinetic description of a plasma.

Some alternate forms of the general plasma kinetic equation (5.13) are also useful. First, we derive a "conservative" form of it. Since $\mathbf{x}$ and $\mathbf{v}$ are independent Eulerian phase space coordinates, using the vector identity (**??**) we find

$$\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v} f = \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + f\left(\frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v}\right) = \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}}.$$

Similarly, for the velocity derivative we have

$$\frac{\partial}{\partial \mathbf{v}} \cdot \frac{q}{m}[\mathbf{E} + \mathbf{v}\times\mathbf{B}]\, f = \frac{q}{m}[\mathbf{E} + \mathbf{v}\times\mathbf{B}] \cdot \frac{\partial f}{\partial \mathbf{v}},$$

since $\partial/\partial\mathbf{v} \cdot [\mathbf{E} + \mathbf{v}\times\mathbf{B}] = 0$ because $\mathbf{E}, \mathbf{B}$ are both independent of $\mathbf{v}$, and $\partial/\partial\mathbf{v} \cdot \mathbf{v}\times\mathbf{B} = 0$ using vector identities (**??**) and (**??**). Using these two results we can write the plasma kinetic equation as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot [\mathbf{v} f] + \frac{\partial}{\partial \mathbf{v}} \cdot \left[\frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B})\, f\right] \;=\; \mathcal{C}(f), \quad \text{conservative form of plasma kinetic equation,} \tag{5.18}$$

which is similar to the corresponding neutral gas kinetic equation (**??**). Like for the kinetic theory of gases, we can put the left side of the plasma kinetic equation in a conservative form because (in the absence of collisions) motion (of particles or along the characteristics) is incompressible in the six-dimensional phase space $\mathbf{x}, \mathbf{v}$: $\partial/\partial\mathbf{x} \cdot (d\mathbf{x}/dt) + \partial/\partial\mathbf{v} \cdot (d\mathbf{v}/dt) = \partial/\partial\mathbf{x} \cdot \mathbf{v} + \partial/\partial\mathbf{v} \cdot (q/m)[\mathbf{E} + \mathbf{v}\times\mathbf{B}] = 0$ — see (**??**).

In a magnetized plasma with small gyroradii compared to perpendicular gradient scale lengths ($\varrho\boldsymbol{\nabla}_\perp << 1$) and slow processes compared to the gyrofrequency ($\partial/\partial t << \omega_c$), it is convenient to change the independent phase space variables from $\mathbf{x}, \mathbf{v}$ phase space to the guiding center coordinates $\mathbf{x}_g, \varepsilon_g, \mu$. (The third velocity-space variable would be the gyromotion angle φ, but that is averaged over to obtain the guiding center motion equations — see Section 4.4.) Recalling the role of the particle equations of motion (5.4) in obtaining the Klimintovich equation, we see that in terms of the guiding center coordinates the plasma kinetic equation becomes $\partial f/\partial t + d\mathbf{x}_g/dt \cdot \boldsymbol{\nabla} f + (d\mu/dt)\,\partial f/\partial\mu + (d\varepsilon_g/dt)\,\partial f/\partial\varepsilon_g = \mathcal{C}(f)$. The gyroaverage of the time derivative of the magnetic moment and $\partial f/\partial\mu$ are both small in the small gyroradius expansion; hence their product can be neglected in this otherwise first order (in a small gyroradius expansion) plasma kinetic equation. The time derivative of the energy can be calculated to lowest order (neglecting the drift velocity $\mathbf{v}_D$) using the guiding center equation (**??**), writing the electric field in its general form $\mathbf{E} = -\boldsymbol{\nabla}\Phi - \partial\mathbf{A}/\partial t$ and $d/dt = \partial/\partial t + d\mathbf{x}_g/dt \cdot \partial/\partial\mathbf{x} \simeq \partial/\partial t + v_\parallel \nabla_\parallel$:

$$\frac{d\varepsilon_g}{dt} \;=\; \frac{d}{dt}\left(\frac{m v_\parallel^2}{2}\right) + q\frac{d\Phi}{dt} + \mu\frac{dB}{dt} \;\simeq\; q\frac{\partial\Phi}{\partial t} + \mu\frac{\partial B}{\partial t} - q v_\parallel \hat{\mathbf{b}} \cdot \frac{\partial \mathbf{A}}{\partial t}. \tag{5.19}$$

Thus, after averaging the plasma kinetic equation over the gyromotion angle φ, the plasma kinetic equation for the gyro-averaged, guiding-center distribution function f_g can be written in terms of the guiding center coordinates (to lowest order — neglecting $v_{D\|}$) as

$$\frac{\partial f_g}{\partial t} + v_\| \,\hat{\mathbf{b}}\cdot\boldsymbol{\nabla} f_g + \mathbf{v}_{D\perp}\cdot\boldsymbol{\nabla} f_g + \frac{d\varepsilon_g}{dt}\frac{\partial f_g}{\partial \varepsilon_g} \quad = \quad \langle \mathcal{C}(f_g)\rangle_\varphi, \quad f_g = f(\mathbf{x}_g, \varepsilon_g, \mu, t),$$

drift-kinetic equation, (5.20)

in which the collision operator is averaged over gyrophase [see discussion before (**??**)] and the spatial gradient is taken at constant ε_g, μ, t, i.e., $\boldsymbol{\nabla} \equiv \partial/\partial\mathbf{x}\,|_{\varepsilon_g,\mu,t}$. This lowest order drift-kinetic equation is sufficient for most applications. However, like the guiding center orbits it is based on, it is incorrect at second order in the small gyroradius expansion [for example, it cannot be put in the conservative form of (5.18) or (**??**)]. More general and accurate "gyrokinetic" equations that include finite gyroradius effects ($\varrho\boldsymbol{\nabla}_\perp \sim 1$) have also been derived; they are used when more precise and complete equations are needed.

For many plasma processes we will be interested in short time scales during which Coulomb collision effects are negligible. For these situations the plasma kinetic equation becomes

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}[\mathbf{E} + \mathbf{v}\times\mathbf{B}]\cdot\frac{\partial f}{\partial \mathbf{v}} = 0,$$

Vlasov equation. (5.21)

This equation, which is also called the collisionless plasma kinetic equation, was originally derived by Vlasov[5] by neglecting the particle discreteness effects that give rise to the Coulomb collisional effects — see Problem 5.2. Because the Vlasov equation has no discrete particle correlation (Coulomb collision) effects in it, it is completely reversible (in time) and its solutions follow the collisionless single particle orbits in the six-dimensional phase space. Thus, its distribution function solutions are entropy conserving (there is no irreversible relaxation of irregularities in the distribution function), and, like the particle orbits, incompresssible in the six-dimensional phase space — see Section 13.1.

The nominal condition for the neglect of collisional effects is that the frequency of the relevant physical process(es) be much larger than the collision frequency: $d/dt \sim -i\omega >> \nu$, in which ν is the Lorentz collision frequency (**??**). Here, the frequency ω represents whichever of the various fundamental frequencies (e.g., ω_p, plasma; kc_S, ion acoustic; ω_c, gyrofrequency; ω_b, bounce; ω_D, drift) are relevant for a particular plasma application. However, since the Coulomb collision process is diffusive in velocity space (see Section 2.1 and Chapter 11), for processes localized to a small region of velocity space $\delta\vartheta \sim \delta v_\perp/v << 1$, the effective collision frequency (for scattering out of this narrow region of velocity space) is $\nu_{\text{eff}} \sim \nu/\delta\vartheta^2 >> \nu$. For this situation the relevant condition for validity of the Vlasov equation becomes $\omega >> \nu_{\text{eff}}$. Often,

[5]A.A. Vlasov, J. Phys. (U.S.S.R.) **9**, 25 (1945).

the Vlasov equation applies over most of velocity space, but collisions must be taken into account to resolve singular regions where velocity-space derivatives of the collisionless distribution function are large.

Finally, we briefly consider equilibrium solutions of the plasma kinetic and Vlasov equations. When the collision operator is dominant in the plasma kinetic equation (i.e., $\nu >> \omega$), the lowest order distribution is the Maxwellian distribution [see Chapter 11 and (**??**)]:

$$f_M(\mathbf{x},\mathbf{v},t) = n\left(\frac{m}{2\pi T}\right)^{3/2}\exp\left(-\frac{m|\mathbf{v}_r|^2}{2T}\right) = \frac{n e^{-v_r^2/v_T^2}}{\pi^{3/2}v_T^3}, \quad \mathbf{v}_r \equiv \mathbf{v}-\mathbf{V},$$
$$\text{Maxwellian distribution function.} \qquad (5.22)$$

Here, $v_T \equiv \sqrt{2T/m}$ is the thermal velocity, which is the most probable speed [see (**??**)] in the Maxwellian distribution. Also, $n(\mathbf{x},t)$ is the density (units of $\#/m^3$), $T(\mathbf{x},t)$ is the temperature (J or eV) and $\mathbf{V}(\mathbf{x},t)$ the macroscopic flow velocity (m/s) of the species of charged particles being considered. Note that the $\mathbf{v}_r$ in (5.22) represents the velocity of a particular particle in the Maxwellian distribution relative to the average macroscopic flow velocity of the entire distribution of particles: $\mathbf{V} \equiv \int d^3v\, \mathbf{v} f_M/n$. It can be shown (see Chapter 13) that the collisionally relaxed Maxwellian distribution has no free energy in velocity space to drive (kinetic) instabilities (collective fluctuations whose magnitude grows monotonically in time) in a plasma; however, its spatial gradients (e.g., $\boldsymbol{\nabla} n$ and $\boldsymbol{\nabla} T$) provide spatial free energy sources that can drive fluidlike (as opposed to kinetic) instabilities — see Chapters 21–23.

If collisions are negligible for the processes being considered (i.e., $\omega >> \nu_{\rm eff}$), the Vlasov equation is applicable. When there exist constants of the single particle motion c_i (e.g., energy $c_1 = \varepsilon$, magnetic moment $c_2 = \mu$, etc. which satisfy $dc_i/dt = 0$), solutions of the Vlasov equation can be written in terms of them:

$$\begin{aligned} f &= f(c_1, c_2, \cdots), \quad c_i = \text{constants of motion}, \quad \text{Vlasov equation solution}, \\ \Longrightarrow &\quad \frac{df}{dt} = \sum_i \frac{dc_i}{dt}\frac{\partial f}{\partial c_i} = 0. \end{aligned} \qquad (5.23)$$

A particular Vlasov solution of interest is when the energy ε is a constant of the motion and the equilibrium distribution function depends only on it: $f_0 = f_0(\varepsilon)$. If such a distribution is a monotonically decreasing function of the energy (i.e., $df_0/d\varepsilon < 0$), then one can readily see from physical considerations and show mathematically (see Section 13.1) that this equilibrium distribution function has no free energy available to drive instabilities — because all possible rearrangements of the energy distribution, which must be area-preserving in the six-dimensional phase space because of the Vlasov equation $df/dt = 0$, would raise the system energy $\int d^3x \int d^3v\,(mv^2/2) f(\varepsilon)$ leaving no free energy available to excite unstable electric or magnetic fluctuations. Thus, we have the statement

$$f_0 = f_0(\varepsilon), \text{ with } df_0/d\varepsilon < 0, \text{ is a kinetically stable distribution.} \qquad (5.24)$$

Note that the Maxwellian distribution in (5.22) satisfies these conditions if there are no spatial gradients in the plasma density, temperature or flow velocity. However, "confined" plasmas must have additional dependencies on spatial coordinates[6] or constants of the motion — so they can be concentrated in regions within and away from the plasma boundaries. Thus, most plasmas of interest do not satisfy (5.24). The stability of such plasmas has to be investigated mostly on a case-by-case basis. When instabilities occur they usually provide the dominant mechanisms for relaxing plasmas toward a stable (but unconfined plasma) distribution function of the type given in (5.24).

5.3 Fluid Moments*

For many plasma applications, fluid moment (density, flow velocity, temperature) descriptions of a charged particle species in a plasma are sufficient. This is generally the case when there are no particular velocities or regions of velocity space where the charged particles behave differently from the typical thermal particles of that species. In this section we derive fluid moment evolution equations by calculating the physically most important velocity-space moments of the plasma kinetic equation (density, momentum and energy) and discuss the "closure moments" needed to close the fluid moment hierarchy of equations. This section is mathematically intensive with many physical details for the various fluid moments; it can skipped since the key features of fluid moment equations for electrons and ions are summarized at the beginning of the section after the next one.

Before beginning the derivation of the fluid moment equations, it is convenient to define the various velocity moments of the distribution function we will need. The various moments result from integrating low order powers of the velocity $\mathbf{v}$ times the distribution function f over velocity space in the laboratory frame: $\int d^3v\, \mathbf{v}^j f$, $j = 0, 1, 2$. The integrals are all finite because the distribution function must fall off sufficiently rapidly with speed so that these low order, physical moments (such as the energy in the species) are finite. That is, we cannot have large numbers of particles at arbitrarily high energy because then the energy in the species would be unrealistically large or divergent. [Note that velocity integrals of all algebraic powers of the velocity times the Maxwellian distribution (5.22) converge — see Section C.2.] The velocity moments of the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ of physical interest are

$$\text{density } (\#/\text{m}^3): \qquad n \equiv \int d^3v\, f, \tag{5.25}$$

$$\text{flow velocity (m/s)}: \qquad \mathbf{V} \equiv \frac{1}{n}\int d^3v\, \mathbf{v} f, \tag{5.26}$$

[6] One could use the potential energy term $q\phi(\mathbf{x})$ in the energy to confine a particular species of plasma particles — but the oppositely charged species would be repelled from the confining region and thus the plasma would not be quasineutral. However, nonneutral plasmas can be confined by a potential ϕ.

$$\text{temperature (J, eV)}: \qquad T \equiv \frac{1}{n}\int d^3v\, \frac{mv_r^2}{3} f = \frac{mv_T^2}{2}, \tag{5.27}$$

$$\text{conductive heat flux (W/m}^2\text{)}: \qquad \mathbf{q} \equiv \int d^3v\, \mathbf{v}_r \left(\frac{mv_r^2}{2}\right) f, \tag{5.28}$$

$$\text{pressure (N/m}^2\text{)}: \qquad p \equiv \int d^3v\, \frac{mv_r^2}{3} f = nT, \tag{5.29}$$

$$\text{pressure tensor (N/m}^2\text{)}: \qquad \mathsf{P} \equiv \int d^3v\, m\mathbf{v}_r\mathbf{v}_r f = p\mathsf{I} + \boldsymbol{\pi}, \tag{5.30}$$

$$\text{stress tensor (N/m}^2\text{)}: \qquad \boldsymbol{\pi} \equiv \int d^3v\, m\left(\mathbf{v}_r\mathbf{v}_r - \frac{v_r^2}{3}\mathsf{I}\right) f, \tag{5.31}$$

in which we have defined and used

$$\mathbf{v}_r \equiv \mathbf{v} - \mathbf{V}(\mathbf{x}, t), \quad v_r \equiv |\mathbf{v}_r|, \quad \text{relative (subscript } r\text{) velocity, speed.} \tag{5.32}$$

By definition, we have $\int d^3v\, \mathbf{v}_r f = n(\mathbf{V} - \mathbf{V}) = \mathbf{0}$. For simplicity, the species subscript $s = e, i$ is omitted here and thoughout most of this section; it is inserted only when needed to clarify differences in properties of electron and ion fluid moments.

All these fluid moment properties of a particular species s of charged particles in a plasma are in general functions of spatial position $\mathbf{x}$ and time t: $n = n(\mathbf{x}, t)$, etc. The density n is just the smoothed average of the microscopic density (5.3). The flow velocity $\mathbf{V}$ is the macroscopic flow velocity of this species of particles. The temperature T is the average energy of this species of particles, and is measured in the rest frame of this species of particles — hence the integrand is $(mv_r^2/2)f$ instead of $(mv^2/2)f$. The conductive heat flux $\mathbf{q}$ is the flow of energy density, again measured in the rest frame of this species of particles. The pressure p is a scalar function that represents the isotropic part of the expansive stress ($p\mathsf{I}$ in P in which I is the identity tensor) of particles since their thermal motion causes them to expand isotropically (in the species rest frame) away from their initial positions. This is an isotropic expansive stress on the species of particles because the effect of the thermal motion of particles in an isotropic distribution is to expand uniformly in all directions; the net force (see below) due to this isotropic expansive stress is $-\boldsymbol{\nabla}\cdot p\mathsf{I} = -\mathsf{I}\cdot\boldsymbol{\nabla}p - p\boldsymbol{\nabla}\cdot\mathsf{I} = -\boldsymbol{\nabla}p$ (in the direction from high to low pressure regions), in which the vector, tensor identities (**??**), (**??**) and (**??**) have been used. The pressure tensor P represents the overall pressure stress in the species, which can have both isotropic and anisotropic (e.g., due to flows or magnetic field effects) stress components. Finally, the stress tensor $\boldsymbol{\pi}$ is a traceless, six-component symmetric tensor that represents the anisotropic components of the pressure tensor.

In addition, we will need the lowest order velocity moments of the Coulomb collision operator $\mathcal{C}(f)$. The lowest order forms of the needed moments can be inferred from our discussion of Coulomb collisions in Section 2.3:

$$\text{density conservation in collisions}: \qquad 0 = \int d^3v\, \mathcal{C}(f), \tag{5.33}$$

$$\text{frictional force density (N/m}^3\text{)}: \qquad \mathbf{R} \equiv \int d^3v\, m\mathbf{v}\, \mathcal{C}(f), \tag{5.34}$$

$$\text{energy exchange density (W/m}^3\text{)}: \qquad Q \equiv \int d^3v\, \frac{mv_r^2}{2}\, \mathcal{C}(f). \tag{5.35}$$

As indicated in the first of these moments, since Coulomb collisions do not create or destroy charged particles, the density moment of the collision operator vanishes. The momentum moment of the Coulomb collision operator represents the (collisional friction) momentum gain or loss per unit volume from a species of charged particles that is flowing relative to another species: $\mathbf{R}_e \simeq -m_e n_e \nu_e(\mathbf{V}_e - \mathbf{V}_i) = n_e e \mathbf{J}/\sigma$ and $\mathbf{R}_i = -\mathbf{R}_e$ from (**??**) and (**??**). Here, rigorously speaking, the electrical conductivity σ is the Spitzer value (**??**). (The approximate equality here means that we are neglecting the typically small effects due to temperature gradients that are needed for a complete, precise theory — see Section 12.2.) The energy moment of the collision operator represents the rate of Coulomb collisional energy exchange per unit volume between two species of charged particles of different temperatures: $Q_i = 3(m_e/m_i)\nu_e n_e(T_e - T_i)$ and $Q_e \simeq J^2/\sigma - Q_i$ from (**??**) and (**??**). In a magnetized plasma, the electrical conductivity along the magnetic field is the Spitzer value [$\sigma_\parallel = \sigma_{\rm Sp}$ from (**??**)], but perpendicular to the magnetic field it is the reference conductivity [$\sigma_\perp = \sigma_0$ from (**??**)] (because the gyromotion induced by the $\mathbf{B}$ field impedes the perpendicular motion and hence prevents the distortion of the distribution away from a flow-shifted Maxwellian — see discussion near the end of Section 2.2 and in Section 12.2). Thus, in a magnetized plasma $\mathbf{R}_e = n_e e(\hat{\mathbf{b}} J_\parallel/\sigma_\parallel + \mathbf{J}_\perp/\sigma_\perp)$ and $Q_e = J_\parallel^2/\sigma_\parallel + J_\perp^2/\sigma_\perp - Q_i$.

As in the kinetic theory of gases, fluid moment equations are derived by taking velocity-space moments of a relevant kinetic equation, for which it is simplest to use the conservative form (5.18) of the plasma kinetic equation:

$$\int d^3v\; g(\mathbf{v}) \left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v} f + \frac{\partial}{\partial \mathbf{v}} \cdot \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) f - \mathcal{C}(f) \right] = 0 \tag{5.36}$$

in which $g(\mathbf{v})$ is the relevant velocity function for the desired fluid moment.

We begin by obtaining the density moment by evaluating (5.36) using $g = 1$. Since the Eulerian velocity space coordinate $\mathbf{v}$ is stationary and hence is independent of time, the time derivative can be interchanged with the integral over velocity space. (Mathematically, the partial time derivative and $\int d^3v$ operators commute, i.e., their order can be interchanged.) Thus, the first integral becomes $(\partial/\partial t) \int d^3v\, f = \partial n/\partial t$. Similarly, since the $\int d^3v$ and spatial derivative $\partial/\partial\mathbf{x}$ operators commute, they can be interchanged in the second term in (5.36) which then becomes $\partial/\partial\mathbf{x} \cdot \int d^3v\, \mathbf{v} f = \partial/\partial\mathbf{x} \cdot n\mathbf{V} \equiv \boldsymbol{\nabla} \cdot n\mathbf{V}$. Since the integrand in the third term in (5.36) is in the form of a divergence in velocity space, its integral can be converted into a surface integral using Gauss' theorem (**??**): $\int d^3v\, \partial/\partial\mathbf{v} \cdot (d\mathbf{v}/dt) f = \int d\mathbf{S}_v \cdot (d\mathbf{v}/dt) f = 0$, which vanishes because there must be exponentially few particles on the bounding velocity space surface $|\mathbf{v}| \to \infty$ — so that all algebraic moments of the distribution function are finite and hence exist. Finally, as indicated in (5.33) the density moment of the Coulomb collision operator vanishes.

Thus, the density moment of the plasma kinetic equation yields the density continuity or what is called simply the "density equation:"

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla}\cdot n\mathbf{V} = 0 \quad\Longrightarrow\quad \frac{\partial n}{\partial t} = -\mathbf{V}\cdot\boldsymbol{\nabla} n - n\boldsymbol{\nabla}\cdot\mathbf{V} \quad\Longrightarrow\quad \frac{dn}{dt} = -n\boldsymbol{\nabla}\cdot\mathbf{V}. \tag{5.37}$$

Here, in obtaining the second form we used the vector identity (**??**) and the last form is written in terms of the total time derivative (local partial time derivative plus that induced by advection[7] — see Fig. 5.2a below) in a fluid moving with flow velocity $\mathbf{V}$:

$$\frac{d}{dt} \equiv \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}} + \mathbf{V}\cdot\boldsymbol{\nabla}, \quad \text{total time derivative in a moving fluid.} \tag{5.38}$$

This total time derivative is sometimes called the "substantive" derivative. From the middle form of the density equation (5.37) we see that at a fixed (Eulerian) position, increases ($\partial n/\partial t > 0$) in the density of a plasma species are caused by advection of the species at flow velocity $\mathbf{V}$ across a density gradient from a region of higher density into the local one with lower density ($\mathbf{V}\cdot\boldsymbol{\nabla} n < 0$), or by compression ($\boldsymbol{\nabla}\cdot\mathbf{V} < 0$, convergence) of the flow. Conversely, the local density decreases if the plasma species flows from a lower into a locally higher density region or if the flow is expanding (diverging). The last form in (5.37) shows that in a frame of reference moving with the flow velocity $\mathbf{V}$ (Lagrangian description) only compression (expansion) of the flow causes the density to increase (decrease) — see Fig. 5.2b below.

The momentum equation for a plasma species is derived similarly by taking the momentum moment of the plasma kinetic equation. Using $g = m\mathbf{v}$ in (5.36), calculating the various terms as in the preceding paragraph and using $\mathbf{v}\mathbf{v} = (\mathbf{v}_r + \mathbf{V})(\mathbf{v}_r + \mathbf{V})$ in evaluating the second term, we find

$$m\,\partial(n\mathbf{V})/\partial t + \boldsymbol{\nabla}\cdot(p\mathsf{I} + \boldsymbol{\pi} + mn\mathbf{V}\mathbf{V}) - nq\,[\mathbf{E} + \mathbf{V}\times\mathbf{B}] - \mathbf{R} = \mathbf{0}. \tag{5.39}$$

In obtaining the next to last term we have used vector identity (**??**) to write $\mathbf{v}\,\partial/\partial\mathbf{v}\cdot[(d\mathbf{v}/dt)f] = \partial/\partial\mathbf{v}\cdot[\mathbf{v}(d\mathbf{v}/dt)f] - (d\mathbf{v}/dt)f\cdot(\partial\mathbf{v}/\partial\mathbf{v})$, which is equal to $\partial/\partial\mathbf{v}\cdot[\mathbf{v}(d\mathbf{v}/dt)f] - (d\mathbf{v}/dt)f$ since $\partial\mathbf{v}/\partial\mathbf{v} \equiv \mathsf{I}$ and $d\mathbf{v}/dt\cdot\mathsf{I} = d\mathbf{v}/dt$; the term containing the divergence in velocity space again vanishes by conversion to a surface integral, in this case using (**??**). Next we rewrite (5.38) using (5.37) to remove the $\partial n/\partial t$ contribution and $\boldsymbol{\nabla}\cdot mn\mathbf{V}\mathbf{V} = m\mathbf{V}(\boldsymbol{\nabla}\cdot n\mathbf{V}) + mn\mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{V}$ to obtain

$$mn\frac{d\mathbf{V}}{dt} = nq\,[\mathbf{E} + \mathbf{V}\times\mathbf{B}] - \boldsymbol{\nabla} p - \boldsymbol{\nabla}\cdot\boldsymbol{\pi} + \mathbf{R} \tag{5.40}$$

in which the total time derivative d/dt in the moving fluid is that defined in (5.38). Equation (5.40) represents the average of Newton's second law ($m\mathbf{a} = \mathbf{F}$)

[7] Many plasma physics books and articles call this convection. In fluid mechanics advection means transport of any quantity by the flow velocity $\mathbf{V}$ and convection refers only to the heat flow $(5/2)nT\mathbf{V}$ induced by the fluid flow. This book adopts the terminology of fluid mechanics.

over an entire distribution of particles. Thus, the $mn\, d\mathbf{V}/dt$ term on the left represents the inertial force per unit volume in this moving (with flow velocity $\mathbf{V}$) charged particle species. The first two terms on the right give the average (over the distribution function) force density on the species that results from the Lorentz force $q\,[\mathbf{E} + \mathbf{v}\times\mathbf{B}]$ on the charged particles. The next two terms represent the force per unit volume on the species that results from the pressure tensor $\mathsf{P} = p\mathsf{I} + \boldsymbol{\pi}$, i.e,, both that due to the isotropic expansive pressure p and the anisotropic stress $\boldsymbol{\pi}$. The $\mathbf{R}$ term represents the frictional force density on this species that results from Coulomb collisional relaxation of its flow $\mathbf{V}$ toward the flow velocities of other species of charged particles in the plasma.

Finally, we obtain the energy equation for a plasma species by taking the energy moment of the plasma kinetic equation. Using $g = mv^2/2$ in (5.36) and proceeding as we did for the momentum moment, we obtain (see Problem 5.**??**)

$$\frac{\partial}{\partial t}\left(\frac{3}{2}nT + \frac{1}{2}mnV^2\right) + \boldsymbol{\nabla\cdot}\left[\mathbf{q} + \left(\frac{5}{2}nT + \frac{1}{2}nmV^2\right)\mathbf{V} + \mathbf{V}\boldsymbol{\cdot}\boldsymbol{\pi}\right] - nq\mathbf{V}\boldsymbol{\cdot}\mathbf{E} - Q - \mathbf{V}\boldsymbol{\cdot}\mathbf{R} = 0. \qquad (5.41)$$

Using the dot product of the momentum equation (5.40) with $\mathbf{V}$ to remove the $\partial V^2/\partial t$ term in this equation and using the density equation (5.37), this equation can be simplified to

$$\frac{3}{2}\frac{\partial p}{\partial t} = -\boldsymbol{\nabla\cdot}\left(\mathbf{q} + \frac{5}{2}p\mathbf{V}\right) + \mathbf{V}\boldsymbol{\cdot\nabla}p - \boldsymbol{\pi}\,\boldsymbol{:}\,\boldsymbol{\nabla}\mathbf{V} + Q,$$

$$\text{or,} \qquad \frac{3}{2}\frac{dp}{dt} + \frac{5}{2}p\boldsymbol{\nabla\cdot}\mathbf{V} = -\boldsymbol{\nabla\cdot}\mathbf{q} - \boldsymbol{\pi}\,\boldsymbol{:}\,\boldsymbol{\nabla}\mathbf{V} + Q \qquad (5.42)$$

The first form of the energy equation shows that the local (Eulerian) rate of increase of the internal energy per unit volume of the species $[(3/2)nT = (3/2)p]$ is given by the sum of the net (divergence of the) energy fluxes into the local volume due to heat conduction ($\mathbf{q}$), heat convection $[(5/2)p\mathbf{V}$ — $(3/2)p\mathbf{V}$ internal energy carried along with the flow velocity $\mathbf{V}$ plus $p\mathbf{V}$ from mechanical work done on or by the species as it moves], advection of the pressure from a lower pressure region into the local one of higher pressure ($\mathbf{V}\boldsymbol{\cdot\nabla}p > 0$), and dissipation due to flow-gradient-induced stress in the species ($-\boldsymbol{\pi}\,\boldsymbol{:}\,\boldsymbol{\nabla}\mathbf{V}$) and collisional energy exchange (Q).

The energy equation is often written in the form of an equation for the time derivative of the temperature. This form is obtained by using the density equation (5.37) to eliminate the $\partial n/\partial t$ term implicit in $\partial p/\partial t$ in (5.42) to yield

$$\frac{3}{2}n\frac{dT}{dt} = -nT(\boldsymbol{\nabla\cdot}\mathbf{V}) - \boldsymbol{\nabla\cdot}\mathbf{q} - \boldsymbol{\pi}\,\boldsymbol{:}\,\boldsymbol{\nabla}\mathbf{V} + Q, \qquad (5.43)$$

in which d/dt is the total time derivative for the moving fluid defined in (5.38). This form of the energy equation shows that the temperature T of a plasma species increases (in a Lagrangian frame moving with the flow velocity $\mathbf{V}$) due to a compressive flow ($\boldsymbol{\nabla\cdot}\mathbf{V} < 0$), the divergence of the conductive heat flux ($-\boldsymbol{\nabla}\cdot\mathbf{q}$), and dissipation due to flow-gradient-induced stress in the species ($-\boldsymbol{\pi}:\boldsymbol{\nabla}\mathbf{V}$) and collisional energy exchange (Q).

Finally, it often useful to switch from writing the energy equation in terms of the temperature or pressure to writing it in terms of the collisional entropy. The (dimensionless) collisional entropy s for $f \simeq f_M$ is

$$s \equiv -\frac{1}{n}\int d^3v\, f \ln f \simeq \ln\left(\frac{T^{3/2}}{n}\right) + C = \frac{3}{2}\ln\left(\frac{p}{n^{5/3}}\right) + C, \quad \text{collisional entropy,} \tag{5.44}$$

in which C is an unimportant constant. Entropy represents the state of disorder of a system — see the discussion at the end of Section A.3. Mathematically, it is the logarithm of the number of number of statistically independent states a particle can have in a relevant volume in the six-dimensional phase space. For classical (i.e., non-quantum-mehanical) systems, it is the logarithm of the average volume of the six-dimensional phase space occupied by one particle. That is, it is the logarithm of the inverse of the density of particles in the six-dimensional phase space, which for the collisional equilibrium Maxwellian distribution (5.22) is $\simeq \pi^{3/2}v_T^3/n \propto T^{3/2}/n$. Entropy increases monotonically in time as collisions cause particles to spread out into a larger volume (and thereby reduce their density) in the six-dimensional phase space, away from an originally higher density (smaller volume, more confined) state.

An entropy equation can be obtained directly by using the density and energy equations (5.37) and (5.43) in the total time derivative of the entropy s for a given species of particles:

$$nT\frac{ds}{dt} = \frac{3}{2}n\frac{dT}{dt} - T\frac{dn}{dt} = -\boldsymbol{\nabla}\cdot\mathbf{q} - \boldsymbol{\pi}:\boldsymbol{\nabla}\mathbf{V} + Q. \tag{5.45}$$

Increases in entropy ($ds/dt > 0$) in the moving fluid are caused by net heat flux into the volume, and dissipation due to flow-gradient-induced stress in the species and collisional energy exchange. The evolution of entropy in the moving fluid can be written in terms of the local time derivative of the entropy density ns by making use of the density equation (5.37) and vector identity (**??**):

$$nT\frac{ds}{dt} = T\left[\frac{d(ns)}{dt} - s\frac{dn}{dt}\right] = T\left[\frac{\partial(ns)}{\partial t} + \boldsymbol{\nabla}\cdot ns\mathbf{V}\right]. \tag{5.46}$$

Using this form for the rate of entropy increase and $\boldsymbol{\nabla}\cdot(\mathbf{q}/T) = (1/T)[\boldsymbol{\nabla}\cdot\mathbf{q} - \mathbf{q}\cdot\boldsymbol{\nabla}\ln T]$ in (5.45), we find (5.45) can be written

$$\frac{\partial(ns)}{\partial t} + \boldsymbol{\nabla}\cdot\left(ns\mathbf{V} + \frac{\mathbf{q}}{T}\right) = \theta \equiv -\frac{1}{T}(\mathbf{q}\cdot\boldsymbol{\nabla}\ln T + \boldsymbol{\pi}:\boldsymbol{\nabla}\mathbf{V} - Q). \tag{5.47}$$

In this form we see that local temporal changes in the entropy density $[\partial(ns)/\partial t]$ plus the net (divergence of) entropy flow out of the local volume by entropy convection ($ns\mathbf{V}$) and heat conduction ($\mathbf{q}/T$) are induced by the dissipation in the species (θ), which is caused by temperature-gradient-induced conductive heat flow $[-\mathbf{q}\cdot\boldsymbol{\nabla}\ln T = -(1/T)\mathbf{q}\cdot\boldsymbol{\nabla}T]$, flow-gradient-induced stress ($-\boldsymbol{\pi}:\boldsymbol{\nabla}\mathbf{V}$), and collisional energy exchange (Q).

The fluid moment equations for a charged plasma species given in (5.37), (5.40) and (5.43) are similar to the corresponding fluid moment equations obtained from the moments of the kinetic equation for a neutral gas — (**??**)–(**??**). The key differences are that: 1) the average force density $n\bar{\mathbf{F}}$ on a plasma species is given by the Lorentz force density $n[\mathbf{E}+\mathbf{V}\times\mathbf{B}]$ instead the gravitational force $-m\boldsymbol{\nabla}V_G$; and 2) the effects of Coulomb collisions between different species of charged particles in the plasma lead to frictional force ($\mathbf{R}$) and energy exchange (Q) additions to the momentum and energy equations. For plasmas there is of course the additional complication that the densities and flows of the various species of charged particles in a plasma have to be added according to (5.1) to yield the charge ρ_q and current $\mathbf{J}$ density sources for the Maxwell equations that then must be solved to obtain the $\mathbf{E}, \mathbf{B}$ fields in the plasma, which then determine the Lorentz force density on each species of particles in the plasma.

It is important to recognize that while each fluid moment of the kinetic equation is an exact equation, the fluid moment equations represent a hierarchy of equations which, without further specification, is not a complete (closed) set of equations. Consider first the lowest order moment equation, the density equation (5.37). In principle, we could solve it for the evolution of the density n in time, if the species flow velocity $\mathbf{V}$ is specified. In turn, the flow velocity is determined from the next order equation, the momentum equation (5.40). However, to solve this equation for $\mathbf{V}$ we need to know the species pressure ($p = nT$) and hence really the temperature T, and the stress tensor $\boldsymbol{\pi}$. The temperature is obtained from the isotropic version of the next higher order moment equation, the energy equation (5.43). However, this equation depends on the heat flux $\mathbf{q}$.

Thus, the density, momentum and energy equations are not complete because we have not yet specified the highest order, "closure" moments in these equations — the heat flux $\mathbf{q}$ and the stress tensor $\boldsymbol{\pi}$. To determine them, we could imagine taking yet higher order moments of the kinetic equation [$g = \mathbf{v}(mv_r^2/2)$ and $m(\mathbf{v}_r\mathbf{v}_r - (v_r^2/3)\mathbf{I})$ in (5.36)] to obtain evolution equations for $\mathbf{q}, \boldsymbol{\pi}$. However, these new equations would involve yet higher order moments ($\mathbf{vvv}$, $v^2\mathbf{vv}$), most of which do not have simple physical interpretations and are not easily measured. Will this hierarchy never end?! Physically, the even higher order moments depend on ever finer scale details of the distribution function f; hence, we might hope that they are unimportant or negligible, particularly taking account of the effects of Coulomb collisions in smoothing out fine scale features of the distribution function in velocity-space. Also, since the fluid moment equations we have derived so far provide evolution equations for the physically most important (and measurable) properties $(n, \mathbf{V}, T)$ of a plasma species, we would like to somehow close the hierarchy of fluid moment equations at this level.

5.4 Closure Moments*

The general procedure for closing a hierarchy of fluid moment equations is to obtain the needed closure moments, which are sometimes called constitutive relations, from integrals of the kinetic distribution function f — (5.28) and (5.31) for $\mathbf{q}$ and $\boldsymbol{\pi}$. The distribution function must be solved from a kinetic equation that takes account of the evolution of the lower order fluid moments $n(\mathbf{x},t), \mathbf{V}(\mathbf{x},t), T(\mathbf{x},t)$ which are the "parameters" of the lowest order "dynamic" equilibrium Maxwellian distribution f_M specified in (5.22). The resultant kinetic equation and procedure for determining the distribution function and closure moments are known as the Chapman-Enskog[8] approach. In this approach, kinetic distortions of the distribution function are driven by the thermodynamic forces $\boldsymbol{\nabla}T$ and $\boldsymbol{\nabla}\mathbf{V}$ — gradients of the parameters of the lowest order Maxwellian distribution, the temperature (for $\mathbf{q}$) and the flow velocity (for $\boldsymbol{\pi}$), see (**??**) in Appendix A.4. For situations where collisional effects are dominant ($\partial/\partial t \sim -i\omega << \nu$, $\lambda\boldsymbol{\nabla} << 1$), the resultant kinetic equation can be solved asymptotically via an ordering scheme and the closure moments $\mathbf{q}$, $\boldsymbol{\pi}$ represent the diffusion of heat and momentum induced by the (microscopic) collisions in the medium. This approach is discussed schematically for a collisional neutral gas in Section A.4. It has been developed in detail for a collisional, magnetized plasma by Braginskii[9] — see Section 12.2. While these derivations of the needed closure relations are beyond the scope of the present discussion, we will use their results. In the following paragraphs we discuss the physical processes (phenomenologies) responsible for the generic scaling forms of their results.

In a Coulomb-collision-dominated plasma the heat flux $\mathbf{q}$ induced by a temperature gradient $\boldsymbol{\nabla}T$ will be determined by the microscopic (hence the superscript m on κ) random walk collisional diffusion process (see Section A.5):

$$\mathbf{q} \equiv -\kappa^m \boldsymbol{\nabla}T = -n\chi\boldsymbol{\nabla}T, \quad \chi \equiv \frac{\kappa^m}{n} \sim \frac{(\Delta x)^2}{2\Delta t}, \quad \text{Fourier heat flux}, \tag{5.48}$$

in which Δx is the random spatial step taken by particles in a time Δt. For Coulomb collisional processes in an unmagnetized plasma, $\Delta x \sim \lambda$ (collision length) and $\Delta t \sim 1/\nu$ (collision time); hence, the scaling of the heat diffusivity is $\chi \sim \nu\lambda^2 = v_T^2/\nu \propto T^{5/2}/n$. [The factor of 2 in the diffusion coefficient is usually omitted in these scaling relations — because the correct numerical coefficients ("headache factors") must be obtained from a kinetic theory.] In a magnetized plasma this collisional process still happens freely along a magnetic field (as long as $\lambda\nabla_\parallel << 1$), but perpendicular to the magnetic field the gyromotion limits the perpendicular step size Δx to the gyroradius ϱ. Thus, in a collisional,

[8] Chapman and Cowling, *The Mathematical Theory of Non-Uniform Gases* (1952).

[9] S.I. Braginskii, "Transport Processes in a Plasma," in *Reviews of Plasma Physics*, M.A. Leontovich, Ed. (Consultants Bureau, New York, 1965), Vol. 1, p. 205.

magnetized plasma we have

$$\begin{aligned} \mathbf{q}_\parallel &= -n\chi_\parallel \hat{\mathbf{b}}\nabla_\parallel T, \quad \chi_\parallel \sim \nu\lambda^2, \quad \text{parallel heat conduction,} \\ \mathbf{q}_\perp &= -n\chi_\perp \boldsymbol{\nabla}_\perp T, \quad \chi_\perp \sim \nu\varrho^2, \quad \text{perpendicular heat conduction.} \end{aligned} \tag{5.49}$$

Here, as usual, $\nabla_\parallel \equiv \hat{\mathbf{b}}\boldsymbol{\cdot}\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \hat{\mathbf{b}}\nabla_\parallel = -\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\boldsymbol{\nabla})$ with $\hat{\mathbf{b}} \equiv \mathbf{B}/B$. The ratio of the perpendicular to parallel heat diffusion is

$$\chi_\perp/\chi_\parallel \sim (\varrho/\lambda)^2 \sim (\nu/\omega_c)^2 << 1, \tag{5.50}$$

which is by definition very small for a magnetized plasma — see (**??**). Thus, in a magnetized plasma collisional heat diffusion is much smaller across magnetic field lines than along them, for both electrons and ions. This is of course the basis of magnetic confinement of plasmas.

Next we compare the relative heat diffusivities of electrons and ions. From formulas developed in Chapters 2 and 4 we find that for electrons and ions with approximately the same temperatures, the electron collision frequency is higher $[\nu_e/\nu_i \sim (m_i/m_e)^{1/2} \gtrsim 43 >> 1]$, the collision lengths are comparable $(\lambda_e \sim \lambda_i)$, and the ion gyroradii are larger $[\varrho_i/\varrho_e \sim (m_i/m_e)^{1/2} \gtrsim 43 >> 1]$. Hence, for comparable electron and ion temperatures we have

$$\frac{\chi_{\parallel e}}{\chi_{\parallel i}} \sim \left(\frac{m_i}{m_e}\right)^{1/2} \gtrsim 43 >> 1, \quad \frac{\chi_{\perp e}}{\chi_{\perp i}} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \lesssim \frac{1}{43} << 1. \tag{5.51}$$

Thus, along magnetic field lines collisions cause electrons to diffuse their heat much faster than ions but perpendicular to field lines ion heat diffusion is the dominant process.

Similarly, the "viscous" stress tensor $\boldsymbol{\pi}$ caused by the random walk collisional diffusion process in an unmagnetized plasma in the presence of the gradient in the species flow velocity $\mathbf{V}$ is (see Section 12.2)

$$\boldsymbol{\pi} \equiv -2\mu^m \mathsf{W}, \quad \frac{\mu^m}{nm} \sim \frac{(\Delta x)^2}{2\Delta t}, \quad \text{viscous stress tensor.} \tag{5.52}$$

Here, W is the symmeterized form of the gradient of the species flow velocity:

$$\mathsf{W} \equiv \frac{1}{2}\left[\boldsymbol{\nabla}\mathbf{V} + (\boldsymbol{\nabla}\mathbf{V})^\mathsf{T}\right] - \frac{1}{3}\mathsf{I}(\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V}), \quad \text{rate of strain tensor,} \tag{5.53}$$

in which the superscript T indicates the transpose. Like for the heat flux, the momentum diffusivity coefficient for an unmagnetized, collisional plasma scales as $\mu^m/nm \sim \nu\lambda^2$. Similarly for a magnetized plasma we have

$$\boldsymbol{\pi}_\parallel = -2\mu_\parallel^m \mathsf{W}_\parallel,\ \mu_\parallel^m/nm \sim \nu\lambda^2, \quad \boldsymbol{\pi}_\perp = -2\mu_\perp^m \mathsf{W}_\perp,\ \mu_\perp^m/nm \sim \nu\varrho^2. \tag{5.54}$$

Since the thermodynamic drives $\mathsf{W}_\parallel \equiv \hat{\mathbf{b}}(\hat{\mathbf{b}}\boldsymbol{\cdot}\mathsf{W}\boldsymbol{\cdot}\hat{\mathbf{b}})\hat{\mathbf{b}}$ and $\mathsf{W}_\perp$ are tensor quantities, they are quite complicated, particularly in inhomogeneous magnetic fields — see Section 12.2. Like for heat diffusion, collisional diffusion of momentum along magnetic field lines is much faster than across them. Because of the mass factor in the viscosity coefficient μ^m, for comparable electron and ion temperatures the ion viscosity effects are dominant both parallel and perpendicular to **B**:

$$\frac{\mu^m_{\parallel e}}{\mu^m_{\parallel i}} \sim \left(\frac{m_e}{m_i}\right)^{1/2} \lesssim \frac{1}{43} << 1, \quad \frac{\mu^m_{\perp e}}{\mu^m_{\perp i}} \sim \left(\frac{m_e}{m_i}\right)^{3/2} \lesssim 1.3 \times 10^{-5} <<<<< 1. \tag{5.55}$$

Now that the scalings of the closure moments have been indicated, we can use (5.45) to estimate the rate at which entropy increases in a collisional magnetized plasma. The contribution to the entropy production rate ds/dt from the divergence of the heat flux can be estimated by $-(\boldsymbol{\nabla}\cdot\mathbf{q})/nT \sim (\chi_\parallel \nabla^2_\parallel T + \chi_\perp \nabla^2_\perp T)/T \sim (\nu/T)(\lambda^2\nabla^2_\parallel + \varrho^2\nabla^2_\perp)T$. Similarly, the estimated rate of entropy increase from the viscous heating is $-(\boldsymbol{\pi} : \boldsymbol{\nabla}\mathbf{V})/nT \sim (\mu^m_\parallel|\nabla_\parallel \mathbf{V}|^2 + \mu^m_\perp|\boldsymbol{\nabla}_\perp\mathbf{V}|^2)/nT \sim \nu(|\lambda\nabla_\parallel\mathbf{V}/v_T|^2 + |\varrho\boldsymbol{\nabla}_\perp\mathbf{V}/v_T|^2)$. Finally, the rate of entropy increase due to collisional energy exchange can be esimated from $Q_i/n_iT_i \sim \nu_e(m_e/m_i)$ and $Q_e/n_eT_e \sim \nu_e(m_e/m_i) + J^2/\sigma \sim \nu_e[m_e/m_i + (V_{\parallel e} - V_{\parallel i})^2/v^2_{Te}]$. For many plasmas the gyroradius ϱ is much smaller than the perpendicular scale lengths for the temperature and flow gradients; hence, the terms proportional to the gyroradius are usually negligible compared to the remaining terms. This is particularly true for electrons since the electron gyroradius is so much smaller than the ion gyroradius. We will see in the next section that in the small gyroradius approximation the flows are usually small compared to their respective thermal speeds; hence the flow terms are usually negligibly small except perhaps for the ion ones. Thus, the rates of electron and ion entropy production for a collision-dominated magnetized plasma are indicated schematically by

$$\frac{ds_e}{dt} \simeq \nu_e \max\left\{\frac{\lambda_e^2\nabla^2_\parallel T_e}{T_e}, \left|\frac{\lambda_e\nabla_\parallel\mathbf{V}_e}{v_{Te}}\right|^2, \frac{m_e}{m_i}, \left(\frac{V_{\parallel e}-V_{\parallel i}}{v_{Te}}\right)^2\right\} << \nu_e, \tag{5.56}$$

$$\frac{ds_i}{dt} \simeq \nu_i \max\left\{\frac{\lambda_i^2\nabla^2_\parallel T_i}{T_i}, \left|\frac{\lambda_i\nabla_\parallel\mathbf{V}_i}{v_{Ti}}\right|^2, \left|\frac{\varrho_i\boldsymbol{\nabla}_\perp\mathbf{V}_i}{v_{Ti}}\right|^2, \left(\frac{m_e}{m_i}\right)^{1/2}\right\} << \nu_i. \tag{5.57}$$

As shown by the final inequalities, these contributions to entropy production are all small in the small gyroradius and collision-dominated limits in which they are derived. Hence, the maximum entropy production rates for electrons and ions are bounded by their respective Coulomb collision frequencies. For more collisionless situations or plasmas, the condition $\lambda\nabla_\parallel << 1$ is usually the first condition to be violated; then, the "collisionless" plasma behavior along magnetic field lines must be treated kinetically and new closure relations derived. Even with kinetically-derived closure relations, apparently the entropy production rates for fluidlike electrons and ion species are still approximately bounded by their respective electron and ion collision frequencies ν_e and ν_i. However, in truly kinetic situations with important fine-scale features in velocity space (localized to $\delta\vartheta \sim \delta v_\perp/v << 1$), the entropy production rate can be much faster ($ds/dt \sim \nu_{\text{eff}} \sim \nu/\delta\vartheta^2$), at least transiently.

When there is no significant entropy production on the time scale of interest (e.g., for waves with radian frequency $\omega >> ds/dt$), entropy is a "constant of the fluid motion." Then, we obtain the "adiabatic" (in the thermodynamic sense) equation of state (relation of pressure p and hence temperature T to density n) for the species:

$$\frac{ds}{dt} \equiv \frac{1}{\Gamma - 1}\,\frac{d}{dt}\ln\frac{p}{n^\Gamma} \simeq 0 \quad \Longleftrightarrow \quad p \propto n^\Gamma, \quad T \propto n^{\Gamma-1}, \qquad \text{isentropic equation of state.} \tag{5.58}$$

Here, we have defined

$$\Gamma = (N+2)/N, \tag{5.59}$$

in which N is the number of degrees of freedom (dimensionality of the system). We have been treating the fully three dimensional case for which $N = 3$, $\Gamma = 5/3$ and $\Gamma - 1 = 2/3$ — see (5.44). Corresponding entropy functionals and equations of state for one- and two-dimenional systems are explored in Problems 5.11 and 5.12. Other equations of state used in plasma physics are

$$p \propto n, \quad T = \text{constant}, \qquad \text{isothermal equation of state } (\Gamma = 1), \tag{5.60}$$

$$p \simeq 0, \quad T \simeq 0, \qquad \text{cold species equation of state,} \tag{5.61}$$

$$\boldsymbol{\nabla}\cdot\mathbf{V} = 0, \quad n = \text{constant}, \qquad \text{incompressible species flow } (\Gamma \to \infty). \tag{5.62}$$

The last equation of state requires some explanation. Setting ds/dt in (5.58) to zero and using the density equation (5.37), we find

$$\frac{1}{p}\frac{dp}{dt} = -\Gamma\frac{1}{n}\frac{dn}{dt} = \Gamma\,\boldsymbol{\nabla}\cdot\mathbf{V} \quad \Longleftrightarrow \quad \boldsymbol{\nabla}\cdot\mathbf{V} = \frac{1}{\Gamma}\,\frac{1}{p}\frac{dp}{dt}. \tag{5.63}$$

From the last form we see that for $\Gamma \to \infty$ the flow will be incompressible ($\boldsymbol{\nabla}\cdot\mathbf{V} = 0$), independent of the pressure evolution in the species. Then, the density equation becomes $dn/dt = \partial n/\partial t + \mathbf{V}\cdot\boldsymbol{\nabla} n = -n(\boldsymbol{\nabla}\cdot\mathbf{V}) = 0$. Hence, the density is constant in time on the moving fluid element (Lagrangian picture) for an incompressible flow; however, the density does change in time in an Eulerian picture due to the advection (via the $\mathbf{V}\cdot\boldsymbol{\nabla} n$ term) of the fluid into spatial regions with different densities. Since the pressure (or temperature) is not determined by the incompressible flow equation of state, it still needs to be solved for separately in this model.

When one of the regular equations of state [(5.58), (5.60),or (5.61)] is used, it provides a closure relation relating the pressure p or temperature T to the density n; hence, it replaces the energy or entropy equation for the species. When the incompresssible flow equation of state (5.62) is used, it just acts as a constraint condition on the flow; for this case a relevant energy or entropy equation must still be solved to obtain the evolution of the pressure p or temperature T of the species in terms of its density n and other variables.

5.5 Two-Fluid Plasma Description

The density, momentum (mom.) and energy or equation of state equations derived in the preceding section for a given plasma species can be specialized to a "two-fluid" set of equations for the electron ($q_e = -e$) and ion ($q_i = Z_i e$) species of charged particles in a plasma:

Electron Fluid Moment Equations ($d_e/dt \equiv \partial/\partial t + \mathbf{V}_e \cdot \boldsymbol{\nabla}$):

$$\text{density:} \quad \frac{d_e n_e}{dt} = -n_e(\boldsymbol{\nabla} \cdot \mathbf{V}_e) \quad \Longleftrightarrow \quad \frac{\partial n_e}{\partial t} + \boldsymbol{\nabla} \cdot n_e \mathbf{V}_e = 0, \tag{5.64}$$

$$\text{mom.:} \quad m_e n_e \frac{d_e \mathbf{V}_e}{dt} = -n_e e\,[\mathbf{E} + \mathbf{V}_e \times \mathbf{B}] - \boldsymbol{\nabla} p_e - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_e + \mathbf{R}_e, \tag{5.65}$$

$$\text{energy:} \quad \frac{3}{2} n_e \frac{d_e T_e}{dt} = -n_e T_e(\boldsymbol{\nabla} \cdot \mathbf{V}_e) - \boldsymbol{\nabla} \cdot \mathbf{q}_e - \boldsymbol{\pi}_e : \boldsymbol{\nabla} \mathbf{V}_e + Q_e, \tag{5.66}$$

$$\text{or eq. of state:} \quad T_e \propto n_e^{\Gamma - 1}. \tag{5.67}$$

Ion Fluid Moment Equations ($d_i/dt \equiv \partial/\partial t + \mathbf{V}_i \cdot \boldsymbol{\nabla}$):

$$\text{density,} \quad \frac{d_i n_i}{dt} = -n_i(\boldsymbol{\nabla} \cdot \mathbf{V}_i) \quad \Longleftrightarrow \quad \frac{\partial n_i}{\partial t} + \boldsymbol{\nabla} \cdot n_i \mathbf{V}_i = 0, \tag{5.68}$$

$$\text{mom.,} \quad m_i n_i \frac{d_i \mathbf{V}_i}{dt} = n_i Z_i e\,[\mathbf{E} + \mathbf{V}_i \times \mathbf{B}] - \boldsymbol{\nabla} p_i - \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_i + \mathbf{R}_i, \tag{5.69}$$

$$\text{energy,} \quad \frac{3}{2} n_i \frac{d_i T_i}{dt} = -n_i T_i(\boldsymbol{\nabla} \cdot \mathbf{V}_i) - \boldsymbol{\nabla} \cdot \mathbf{q}_i - \boldsymbol{\pi}_i : \boldsymbol{\nabla} \mathbf{V}_i + Q_i, \tag{5.70}$$

$$\text{or eq. of state:} \quad T_i \propto n_i^{\Gamma - 1}. \tag{5.71}$$

The physics content of the two-fluid moment equations is briefly as follows. The first forms of (5.64) and (5.68) show that in the (Lagrangian) frame of the moving fluid element the electron and ion densities increase or decrease according to whether their respective flows are compressing ($\boldsymbol{\nabla} \cdot \mathbf{V} < 0$) or expanding ($\boldsymbol{\nabla} \cdot \mathbf{V} > 0$). The second forms of the density equations can also be written as $\partial n/\partial t|_{\mathbf{x}} = -\mathbf{V} \cdot \boldsymbol{\nabla} n - n \boldsymbol{\nabla} \cdot \mathbf{V}$ using the vector identity (**??**); thus, at a given (Eulerian) point in the fluid, in addition to the effect of the compression or expansion of the flows, the density advection[10] by the flow velocity $\mathbf{V}$ increases the local density if the flow into the local region is from a higher density region ($-\mathbf{V} \cdot \boldsymbol{\nabla} n > 0$). Density increases by advection and compression are illustrated in Fig. 5.2. In the force balance (momentum) equations (5.65) and (5.69) the inertial forces on the electron and ion fluid elements (on the left) are balanced by the sum of the forces on the fluid element (on the right) — Lorentz force density ($nq[\mathbf{E} + \mathbf{V} \times \mathbf{B}]$), that due to the expansive isotropic pressure ($-\boldsymbol{\nabla} p$) and anisotropic stress in the fluid ($-\boldsymbol{\nabla} \cdot \boldsymbol{\pi}$), and finally the frictional force density due to Coulomb collisional relaxation of flow relative to the other species ($\mathbf{R}$). Finally, (5.66) and (5.70) show that temperatures of electrons and ions increase due to compressional work ($\boldsymbol{\nabla} \cdot \mathbf{V} < 0$) by their respective flows, the net (divergence of the) heat flux into the local fluid element ($-\boldsymbol{\nabla} \cdot \mathbf{q}$), viscous

[10] See footnote at bottom of page 15.

Figure 5.2: The species density n can increase due to: a) advection of a fluid element by flow velocity $\mathbf{V}$ from a higher to a locally lower density region, or b) compression by the flow velocity $\mathbf{V}$.

dissipation ($-\boldsymbol{\pi} : \nabla \mathbf{V}$) and collisional heating (Q) from the other species. Alternatively, when appropriate, the electron or ion temperature can be obtained from an equation of state: isentropic ($\Gamma = 5/3$), isothermal ($\Gamma = 1$) or "cold" species ($T \simeq 0$).

As written, the two-fluid moment description of a plasma is exact. However, the equations are incomplete until we specify the collisional moments $\mathbf{R}$ and Q, and the closure moments $\mathbf{q}$ and $\boldsymbol{\pi}$. Neglecting the usually small temperature gradient effects, the collisional moments are, from Section 2.3:

$$\text{Electrons:} \quad \mathbf{R}_e \simeq -m_e n_e \nu_e (\mathbf{V}_e - \mathbf{V}_i) = n_e e \mathbf{J}/\sigma, \quad Q_e \simeq J^2/\sigma - Q_i, \tag{5.72}$$

$$\text{Ions:} \quad \mathbf{R}_i = -\mathbf{R}_e, \quad Q_i = 3\frac{m_e}{m_i} \nu_e n_e (T_e - T_i). \tag{5.73}$$

For an unmagnetized plasma, the electrical conductivity σ is the Spitzer electrical conductivity $\sigma_{\rm Sp}$ defined in (**??**) and (**??**). In a magnetized plasma the electrical conductivity is different along and perpendicular to the magnetic field. The general frictional force $\mathbf{R}$ and Q_e for a magnetized plasma is written as

$$\mathbf{R} = -nq\left(\frac{\mathbf{J}_\parallel}{\sigma_\parallel} + \frac{\mathbf{J}_\perp}{\sigma_\perp}\right), \quad Q_e = \frac{J_\parallel^2}{\sigma_\parallel} + \frac{J_\perp^2}{\sigma_\perp} - Q_i, \quad \text{magnetized plasma}, \tag{5.74}$$

in which nq is $-n_e e$ (electrons) or $n_i Z_i e = n_e e$ (ions), $\mathbf{J}_\parallel \equiv J_\parallel \hat{\mathbf{b}} = (\mathbf{B} \cdot \mathbf{J}/B^2)\mathbf{B}$, $\mathbf{J}_\perp \equiv \mathbf{J} - J_\parallel \hat{\mathbf{b}} = -\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{J})$, $\sigma_\parallel \equiv \sigma_{\rm Sp}$ and $\sigma_\perp \equiv \sigma_0$. Here, σ_0 is the reference electrical conductivity which is defined in (**??**): $\sigma_0 \equiv n_e e^2/m_e \nu_e = 1/\eta$, where η is the plasma resistivity.

The closure moments $\mathbf{q}$ and $\boldsymbol{\pi}$ are calculated from moments of the distribution function as indicated in (5.28) and (5.31). The distribution function f must be determined from an appropriate kinetic theory. The closure moments can be calculated rigorously for only a few special types of plasmas, such as for plasmas where Coulomb collision effects dominate ($\partial/\partial t \sim -i\omega << \nu$, $\lambda \nabla << 1$ in general together with $\nu << \omega_c$, $\varrho \nabla_\perp << 1$ for magnetized plasmas) — see Section 12.2. Then, they represent the diffusive transport processes induced by the (microscopic) Coulomb collision processes in a plasma. For such a plasma the parametric dependences of the closure moments $\mathbf{q}$, $\boldsymbol{\pi}$ on the collision frequency ν and length λ, and gyroradius ϱ are indicated in (5.48)–(5.55) above for both unmagnetized and magnetized plasmas.

We will now illustrate some of the wide range of phenomena that are included in the two-fluid model by using these equations to derive various fundamental

Figure 5.3: Density distributions of electrons and ions in adiabatic response to a potential $\phi(x)$.

plasma responses to perturbations. The procedure we will use is to identify the relevant equation for the desired response, discuss the approximations used to simplify it and then finally use the reduced form to obtain the desired response. Since most of these phenomena can occur for either species of charged particles in a plasma, the species subscript is omitted in most of this discussion.

We begin by considering unmagnetized ($\mathbf{B} = \mathbf{0}$) plasmas. First, consider the "Boltzmann relation" adiabatic response (**??**) to an electrostatic perturbation, which was used in deriving Debye shielding in Section 1.1. It can be obtained from the momentum equation (5.40), (5.65) or (5.69). Physically an adiabatic description is valid when the thermal motion (pressure in the two-fluid model) is rapid compared to temporal evolution and dissipative processes — $\omega, \nu << v_T/\delta x \sim k v_T$ in the language of Section 1.6. Dividing the momentum equation by mnv_T and assuming for scaling purposes that $|\mathbf{V}| \sim v_T$, $d/dt \sim -i\omega$, $q\phi \sim T$, $|\boldsymbol{\nabla}| \sim 1/\delta x \sim k$, its various terms are found to scale as ω (inertia), kv_T ($\mathbf{E} = -\boldsymbol{\nabla}\phi$ electrostatic field force), kv_T (pressure force), $\nu(k\lambda)^2$ (stress force), and ν (frictional force). Thus, for $\omega, \nu << kv_T$ (adiabatic regime) and $k\lambda << 1$ (collisional species), the lowest order momentum equation is obtained by neglecting the inertial force ($mn\, d\mathbf{V}/dt$) and dissipative forces due to viscous stress ($\boldsymbol{\nabla\cdot\pi}$) and collisional friction ($\mathbf{R}$):

$$0 = -nq\boldsymbol{\nabla}\phi - \boldsymbol{\nabla}p. \tag{5.75}$$

If we assume an isothermal species [$\Gamma = 1$ in (5.60), (5.67) or (5.71)], the temperature is constant and hence $\boldsymbol{\nabla}p = T\,\boldsymbol{\nabla}n$. Then, we can write the adiabatic force density balance equation in the form $\boldsymbol{\nabla}[(q\phi/T) + \ln n] = 0$, which in complete and perturbed form yields

$$n(\mathbf{x}) = n_0 e^{-q\phi(\mathbf{x})/T_0}, \quad \frac{\tilde{n}}{n_0} = -\frac{q\tilde{\phi}}{T_0}, \quad \text{isothermal adiabatic response}, \tag{5.76}$$

This is the usual Boltzmann relation: (**??**), (**??**) or (**??**). As indicated in Fig. 5.3, in an adiabatic response a potential $\phi(x)$ causes the electron ($q_e = -e < 0$) density to peak where the potential is highest and the ion ($q_i = Z_i e > 0$) density to be at its minimum there. Thus, for an adiabatic response a potential hill confines electrons but repels ions, whereas a potential valley confines ions but repels electrons. The adiabatic response for a general isentopic equation of state [(5.58), (5.67) or (5.71)] is somehat different, although the perturbed response is the same as (5.76) with the temperature changed to ΓT_0 — see Problem 5.13. In addition, the density equation [(5.37), (5.64) or (5.68)] shows that perturbed flows are nearly incompressible ($\boldsymbol{\nabla\cdot}\mathbf{V} \simeq 0$) in the (adiabatic) limit of slow changes.

Next, we consider the inertial response, which in the two-fluid context is usually called the fluid response. It is obtained from a combination of the density and momentum equations. Physically, an inertial response obtains for fast (short time scale) processes ($\omega >> v_T/\delta x \sim kv_T$) for which the response to forces is limited by the inertial force $nm\, d\mathbf{V}/dt$. Using the same ordering of the contributions to the momentum equation as in the preceding paragraph, but now assuming $\omega \gtrsim kv_T >> \nu$, the lowest order perturbed (linearized) momentum equation becomes $mn_0\, \partial\tilde{\mathbf{V}}/\partial t = -n_0 q \boldsymbol{\nabla}\tilde{\phi} - \boldsymbol{\nabla}\tilde{p}$. For a plasma species with a spatially homogeneous density (i.e., $\boldsymbol{\nabla} n_0 = \mathbf{0}$), the perturbed density equation [(5.37), (5.64) or (5.68)] becomes $\partial\tilde{n}/\partial t = -n_0 \boldsymbol{\nabla\cdot}\tilde{\mathbf{V}}$. Thus, in the dissipationless, inertial (fluid) limit the density and momentum equations for a homogeneous plasma species become

$$\frac{\partial\tilde{n}}{\partial t} = -n_0\boldsymbol{\nabla\cdot}\tilde{\mathbf{V}}, \quad mn_0\frac{\partial\tilde{\mathbf{V}}}{\partial t} = -n_0 q\boldsymbol{\nabla}\tilde{\phi} - \boldsymbol{\nabla}\tilde{p}. \tag{5.77}$$

These equations can be combined into a single density response equation by taking the partial time derivative of the density equation and substituting in the perturbed momentum equation to yield

$$\frac{\partial^2\tilde{n}}{\partial t^2} = -\boldsymbol{\nabla\cdot} n_0\frac{\partial\tilde{\mathbf{V}}}{\partial t} = \frac{n_0 q}{m}\nabla^2\tilde{\phi} + \frac{1}{m}\nabla^2\tilde{p}, \quad \text{inertial (fluid) response.} \tag{5.78}$$

The potential fluctuation term represents the inertial polarization charge density derived earlier in (**??**): $\partial^2\tilde{\rho}_{\rm pol}/\partial t^2 = -(n_0 q^2/m)\boldsymbol{\nabla\cdot}\tilde{\mathbf{E}} = -\epsilon_0\omega_p^2\boldsymbol{\nabla\cdot}\tilde{\mathbf{E}}$. The second term on the right of (5.77) represents the modification of this polarization response due to the thermal motion (pressure) of the species — see Problem 5.15. Alternatively, if we neglect the polarization response, and use a general equation of state [(5.58), (5.67) or (5.71)], then (5.77) becomes $\partial^2\tilde{n}/\partial t^2 - (\Gamma p_0/n_0)\nabla^2\tilde{n} = 0$ which represents a sound wave with a sound wave speed $c_S \equiv (\Gamma p_0/n_0)^{1/2}$ — see (**??**), (**??**). Note that in the inertial (fluid) limit the perturbed density response is due to the compressibility of the perturbed flow ($\boldsymbol{\nabla\cdot}\tilde{\mathbf{V}} \neq 0$).

We next consider plasma transport processes in a collision-dominated limit. Specifically, we consider the electron momentum equation (5.65) in a limit where the electric field force is balanced by the frictional force ($\mathbf{R}$) and the pressure force:

$$0 = -\,n_e e\mathbf{E} - \boldsymbol{\nabla} p_e - m_e n_e \nu_e(\mathbf{V}_e - \mathbf{V}_i) = -\,n_e e\mathbf{E} - \boldsymbol{\nabla} p_e + n_e e\mathbf{J}/\sigma. \tag{5.79}$$

Here, we have neglected the inertia and viscous stress in the collisional limit by assuming $d/dt \sim -i\omega << \nu_e$ and $\lambda_e\boldsymbol{\nabla} << 1$. In a cold electron limit ($T_e \to 0$) the last form of this equation becomes

$$\mathbf{J} = \sigma\mathbf{E}, \quad \text{Ohm's law.} \tag{5.80}$$

Neglecting the ion flow $\mathbf{V}_i$ and using an isothermal equation of state [$\Gamma = 1$ in (5.67)], we can obtain the electron particle flux (units of #/m^2 s) from the first

Figure 5.4: Unmagnetized plasma particle flux components due to electron diffusion (D_e) and mobility (for $\mathbf{E} = -\boldsymbol{\nabla}\phi$).

form in (5.79):

$$\boldsymbol{\Gamma}_e \equiv n_e \mathbf{V}_e = -\, D_e \boldsymbol{\nabla} n_e + \mu_e^M n_e \mathbf{E}, \quad D_e \equiv \frac{T_e}{m_e \nu_e}, \quad \mu_e^M \equiv -\,\frac{e}{m_e \nu_e}, \quad \text{electron diffusion, mobility particle fluxes.} \tag{5.81}$$

The first term represents the particle flux due to the density gradient which is in the form of a Fick's law (**??**) with a diffusion coefficient $D_e \equiv T_e/m_e\nu_e = v_{Te}^2/2\nu_e = \nu_e\lambda_e^2/2$. The contribution to the particle flux induced by the electric field is known as the mobility flux (superscript M). The directions of these diffusive and mobility particle flux components for an equilibrium ($\boldsymbol{\Gamma}_e \simeq \mathbf{0}$) electron species are shown in Fig. 5.4. Note that the electron collision length $\lambda_e = v_{Te}/\nu_e$ must be small compared to the gradient scale length (i.e., $|\lambda_e \boldsymbol{\nabla} \ln n_e| << 1$) for this collisional plasma analysis to be valid. In general, the ratio of the diffusion coefficient to the mobility coefficient is known as the Einstein relation: $D/\mu^M = T/q \rightarrow D_e/\mu_e^M = -\,T_e/e$. The Einstein relation is valid for many types of collisional random walk processes besides Coulomb collisions.

Finally, we consider the transport properties embodied in the energy equation for an unmagnetized plasma. Neglecting flows and temperature equilibration between species, the energy equation [(5.43), (5.66) or (5.70)] becomes

$$\frac{3}{2} n \frac{\partial T}{\partial t} = -\boldsymbol{\nabla \cdot} \mathbf{q}, \quad \Longrightarrow \quad \frac{\partial T}{\partial t} = \frac{2}{3}\chi \nabla^2 T, \quad \text{temperature diffusion.} \tag{5.82}$$

Here, in the second form we have used the general Fourier heat flux closure relation (5.48) and for simplicity assumed that the species density and diffusivity are constant in space ($\boldsymbol{\nabla} n = \mathbf{0}$, $\boldsymbol{\nabla}\chi = \mathbf{0}$). In a single dimension this equation becomes a one-dimensional diffusion equation (**??**) for the temperature T with diffusion coefficient $D = 2\chi/3 \sim \nu\lambda^2$. Diffusion equations relax gradients in the species parameter operated on by the diffusion equation — here the temperature gradient for which L_T is the temperature gradient scale length defined by $1/L_T \equiv (1/T)|dT/dx|$. From (**??**) or (5.82) in the form $T/\tau \sim \chi T/L_T^2$ we infer that the transport time scale τ on which a temperature gradient in a collisional plasma ($\lambda << L_T$) will be relaxed is $\tau \sim (L_T/\lambda)^2/\nu >> 1/\nu$.

As we have seen, the two-fluid equations can be used to describe responses in both the adiabatic ($\omega << kv_T$) and inertial ($\omega >> kv_T$) limits. In between, where $\omega \sim kv_T$, neither of these limits apply and in general we must use a kinetic equation to describe the responses. Also, we have illustrated the responses for a collisional species. When Coulomb collision lengths become of order or longer

Figure 5.5: Flow components in a magnetized plasma.

than the gradient scale lengths ($\lambda\boldsymbol{\nabla} \gtrsim 1$), the heat flux and viscous stress can no longer be neglected. However, simultaneously the conditions for the derivation of these closure relations break down. Thus, for $\lambda\boldsymbol{\nabla} \gtrsim 1$ we usually need to use a kinetic equation or theory — at least to derive new forms for the closure relations.

5.6 Two-Fluid Magnetized-Plasma Properties*

We next explore the natural responses of a magnetized plasma using the two-fluid model. Because the magnetic field causes much different particle motions along and across it, the responses parallel and perpendicular to magnetic field lines are different and must be examined separately. The equation for the evolution of the parallel flow $V_\parallel \equiv \hat{\mathbf{b}}\cdot\mathbf{V}$ is obtained by taking the dot product of the momentum equation [(5.40), (5.65) or (5.69)] with $\hat{\mathbf{b}} \equiv \mathbf{B}/B$ and using $\hat{\mathbf{b}}\cdot d\mathbf{V}/dt = dV_\parallel/dt - \mathbf{V}\cdot d\hat{\mathbf{b}}/dt$:

$$mn\frac{dV_\parallel}{dt} = nqE_\parallel - \nabla_\parallel p - \hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi} - R_\parallel + mn\mathbf{V}\cdot\frac{d\hat{\mathbf{b}}}{dt}. \qquad (5.83)$$

Here, the parallel ($\parallel$) subscript indicates the component parallel to the magnetic field: i.e., $E_\parallel \equiv \hat{\mathbf{b}}\cdot\mathbf{E}$, $\nabla_\parallel p \equiv \hat{\mathbf{b}}\cdot\boldsymbol{\nabla}p$, $R_\parallel \equiv \hat{\mathbf{b}}\cdot\mathbf{R} = -nqJ_\parallel/\sigma_\parallel$. The responses along the magnetic field are mostly just one-dimensional (parallel direction) forms of the responses we derived for unmagnetized plasmas. However, many plasmas of practical interest are relatively "collisionless" along magnetic field lines ($\lambda\nabla_\parallel \ln B \gtrsim 1$); for them appropriate parallel stress tensor and heat flux closure relations must be derived and taken into account, or else a kinetic description needs to be used for the parallel responses. [See the discussion in the paragraphs after (**??**) and (**??**) in Section 6.1 for an example: the effects of "neoclassical" closures for axisymmetric toroidal magnetic systems.]

When the magnetic field is included in the momentum equation [(5.40), (5.65) or (5.69)], the $nq\mathbf{V}\times\mathbf{B}$ term it adds scales (by dividing by mnv_T) to be of order ω_c; hence, it is the largest term in the equation for a magnetized plasma in which $\omega_c >> \omega, \nu, kv_T$. Thus, like for the determination of the perpendicular guiding center drifts in Section 4.4*, the perpendicular flow responses are obtained by taking the cross product of the momentum equation [(5.40), (5.65) or (5.69)] with the magnetic field $\mathbf{B}$. Adding the resultant perpendicular flows to the parallel flow, the total flow can be written (see Fig. 5.5)

$$\mathbf{V} = \mathbf{V}_\parallel + \epsilon\,\mathbf{V}_\wedge + \epsilon^2\mathbf{V}_\perp, \quad \text{with} \qquad (5.84)$$

$$\mathbf{V}_\parallel = V_\parallel \hat{\mathbf{b}} \equiv \frac{(\mathbf{B}\cdot\mathbf{V})\mathbf{B}}{B^2}, \tag{5.85}$$

$$\mathbf{V}_\wedge = \mathbf{V}_E + \mathbf{V}_* \equiv \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{B}\times\boldsymbol{\nabla}p}{nqB^2}, \tag{5.86}$$

$$\mathbf{V}_\perp = \mathbf{V}_p + \mathbf{V}_\eta + \mathbf{V}_\pi \equiv \frac{\mathbf{B}\times mn\, d\mathbf{V}/dt}{nqB^2} + \frac{\mathbf{R}\times\mathbf{B}}{nqB^2} + \frac{\mathbf{B}\times\boldsymbol{\nabla}\cdot\boldsymbol{\pi}}{nqB^2}. \tag{5.87}$$

Here, the ϵ indicates the ordering of the various flow components in terms of the small gyroradius expansion parameter $\epsilon \sim \varrho\boldsymbol{\nabla}_\perp \sim (\omega,\nu)/\omega_c << 1$ — see (**??**) and (**??**). As indicated, the "cross" (subscript $\wedge$) flow is first order in the small gyroradius expansion, while the "perpendicular" (subscript $\perp$) flow is second order — compared to the thermal speed v_T of the species. For example,

$$\frac{\mathbf{V}_*}{v_T} = \frac{\mathbf{B}\times\boldsymbol{\nabla}p}{nqB^2 v_T} \sim \frac{T/m}{(qB/m)v_T}\frac{\boldsymbol{\nabla}_\perp p}{p} \sim \varrho\boldsymbol{\nabla}_\perp \ln p \sim \epsilon << 1. \tag{5.88}$$

For the scaling of the other contributions to $\mathbf{V}_\wedge$ and $\mathbf{V}_\perp$, see Problems 5.19 and 5.20.

The first order flow $\mathbf{V}_\wedge \equiv \mathbf{V}_E + \mathbf{V}_*$ is composed of $\mathbf{E}\times\mathbf{B}$ and diamagnetic flows. The very important $\mathbf{E}\times\mathbf{B}$ flow is the result of all the particles in a given species drifting with the same $\mathbf{E}\times\mathbf{B}$ drift velocity (**??**):

$$\mathbf{V}_E \equiv \frac{\mathbf{E}\times\mathbf{B}}{B^2} \overset{\mathbf{E}=-\boldsymbol{\nabla}\Phi}{\Longrightarrow} \frac{\mathbf{B}\times\boldsymbol{\nabla}\Phi}{B^2} \simeq \frac{1}{B_0}\frac{d\Phi}{dx}\hat{\mathbf{e}}_\mathrm{y}, \quad \mathbf{E}\times\mathbf{B} \text{ flow velocity.} \tag{5.89}$$

Here and below, the approximate equality indicates evaluation in the sheared slab model of Section 3.1 with $\mathbf{B} \simeq B_0\hat{\mathbf{e}}_\mathrm{z}$ and for which plasma parameters (and the potential Φ) only vary in the x direction. The diamagnetic flow $\mathbf{V}_*$ is

$$\mathbf{V}_* \equiv \frac{\mathbf{B}\times\boldsymbol{\nabla}p}{nqB^2} \simeq \frac{T}{qB_0}\left(\frac{1}{p}\frac{dp}{dx}\right)\hat{\mathbf{e}}_\mathrm{y} = -\frac{T(\mathrm{eV})}{(q/e)B_0L_p}\hat{\mathbf{e}}_\mathrm{y}, \quad \text{diamagnetic flow velocity,} \tag{5.90}$$

in which

$$L_p \equiv -p/(dp/dx), \quad \text{pressure-gradient scale length,} \tag{5.91}$$

which is typically approximately equal to the plasma radius in a cylindrical model. (The definition of the pressure gradient scale length has a minus sign in it because the plasma pressure usually decreases with radius or x for a confined plasma.) The last form in (5.90) gives a formula for numerical evaluation (in SI units, except for T in eV). The $\mathbf{V}_*$ flow is called the diamagnetic flow because the current density $nq\mathbf{V}_*$ it produces causes a magnetic field that reduces the magnetic field strength in proportion to the species pressure p [see Problem 5.**??**], which is a diamagnetic effect. Note that the diamagnetic flows of electrons and ions are comparable in magnitude and in opposite directions. Hoewever, the electrical current densities they produce are in the same direction. These diamagnetic currents in the cross ($\hat{\mathbf{e}}_\mathrm{y}$ in slab model) direction cause charge buildups and polarization of the plasma, which are very important in inhomogeneous magnetized plasmas.

Of particular importance is the electron diamagnetic flow obtained from (5.90) with $q = -e$:

$$\mathbf{V}_{*e} = -\frac{\mathbf{B}\times\boldsymbol{\nabla} p_e}{n_e e B^2} \simeq -\frac{T_e}{eB_0}\left(\frac{1}{p_e}\frac{dp_e}{dx}\right)\hat{\mathbf{e}}_{\rm y} = \frac{T_e(\text{eV})}{B_0 L_{p_e}}\hat{\mathbf{e}}_{\rm y}, \quad \text{electron diamagnetic flow velocity.} \tag{5.92}$$

This is a fundamental flow in a plasma; flows in a plasma are usually quoted relative to its direction.

The $\mathbf{E}\times\mathbf{B}$ and diamagnetic flows are called "cross" flows because they flow in a direction given by the cross product of the magnetic field and the "radial" gradients of plasma quantities. Thus, they flow in what tends to be the ignorable coordinate direction — the $\hat{\mathbf{e}}_{\rm y}$ direction in the sheared slab model, the azimuthal direction in a cylindrical model, or perpendicular to $\mathbf{B}$ but within magnetic flux surfaces in mirror and toroidal magnetic field systems. Since they have no component in the direction of the electric field and pressure gradient forces (i.e., $\mathbf{V}_E \cdot \mathbf{E} = 0$ and $\mathbf{V}_* \cdot \boldsymbol{\nabla} p = 0$), they do no work and hence produce no increase in internal energy of the plasma [i.e., no contributions to (5.41) or (5.42)].

The presence of the $\mathbf{E}\times\mathbf{B}$ and diamagnetic flows in a plasma introduces two important natural frequencies for waves in an inhomogeneneous plasma:

$$\omega_E \equiv \mathbf{k}\cdot\mathbf{V}_E \simeq \frac{k_y}{B_0}\frac{d\Phi}{dx} \simeq \frac{k_y T}{qB_0}\frac{d}{dx}\left(\frac{q\Phi}{T}\right), \qquad \mathbf{E}\times\mathbf{B} \text{ frequency,} \tag{5.93}$$

$$\omega_* \equiv \mathbf{k}\cdot\mathbf{V}_* \simeq \frac{k_y T}{qB_0}\left(\frac{1}{p}\frac{dp}{dx}\right) = -k_y \varrho \frac{v_T}{2L_p}, \quad \text{diamagnetic frequency.} \tag{5.94}$$

The last approximate form of ω_E is for $T =$ constant. In the last form of ω_* we have used the definitions of the thermal speed $v_T \equiv \sqrt{2T/m}$ and gyroradius $\varrho \equiv v_T/\omega_c$ (**??**). The electron diamagnetic frequency is often written as

$$\omega_{*e} \simeq -\frac{k_y T_e}{eB_0}\left(\frac{1}{p_e}\frac{dp_e}{dx}\right) = k_y \varrho_S \frac{c_S}{L_{pe}} = \frac{k_y T_e(\text{eV})}{B_0 L_{pe}}, \quad \text{electron diamagnetic frequency.} \tag{5.95}$$

in which $c_S \equiv \sqrt{T_e/m_i}$ is the ion acoustic speed (**??**) and $\varrho_S \equiv c_S/\omega_{ci}$.

The significance of the $\mathbf{E}\times\mathbf{B}$ frequency is that it is the Doppler shift frequency for waves propagating in the cross direction in a plasma. The significance of the electron diamagnetic frequency is that it is the natural frequency for an important class of waves in inhomogeneous plasmas called drift waves (see Section 7.6). Both electron and ion diamagnetic frequency drift waves can become unstable for a wide variety of plasma conditions (see Section 23.3). Because drift wave instabilities tend to be ubiquitous in inhomogeneneous plasmas, they are often called "universal instabilities." The presence of the $k_y\varrho$ factor in the diamagnetic frequencies highlights the significance for drift waves of finite gyroradius effects, mostly due to the ions — see (**??**)–(**??**). The maximum frequency

Figure 5.6: The diamagnetic flow velocity $\mathbf{V}_*$ can be interpreted physically as due to either: a) a net $\hat{\mathbf{e}}_y$ flow due to the inhomogeneous distribution of guiding centers because $p = p(x)$, or b) the combination of the particle guiding center drifts and magnetization current due to the magnetic moments of the entire species.

of drift waves is usually limited by finite ion gyroradius effects. For example, for electron drift waves $\max\{\omega\} \simeq v_{Ti}/(4\sqrt{\pi}\, L_{pi})$ for $T_e = T_i$ (see Section 8.6).

Figure 5.6 illustrates two different physical interpretations of the diamagnetic flow. In the "fluid, gyromotion" picture shown in Fig. 5.6a, because the density of guiding centers decreases as the radial variable x increases, in a full distribution of ions executing their gyromotion orbits, more ions are moving downward ($-\hat{\mathbf{e}}_y$ direction) than upward at any given x; hence, $dp/dx < 0$ in a magnetized plasma with $\mathbf{B} \simeq B_0\hat{\mathbf{e}}_z$ induces an ion diamagnetic flow in the $-\hat{\mathbf{e}}_y$ direction — see Problem 5.23. In the "particle" picture shown in Fig. 5.6b, the flow is produced by a combination of the particle drifts in the inhomogeneneous magnetic field and the magnetization current due to the magnetic moments of the charged particles gyrating in the magnetic field, both integrated over the entire distribution of particles in the species. The electrical current induced by the guiding center drift velocity $d\mathbf{x}_g/dt = \mathbf{v}_D \equiv \mathbf{v}_{D\perp} + v_{D\|}\hat{\mathbf{b}}$ from (**??**) and (**??**), integrated over an isotropic Maxwellian distribution function f_M of particles is

$$nq\bar{\mathbf{v}}_D \equiv q\int d^3v\, \mathbf{v}_D f = nq\frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{B}\times p\,(\boldsymbol{\nabla}\ln B + \boldsymbol{\kappa})}{B^2} + \frac{p}{B}\hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}). \tag{5.96}$$

Here, we have used (**??**) in evaluating the two types of velocity-space integrals: $\int d^3v\,(mv_\perp^2/2)f_M = \int d^3v\,(m/2)(v_x^2 + v_y^2)f_M = nT = p$ and $\int d^3v\, mv_\|^2 f_M = nT = p$. The (macroscopic) magnetization due to an entire species of particles with magnetic moments $\boldsymbol{\mu}$ defined in (**??**) is given by

$$\mathbf{M} = \int d^3v\,\boldsymbol{\mu} f_M = -\int d^3v\,\frac{mv_\perp^2}{2B}\hat{\mathbf{b}} f_M = -\frac{p}{B}\hat{\mathbf{b}}. \tag{5.97}$$

The electrical current caused by such a magnetization is

$$\mathbf{J}_M = \boldsymbol{\nabla}\times\mathbf{M} = \frac{\mathbf{B}\times\boldsymbol{\nabla}p}{B^2} + \frac{\mathbf{B}\times p\,(\boldsymbol{\nabla}\ln B + \boldsymbol{\kappa})}{B^2} - \frac{p}{B}\hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}). \tag{5.98}$$

Here, we have used the vector identity (**??**) and $\boldsymbol{\nabla}\times\hat{\mathbf{b}} = \hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}) + \hat{\mathbf{b}}\times\boldsymbol{\kappa}$, which can be proved by splitting $\boldsymbol{\nabla}\times\hat{\mathbf{b}}$ into its parallel and perpendicular (to $\mathbf{B}$) components using (**??**)–(**??**). Comparing these various current components, we find

$$nq(\mathbf{V}_E + \mathbf{V}_*) = nq\bar{\mathbf{v}}_D + \boldsymbol{\nabla}\times\mathbf{M}. \tag{5.99}$$

Thus, for a single species of charged particles in a magnetized plasma, the (fluid, gyromotion picture) cross ($\wedge$) current induced by the sum of the $\mathbf{E}\times\mathbf{B}$ and diamagnetic ($\mathbf{V}_*$) flows is equal to the (drift picture) sum of the currents induced by the guiding center particle drifts and the magnetization induced by the magnetic moments of all the particles in the species. Note that no single particle has a drift velocity that corresponds in any direct way to the diamagnetic flow velocity $\mathbf{V}_*$.[11] Rather, the diamagnetic flow velocity is a macroscopic flow of an entire species of particles that is a consequence of the (radially) inhomogeneous distribution of charged particles in a magnetized plasma. Finally, note that the net flow of current of a species in or out of an infinitesimal volume does not involve the magnetization: $\boldsymbol{\nabla}\cdot nq(\mathbf{V}_E+\mathbf{V}_*)=\boldsymbol{\nabla}\cdot nq\bar{\mathbf{v}}_D$ since $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{M}=0$. Thus, the net flow of (divergence of the) currrent can be calculated from either the fluid or particle picture, whichever is more convenient.

Next, we discuss the components of the second order "perpendicular" flow velocity $\mathbf{V}_\perp \equiv \mathbf{V}_p+\mathbf{V}_\eta+\mathbf{V}_\pi$ defined in (5.87). The polarization flow $\mathbf{V}_p$ represents the effect of the polarization drifts (**??**) of an entire species of particles and to lowest order in ϵ is given by:

$$\mathbf{V}_p=\frac{\mathbf{B}\times mn\,d\mathbf{V}_\wedge/dt}{nqB^2}\simeq-\frac{1}{\omega_c}\frac{\partial}{\partial t}\frac{1}{B_0}\left(\frac{d\Phi}{dx}+\frac{T}{q}\frac{1}{p}\frac{dp}{dx}\right)\hat{\mathbf{e}}_{\mathrm{x}},\quad\text{polarization flow velocity.}\tag{5.100}$$

Simlarly, we use the first order perpendicular flow $\mathbf{V}_\wedge$ in evaluating the frictional-force-induced flow $\mathbf{V}_\eta$ due to the perpendicular component of the frictional force $\mathbf{R}$ defined in (5.74):

$$\begin{aligned}\mathbf{V}_\eta &= \frac{\mathbf{R}\times\mathbf{B}}{nqB^2}=\frac{\mathbf{B}\times\mathbf{J}}{\sigma_\perp B^2}\simeq\frac{\mathbf{B}\times[-n_e e(\mathbf{V}_{\wedge e}-\mathbf{V}_{\wedge i})]}{\sigma_\perp B^2}=-\frac{n_e e\mathbf{B}\times(\mathbf{V}_{*e}-\mathbf{V}_{*i})}{\sigma_\perp B^2}\\ &= -\frac{\boldsymbol{\nabla}_\perp(p_e+p_i)}{\sigma_0 B^2}=-\frac{\nu_e\varrho_e^2}{2}\frac{\boldsymbol{\nabla}_\perp(p_e+p_i)}{n_eT_e}\simeq-\nu_e\varrho_e^2\left(\frac{T_e+T_i}{2T_i}\right)\frac{1}{n_e}\frac{dn_e}{dx}\hat{\mathbf{e}}_{\mathrm{x}},\end{aligned}$$
$$\text{classical transport flow velocity.}\tag{5.101}$$

Here, for simplicity in the evaluation for the sheared slab model form we have assumed that the electron and ion temperatures are uniform in space and only the density varies spatially (in the x direction in the sheared slab model). This flow velocity is in the form of a Fick's diffusion law (**??**) particle flux

$$\boldsymbol{\Gamma}_\perp \equiv n\mathbf{V}_\eta = -D_\perp\boldsymbol{\nabla}_\perp n_e,\quad\text{classical particle flux,}\tag{5.102}$$
$$D_\perp \equiv \nu_e\varrho_e^2\left(\frac{T_e+T_i}{2T_e}\right)\simeq 5.6\times10^{-22}\frac{n_eZ_i}{B^2[T_e(eV)]^{1/2}}\left(\frac{\ln\Lambda}{17}\right)\ \mathrm{m}^2/\mathrm{s},\tag{5.103}$$

This is called "classical" transport because its random walk diffusion process results from and scales with the (electron) gyroradius: $\Delta x\sim\varrho_e$. The scaling

[11]Many plasma physics books and articles call $\mathbf{V}_*$ the "diamagnetic drift velocity." This nomenclature is very unfortunate since no particles "drift" with this velocity. Throughout this book we will call $\mathbf{V}_*$ the diamagnetic flow velocity to avoid confusion about its origin.

of the particle diffusion coefficient $D_\perp$ with collision frequency and gyroradius is the same as that for the perpendicular electron heat diffusion coefficient $\chi_{\perp e}$ — see (5.49). The particle flux in (5.103) leads to a particle density equation of the form $\partial n_e/\partial t = -\boldsymbol{\nabla}\cdot n_e\mathbf{V}_e = D_\perp\nabla^2 n_e$ and hence to perpendicular (to $\mathbf{B}$) diffusion of particles — see (5.82), Fig. 5.4 and (**??**). It is important to note that the particle flux (and consequent transport) is the same for either species of particles (electrons or ions). Therefore, it induces no net charge flow perpendicular to magnetic field lines; hence, it is often said that classical transport is intrinsically ambipolar — electrons and ions diffuse together and induce no polarization or charge buildup perpendicular to $\mathbf{B}$.

The final perpendicular flow component is:

$$\mathbf{V}_\pi \equiv \frac{\mathbf{B}\times\boldsymbol{\nabla}\cdot\boldsymbol{\pi}}{nqB^2}, \quad \text{viscous-stress-induced flow velocity.} \tag{5.104}$$

For a collisional, magnetized species ($\lambda\nabla_\parallel << 1$, $\varrho\boldsymbol{\nabla}_\perp << 1$), this flow is smaller than the classical transport flow velocity $\mathbf{V}_\eta$. However, in more collisionless plasmas where $\lambda\nabla_\parallel \gtrsim 1$ this flow represents "neoclassical" transport due to the effects of particles drifting radially off magnetic flux surfaces and it can be larger than classical transport. For example, for an axisymmetric, large aspect ratio tokamak, collisions of particles on banana dift orbits (see Section 4.8*) induce a radial particle flux similar to (5.103) with $D_r \sim \nu_e\varrho_e^2 q^2\epsilon^{-3/2}$ in which $q \gtrsim 1$ is the toroidal winding number of the magnetic field lines and $\epsilon = r/R_0 << 1$ is the inverse aspect ratio — see Chapter 16.

All of the components of the perpendicular flow $\mathbf{V}_\perp$ have components in the x or radial (across magnetic flux surface) direction. The polarization flow leads to a radial current in the plasma and hence to radial charge buildup and polarization. Because it is due to an inertial force, it is reversible. The radial flows induced by the frictional and viscous stress forces are due to (microscopic) collisions and hence yield entropy-producing radial transport fluxes that tend to relax the plasma toward a (homogeneous) thermodynamic equilibrium.

Finally, it is important to note that like the species flow velocity $\mathbf{V}$, the heat flow $\mathbf{q}$ and stress tensor $\boldsymbol{\pi}$ have similarly ordered parallel, cross (diamagnetic-type) and perpendicular components:

$$\mathbf{q} = \mathbf{q}_\parallel + \epsilon\,\mathbf{q}_\wedge + \epsilon^2\mathbf{q}_\perp, \quad \text{total conductive heat flux,} \tag{5.105}$$

$$\boldsymbol{\pi} = \boldsymbol{\pi}_\parallel + \epsilon\,\boldsymbol{\pi}_\wedge + \epsilon^2\boldsymbol{\pi}_\perp, \quad \text{total stress tensor.} \tag{5.106}$$

The scalings of the parallel and pependicular fluxes $\mathbf{q}_\parallel$, $\mathbf{q}_\perp$ and $\boldsymbol{\pi}_\parallel$, $\boldsymbol{\pi}_\perp$ with collision frequency and gyroradius are indicated in (5.49) and (5.54). The cross heat flux is

$$\mathbf{q}_\wedge \equiv \mathbf{q}_* = \frac{5}{2}\frac{nT\mathbf{B}\times\boldsymbol{\nabla}T}{qB^2} \equiv n\chi_\wedge\hat{\mathbf{b}}\times\boldsymbol{\nabla}T, \quad \chi_\wedge = \frac{5}{2}\frac{T}{qB}, \quad \text{diamagnetic heat flux.} \tag{5.107}$$

Like the diamagnetic flow, this cross heat flux produces no dissipation [see (5.47)] since $\mathbf{q}_\wedge\cdot\boldsymbol{\nabla}T = 0$. Similarly, the cross stress tensor is a diamagnetic-type tensor

Table 5.1: Phenomena, Models For An Unmagnetized Plasma

Physical Process	Time, Length Scales	Species, Plasma Model	Consequences
plasma oscillations	$1/\omega_{pe} \sim 10^{-11}$ s	inertial	$\tilde{\rho}_q \simeq 0$, $\omega < \omega_{pe}$
Debye shielding	$\lambda_D \sim 10^{-5}$ m	adiabatic	$\tilde{\rho}_q \simeq 0$, $k\lambda_D < 1$
cold plasma waves	$\omega/k > v_T$,	two-fluid	oscillations,
	$\omega/k \simeq c_S > v_{Ti}$	$(T \simeq 0,\, \nu = 0)$	dielectric const.
hot plasma waves	$\nu < \mathcal{I}m\{\omega\} \lesssim \omega_p$	Vlasov	dielectric const.
Landau damping	$\nu < \mathcal{I}m\{\omega\} \lesssim \omega_p$	Vlasov	wave damping
velocity-space inst.	$\nu < \mathcal{I}m\{\omega\} \lesssim \omega_p$	Vlasov	NL, via collisions
Coulomb collisions	$\omega \sim \nu$, $k\lambda \sim 1$	plasma	two-fluid model
frequency	$1/\nu \sim 10^{-7}s$	kinetic	
length	$\lambda \sim 0.1$ m	equation	
plasma transport	$\tau \sim (L/\lambda)^2/\nu$	two-fluid	loss of plasma

of the form $\boldsymbol{\pi}_\wedge \sim mn\mu_\wedge^m \hat{\mathbf{b}}\times\boldsymbol{\nabla}\mathbf{V}$ and produces no dissipation [see (5.47)] since $\boldsymbol{\pi}_\wedge : \boldsymbol{\nabla}\mathbf{V} = 0$ — see Section 12.2. The cross stress tensor $\boldsymbol{\pi}_\wedge$ is often called the gyroviscous stress tensor. Since the gyroviscous effects are comparable to those from $\mathbf{V}_*$ and $\mathbf{q}_*$, $\boldsymbol{\pi}_\wedge$ must be retained in the momentum equations when diamagnetic flow effects are investigated using the two-fluid equations.

5.7 Which Plasma Description To Use When?

In this section we discuss which types of plasma descriptions are used for describing various types of plasma processes. This discussion also serves as an introduction to most of the subjects that will be covered in the remainder of the book. The basic logic is that the fastest, finest scale processes require kinetic descriptions, but then over longer time and length scales more fluidlike, macroscopic models become appropriate. Also, the "equilibrium" of the faster time scale processes often provide constraint conditions for the longer time scale, more macroscopic processes.

We begin by discussing the models used to describe an unmagnetized plasma. For specific parameters we consider a plasma-processing-type plasma with $T_e = 3$ eV, $n_e = 10^{18}$ m^{-3} and singly-charged ions ($Z_i = 1$). An outline of the characteristic phenomena, order of magnitude of relevant time and length scales, and models used to describe unmagnetized plasmas is shown in Table 5.1. As indicated in the table, the fastest time scale plasma phenomenon is oscillation at the electron plasma frequency (Section 1.3) which is modeled with an inertial electron response (5.78). The shortest length scale plasma process is Debye shielding (Section 1.1), which is produced by an adiabatic response (5.76).

Cold plasma waves (electon plasma and ion acoustic waves) are modeled by the two-fluid equations by neglecting collisional effects and considering thermal effects to be small and representable by fluid moments. These natural oscillations result from the dielectric medium responses of the plasma — see Chapters 1 and 7. The corresponding hot plasma (kinetic) waves and dielectric functions, which include wave-particle interaction effects, are modeled with the Vlasov equation (5.21) and discussed in Chapter 8. Consequences of this kinetic model of an unmagnetized plasma include the phenomena of "collisionless" Landau damping (Section 8.2) of waves and velocity-space instabilities (Chapter 19). The use of the Vlasov equation is justified because the natural growth or damping rates [$\mathcal{I}m\{\omega\}$] for these phenomena are larger than the effective collision frequency. However, velocity-space diffusion due to collisions is required for irreversibility of the wave-particle interactions involved in Landau damping (see Section 10.2) and to produce a steady state saturation or bounded cyclic behavior during the nonlinear (NL) evolution of velocity-space instabilities (see Sections 10.3, 24.1, 25.1).

On longer time scales ($\omega \lesssim \nu_{\text{eff}}$), Coulomb collisions become important and are modeled using the plasma kinetic equation (5.13). Finally, on transport time scales $\tau \sim (L/\lambda)^2/\nu$ (see Section A.5) long compared to the collision time $1/\nu$ and length scales L long compared to the collision length $\lambda = v_T/\nu$, the electron and ion species can be described by the two-fluid equations (5.64)–(5.71). Plasma radiation (caused by particle acceleration via Coulomb collisions or from atomic line radiation — see Chapter 14) can also beome relevant on the plasma transport time scale. Modeling of plasma particle and energy transport in collisional plasmas is discussed in Section 17.1.

A similar table and discussion of the relevant phenomena and plasma descriptions on various time and length scales for magnetized plasmas is deferred to Section 6.8 in the following chapter — after we have discussed the important fast time scale physical effects in a MHD description of a plasma, and in particular Alfvèn waves.

REFERENCES AND SUGGESTED READING

Plasma physics books that provide discussions of various plasma descriptions are

Schmidt, *Physics of High Temperature Plasmas* (1966,1979), Chapts. 3,4 [**?**]

Krall and Trivelpiece, *Principles of Plasma Physics* (1973), Chapts. 2.3 [**?**]

Nicholson, *Introduction to Plasma Theory* (1983), Chapts. 3-8 [**?**]

Sturrock, *Plasma Physics, An Introduction to the Theory of Astrophysical, Geophysical & Laboratory Plasmas* (1994), Chapts. 11,12 [**?**]

Hazeltine and Waelbroeck, *The Framework of Plasma Physics* (1998), Chapts. 3–6 [**?**]

Plasma books that provide extensive discussions of plasma kinetic theory are

Klimontovich, *The Statistical Theory of Non-equilibrium Processes in a Plasma* (1967) [**?**]

Montgomery and Tidman, *Plasma Kinetic Theory* (1964) [**?**]
Montgomery, *Theory of the Unmagnetized Plasma* (1971) [**?**]

A comprehensive development of the fluid moment equations is given in
S.I. Braginskii, "Transport Processes in a Plasma," in *Reviews of Plasma Physics*, M.A. Leontovich, Ed. (Consultants Bureau, New York, 1965), Vol. 1, p. 205 [**?**]

PROBLEMS

5.1 In the year 2000, single computer processor units (CPUs) were capable of about 10^9 floating point operations per second (FLOPs). Assume a "particle pushing" code needs about 100 FLOPs to advance a single particle a plasma period ($1/\omega_{pe}$) and that the CPU time scales linearly with the number of particles. How long would a year 2000 CPU have to run to simulate 0.03 m^3 of plasma with a density of $n_e = 3 \times 10^{18}$ m^{-3} for 10^{-3} seconds by advancing all the particles in a plasma? Taking account of Moore's (empirical) law which says that CPU speeds double every 18 months, how long will it be before such a simulation can be performed in a reasonable time — say one day — on a single CPU? Do you expect such plasma simulations to be possible in your lifetime? /

5.2 Consider a continuum ("mush") limit of the plasma kinetic equation. In this limit charged particles in a plasma are split in two and distributed randomly while keeping the charge density, mass density and species pressure constant. Then, the particles are split in two again, and the splitting process repeated an infinite number of times. What are the charge, mass, density and temperature of particles in one such split generation relative to the previous one? Show that in this limiting process the plasma frequency and Debye length are unchanged but that the term on the right of the averaged Klimontovich equation (5.12) becomes negligibly small compared to the terms on the left. Use these results to discuss the role of particle discreteness versus continuum effects in the Vlasov equation and the plasma kinetic equation. //

5.3 Show that for a Lorentz collision model the right side of the averaged Klimontovich equation (5.12) becomes the Lorentz collision operator:

$$\mathcal{C}_L(f) = \frac{\partial}{\partial \mathbf{v}} \cdot \frac{\langle \Delta \mathbf{v}\, \Delta \mathbf{v} \rangle}{2\Delta t} \cdot \frac{\partial f}{\partial \mathbf{v}}$$

in which $\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle / \Delta t$ is given by (**??**). [Hint: First subtract the averaged Klimontovich equation (5.12) from the full Klimontovich equation (5.8) and show that $d\delta f^m/dt = -(q/m)\, \delta \mathbf{E}^m \cdot \partial f / \partial \mathbf{v}$. Then, for an ensemble average defined by $\langle g \rangle = n_i \int d^3x\, g = n_i \int v\, dt \int b\, db \int d\varphi\, g$ show that $(q/m)^2 \langle \delta \mathbf{E}^m \int_{-\infty}^{t} dt'\, \delta \mathbf{E}^m \rangle = \langle \Delta \mathbf{v}\, \Delta \mathbf{v} \rangle / 2\Delta t$.] ///

5.4 Use the Lorentz collision operator defined in the preceding problem to show that for a Maxwellian distribution with a small flow ($|\mathbf{V}|/v_T << 1$) the Coulomb collision frictional force density on an electron species in the ion rest frame is

$$\mathbf{R}_e = -m_e n_e \nu_e \mathbf{V}_e. \; //$$

5.5 Show that the partial time derivative of the Maxwellian distribution (5.22) is

$$\frac{\partial f_M}{\partial t} = \left[\frac{1}{n} \frac{\partial n}{\partial t} + \frac{1}{T} \frac{\partial T}{\partial T} \left(\frac{m v_r^2}{2T} - \frac{3}{2} \right) + \frac{m}{T} \mathbf{v}_r \cdot \frac{\partial \mathbf{V}}{\partial t} \right] f_M .$$

Also, derive similar expressions for $\boldsymbol{\nabla} f_M \equiv \partial f_M / \partial \mathbf{x}$ and $\partial f_M / \partial \mathbf{v}$. //

5.6 Write down a one-dimensional Vlasov equation governing the distribution function along a magnetic field line neglecting particle drifts. What are the constants of the motion for this situation? What is the form of the general solution of this Vlasov equation? Discuss what dependences of the distribution function on the constants of the motion are needed to represent electrostatic and magnetic field confinement of the charged particles in a plasma along **B**. //

5.7 Show the integration and other steps needed to obtain the energy equation (5.41). [Hint: For the velocity derivative term derive and use the vector identity

$$\frac{mv^2}{2}\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{A}(\mathbf{v}) = \frac{\partial}{\partial \mathbf{v}}\cdot\frac{mv^2}{2}\mathbf{A}(\mathbf{v}) - m\mathbf{v}\cdot\mathbf{A}(\mathbf{v}).]$$

Also, use the origin of the energy flux $(5/2)nT\mathbf{V}$ to show that it represents a combination of the convection of the internal energy and mechanical work done on or by the species moving with a flow velocity $\mathbf{V}$. //*

5.8 Show the steps in going from the first energy equation (5.41) to the second (5.42). [Hint: Use vector identities (**??**) and (**??**).] //*

5.9 Parallel electron heat conduction often limits the electron temperature that can be obtained in a collisional magnetized plasma that comes into contact with the axial end walls. a) Develop a formula for estimating the equilibrium central electron temperature $T_e(0)$ produced by a power source supplying Q_S watts per unit volume in a plasma of length $2L$ that loses energy to the end walls primarily by parallel electron heat conduction. For simplicity, neglect the variation of the parallel heat conduction with distance ℓ along a magnetic field line and assume a sinusoidal electron temperature distribution along a magnetic field line given by $T_e(\ell) = T_e(0)\cos(\pi\ell/2L)$. b) How does $T_e(0)$ scale with Q_S? c) For a plasma with singly-charged ions and $n_e = 10^{12}$ cm^{-3} in a chamber with an axial length of 1 m, what $T_e(0)$ can be produced by a power source that supplies 0.1 W/cm^3 to the plasma electrons? d) How large would Q_S need to be achieve a $T_e(0)$ of 25 eV? //*

5.10 The irreducible minimum level of perpendicular heat transport is set by classical plasma transport. Consider an infinitely long cylinder of magnetized plasma. Estimate the minimum radius of a 50% deuterium, 50% tritium fusion plasma at $T_e = T_i = 10$ keV, $n_e = 10^{20}$ m^{-3} in a 5 T magnetic field that is required to obtain a plasma energy confinement time of 1 s. //*

5.11 Write down one- and two-dimensional Maxwellian distribution functions. Use the entropy definition in (5.44) to obtain entropy functionals for these two distributions. Show that the entropy functions are as indicated in (5.58). //*

5.12 First, show that in N dimensions the energy equation (5.42) can be written, in the absence of dissipative effects, as

$$\frac{N}{2}\frac{\partial p}{\partial t} = -\boldsymbol{\nabla}\cdot\left(\frac{N+2}{2}\,p\mathbf{V}\right) + \mathbf{V}\cdot\boldsymbol{\nabla}p.$$

Then, show that in combination with the density equation (5.37) this equation can be rearranged to yield the isentropic equation of state in (5.58). //*

5.13 Derive the adiabatic response for an isentropic equation of state. Show that the perturbed adiabatic response is $\tilde{n}/n_0 \simeq -\,q\tilde{\phi}/\Gamma T_0$ in which $T_0 \equiv p_0/n_0$. //

5.14 Use the ion fluid equations (5.68)–(5.71) to derive the ion energy conservation relation (**??**) that was used in the analysis of a plasma sheath in Section 1.2. Discuss the various approximations needed to obtain this result. //

5.15 Use the inertial electron fluid response (5.78) with a general isentropic equation of state to obtain the thermal speed corrections to the electron plasma wave dielectric $\hat{\epsilon}_I$ (**??**). Set the dielectric function to zero and show that the normal modes of oscillation satisfy the dispersion relation

$$\omega^2 = \omega_{pe}^2 + (\Gamma/2)\, k^2 v_{Te}^2. \ //$$

5.16 Use the two-fluid equations (5.64)–(5.71) to obtain the ion sound wave equation (**??**). Also, use the two-fluid equations and an isothermal equation of state for the ions to obtain the ion thermal corrections to the ion acoustic wave dispersion relation (**??**). //

5.17 Show how to use the electron fluid equations to derive the electromagnetic skin depth defined in (**??**). /

5.18 Consider a collisional unmagnetized plasma where the electron density distribution $n_e(\mathbf{x})$ is determined by some external means, for example by a combination of wave heating and ionization of neutrals. Use the equilibrium Ohm's law (electron momentum equation) in (5.79) to determine the potential distribution $\Phi(\mathbf{x})$ (for $\mathbf{E} = -\boldsymbol{\nabla}\Phi$) required to obtain no net current flowing in the plasma. For simplicity assume isothermal electrons. Then, use this potential to show that the equilibrium distribution of isothermal ions of charge Z_i in this plasma is

$$n_i(\mathbf{x})/n_i(0) = [n_e(0)/n_e(\mathbf{x})]^{Z_i T_e/T_i}.$$

What is the role of the potential $\Phi(\mathbf{x})$ here? Explain why the ion density is smallest where the electron density is the largest in this plasma situation. //

5.19 Show that for $q\Phi \sim T$ the $\mathbf{E}\times\mathbf{B}$ flow is order ϵ relative to the thermal speed of the species in the small gyroradius expansion. /

5.20 Show that all the terms in the $\mathbf{V}_\perp$ defined in (5.87) are of order ϵ^2 (or smaller) relative to the thermal speed of the species in the small gyroradius expansion. [Hint: Use the first order $\mathbf{E}\times\mathbf{B}$ and diamagnetic cross flows to estimate the various contributions to $\mathbf{V}_\perp$.] //

5.21 Suppose a drift-wave has a real frequency of $0.5\,\omega_{*i}$ in the $\mathbf{E}\times\mathbf{B}$ rest frame and that $n_i q_i \boldsymbol{\nabla}_\perp\Phi = -2\,\boldsymbol{\nabla}_\perp p_i$, $k_y = 0.1$ cm^{-1} and $d\Phi/dx = 100$ V/cm with a magnetic field of 2.5 T. What is the frequency (in rad/s and Hz) of the wave in the laboratory frame? Does the wave propagate in the electron or ion diamagnetic flow direction in the laboratory frame? /

5.22 Calculate the diamagnetic flow velocity in a uniform magnetic field from a simple kinetic model as follows. First, note that since the relevant constants of the motion are the guiding center position $x_g = x + v_y/\omega_c$ from (**??**) and energy $\mathcal{E}_g$, an appropriate solution of the Vlasov equation is $f = f(x_g, \mathcal{E}_g)$. Assume a Maxwellian energy distribution and expand this distribution in a small gyroradius expansion. Show that the flow velocity in this expanded distribution is the diamagnetic flow velocity (5.90). Discuss how this derivation quantifies the illustration of the diamagnetic flow in Fig. 5.6a. //

5.23 Consider electron and ion pressure profiles peaked about $x = 0$ in a sheared slab magnetic field model with no curvature or shear. a) Sketch the directions of the diamagnetic flows of the electrons and ions. b) Show that the currents they induce are in the same direction. c) Show that these currents have a diamagnetic effect on the magnetic field strength. d) Finally show that for each species the induced diamagnetic change in the magnetic field energy density is proportional to the pressure of the species. /

5.24 Consider a plasma species with an anistropic Maxwellian-type distribution that has different temperatures parallel and perpendicular to the magnetic field but no dependence on the gyrophase angle φ. a) Show that for this anisotropic distribution the pressure tensor is $\mathsf{P} = p_\perp(\mathsf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + p_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}}$. b) Show that for an anisotropic species the diamagnetic flow velocity is

$$\mathbf{V}_* \equiv \frac{\mathbf{B}\times\boldsymbol{\nabla}\cdot\mathsf{P}}{nqB^2} = \frac{\mathbf{B}\times[\boldsymbol{\nabla}p_\perp + (p_\parallel - p_\perp)\boldsymbol{\kappa}]}{nqB^2}.$$

c) Calculate the velocity-space-average drift current $nq\bar{\mathbf{v}}_D$, magnetization $\mathbf{M}$ and magnetization current $\mathbf{J}_M$ for an anisotropic species. d) Show that your results reduce to (5.96)–(5.98) for isotropic pressure. e) Finally, show that (5.99) is also satisfied for a plasma species with an anisotropic pressure. ///

5.25 In the derivation of (5.99) we neglected the guiding center drift due to the direction of the magnetic field changing in time — the $\partial\hat{\mathbf{b}}/\partial t$ contribution. Show how, when this drift is included in $\bar{\mathbf{v}}_D$, (5.99) must be modified by adding the part of the polarization flow $\mathbf{V}_p$ caused by $\mathbf{V}_\parallel$ to its left side to remain valid. (Assume for simplicity that the magnetic field is changing in direction slowly compared to the gyrofrequency $[(1/\omega_c)|\partial\hat{\mathbf{b}}/\partial t| << 1]$ so the small gyroradius expansion used to derive the guiding center orbits is valid.) //

5.26 Show that classical diffusion is automatically ambipolar for a plasma with multiple species of ions. [Hint: Note that because of momentum conservation in Coulomb collisions $\mathbf{R}_e = -\sum_i \mathbf{R}_i$.] //

