Chapter 6

Plasma Descriptions II: MHD

The preceding chapter discussed the microscopic, kinetic and two-fluid decsriptions of a plasma. But we would actually like a simpler model — one that would include most of the macroscopic properties of a plasma in a "one-fluid" model. The simplest such model is magnetohydrodynamics (MHD), which is a combination of a one-fluid (hydrodynamic-type plus Lorentz force effects) model for the plasma and the Maxwell equations for the electromagnetic fields. The main equations, properties and applications of the MHD model are developed in this chapter.

In the first section, we further approximate and combine the two-fluid description in Section 5.5 to obtain a "one-fluid" magnetohydrodynamics (MHD) description of a magnetized plasma. Section 6.2 presents the MHD equations in various forms and discusses their physical content. Subsequent sections discuss general properies of the MHD model – (force-balance) equilibria (Section 6.3), boundary and shock conditions (Section 6.4), dynamical responses (Section 6.5), and the Alfvèn waves (Section 6.6) that result from them. Then, Section 6.7 discusses magnetic field diffusion in the presence of a nonvanishing plasma electrical resistivity. Finally, Section 6.8 discusses the relevant time and length scales on which the kinetic, two-fluid and MHD models of magnetized plasmas are applicable, and hence usable for describing various magnetized plasma phenomena. This chapter thus presents the final steps in the procedures and approximations used to progress from the two-fluid plasma model to a macroscopic description, and discusses the key properties of the resultant MHD plasma model.

6.1 Magnetohydrodynamics Model*

Magnetohydrodynamics (MHD) is the name given to the nonrelativistic single fluid model of a magnetized ($\omega, \nu_i << \omega_{ci}$), small gyroradius ($\varrho_i \boldsymbol{\nabla}_\perp << 1$) plasma. The MHD description is derived in this section by adding appropriately the two-fluid equations [(??)–(??)] to obtain a "one-fluid" description and then making suitable approximations. The philosophy of the "ideal MHD" description is to obtain density, momentum and equation of state equations that govern the macroscopic behavior of a magnetized plasma on "fast" time scales where dissipative processes are negligible and entropy is conserved. Thus, ideal MHD processes are isentropic. The philosophy of "resistive MHD" is to extend the time scale beyond the electron collision time scale ($\sim 1/\nu_e$) by adding to ideal MHD the irreversible, dissipative effects due to the electrical resistivity in the plasma.

The pedagogical approach we will use is to first define the MHD plasma variables and next obtain conservation equations for these quantities. Then, we discuss the approximations used in obtaining the MHD plasma equations, and finally (in the next Section) we summarize the equations that constitute the MHD model of a plasma and its electromagnetic fields. We begin by defining the one-fluid "plasma" variables of MHD:

$$\text{mass density (kg/m}^3\text{):} \qquad \rho_m \equiv \sum_s m_s n_s = m_e n_e + m_i n_i \simeq m_i n_i \tag{6.1}$$

$$\text{mass flow velocity (m/s):} \qquad \mathbf{V} \equiv \frac{\sum_s m_s n_s \mathbf{V}_s}{\sum_s m_s n_s} = \frac{m_e n_e \mathbf{V}_e + m_i n_i \mathbf{V}_i}{\rho_m} \simeq \mathbf{V}_i \tag{6.2}$$

$$\text{current density (A/m}^2\text{):} \qquad \mathbf{J} \equiv \sum_s n_s q_s \mathbf{V}_s = -n_e e(\mathbf{V}_e - \mathbf{V}_i) \tag{6.3}$$

$$\text{plasma pressure (N/m}^2\text{):} \qquad P \equiv \sum_s \left[p_s + \frac{n_s m_s}{3} |\breve{\mathbf{V}}_s|^2 \right] \simeq p_e + p_i \tag{6.4}$$

$$\text{stress tensor (N/m}^2\text{):} \qquad \mathbf{\Pi} = \sum_s \left[\boldsymbol{\pi}_s + n_s m_s \left(\breve{\mathbf{V}}_s \breve{\mathbf{V}}_s - \frac{1}{3}\, \mathsf{I}\, |\breve{\mathbf{V}}_s|^2 \right) \right] \simeq \boldsymbol{\pi}_e + \boldsymbol{\pi}_i, \tag{6.5}$$

in which $\breve{\mathbf{V}}_s \equiv \mathbf{V}_s - \mathbf{V}$ is the species flow velocity relative to the mass flow velocity $\mathbf{V}$ of the entire plasma. Here, the forms on the right indicate first the general form as a sum over the species index s, second the electron-ion two-fluid form, and finally, after an appoximate equality, the usual, approximate forms for $m_e/m_i \lesssim 1/1836 <<< 1$, comparable $\mathbf{V}_e$ and $\mathbf{V}_i$, and $|\mathbf{V}_i| << v_{Ti}$. By construction, the pressure and stress tensor are defined in the flow velocity rest frame, which is often called the center-of-mass (really momentum) frame — see Problem 6.1.

A one-fluid mass density (continuity) equation for the plasma is obtained by multiplying the electron and ion density equations (??) and (??) by their respective masses to yield $\partial \rho_m/\partial t + \boldsymbol{\nabla\cdot} \rho_m \mathbf{V} = 0$. Multiplying the density equations by their respective charges q_s and summing over species yields the charge continuity equation $\partial \rho_q/\partial t + \boldsymbol{\nabla\cdot}\mathbf{J} = 0$. In MHD the plasma is presumed to be quasineutral because we are interested in plasma behavior on time scales long compared to the plasma period ($\omega << \omega_p$) and length scales long compared to the Debye shielding distance ($\lambda_D/\delta x \sim k\lambda_D << 1$). Mathematically, quasineutrality in the plasma means $\rho_q \equiv \sum_s n_s q_s = e(Z_i n_i - n_e) \simeq 0$. Thus, in the MHD model the charge continuity equation simplifies to $\boldsymbol{\nabla}\cdot \mathbf{J} = 0$. Note that this equation is also consistent with the divergence of Ampere's law when the displacement current is neglected — see (**??**). Hence, the charge continuity equation $\boldsymbol{\nabla}\cdot \mathbf{J} = 0$ is also consistent with a nonrelativistic MHD description of particles and waves in a plasma. Since MHD plasmas are quasineutral and have no net charge density ($\rho_q = 0$), the Gauss' law Maxwell equation $\boldsymbol{\nabla}\cdot \mathbf{E} = \rho_q/\epsilon_0$ cannot be used to determine the electric field in the plasma. Rather, since a plasma is a highly polarizable medium, in MHD the electric field $\mathbf{E}$ is determined self-consistently from Ohm's law, Ampere's law and the charge continuity equation ($\boldsymbol{\nabla}\cdot \mathbf{J} = 0$).

A one-fluid momentum equation (equation of motion) for a plasma is obtained by simply adding the electron and ion momentum equations (**??**) and (**??**) (see Problem 6.2 for the structure of the inertia term $\rho_m d\mathbf{V}/dt$):

$$\rho_m \frac{d\mathbf{V}}{dt} = \rho_q \mathbf{E} + \mathbf{J}\times\mathbf{B} - \boldsymbol{\nabla} P - \boldsymbol{\nabla}\cdot \boldsymbol{\Pi}, \tag{6.6}$$

in which $\boldsymbol{\Pi} \simeq \boldsymbol{\pi}_e + \boldsymbol{\pi}_i$ is the total plasma stress tensor in the center-of-mass frame defined in (6.5). The electric field term is eliminated in MHD by the assumption of quasineutrality in the plasma: $\rho_q \simeq 0$. In a collisional plasma the viscosity effects of the ions are dominant in the stress tensor $\boldsymbol{\Pi}$ [see (**??**)]. The dissipative effects due to ion viscosity become important on time scales long compared to the relatively slow ion collision time scale [see (**??**)]. For low collisionality plasmas in axisymmetric toroidal magnetic systems these parallel ion viscosity effects (due to $\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot \boldsymbol{\pi}_{\parallel i}$) represent the viscous drag on the parallel (poloidal) ion flow carried by untrapped ions due to their collisions with the stationary trapped ions, and are included in a model called neoclassical MHD; there they result in damping of the poloidal ion flow at a rate proportional to the ion collision frequency ν_i and consequently to an increased perpendicular inertia and dielectric response for $t >> 1/\nu_i$ — see Chapter 16. In ideal and resistive MHD it is customary to neglect the viscous stress effects and thus set $\boldsymbol{\Pi} = \mathbf{0}$ in (6.6). This assumption is usually valid for time scales shorter than the ion collision time scale: $d/dt \sim -i\omega >> \nu_i$.

Since the magnetic field causes the plasma responses to be very different along and transverse to the magnetic field direction, it is useful to explore the responses in different directions separately. Taking the dot product of $\hat{\mathbf{b}} \equiv \mathbf{B}/B$ with the plasma momentum equation (6.6) and neglecting $\rho_q\mathbf{E}$ (quasineutrality assumption) and the stress tensor $\boldsymbol{\Pi}$, the parallel plasma momentum equation becomes

$$\rho_m \frac{dV_\parallel}{dt} = -\nabla_\parallel P - \rho_m \mathbf{V}\cdot \frac{d\hat{\mathbf{b}}}{dt}. \tag{6.7}$$

in which $\nabla_\parallel \equiv \hat{\mathbf{b}}\cdot\boldsymbol{\nabla} = \partial/\partial\ell$. The last term is important only when the magnetic field direction is changing in time or in inhomogeneous plasmas when the flow velocity $\mathbf{V}$ is large. Neglecting this term, (6.7) in combination with the plasma mass density (continuity) equation leads to compressible flows due to plasma pressure perturbations and hence to sound waves along the magnetic field — see (**??**)–(**??**) in Section A.6 and (6.89) below.

Taking the cross product of $\mathbf{B}$ with the momentum equation and using the *bac* − *cab* vector identity (**??**), again neglecting $\rho_q\mathbf{E}$ and the stress tensor $\mathbf{\Pi}$, we obtain the two perpendicular components of the current:

$$\mathbf{J}_* \equiv \frac{\mathbf{B}\times\boldsymbol{\nabla}P}{B^2}, \quad \text{diamagnetic current density,} \tag{6.8}$$

$$\mathbf{J}_p \equiv \frac{\mathbf{B}\times\rho_m d\mathbf{V}/dt}{B^2}, \quad \text{polarization current density.} \tag{6.9}$$

The diamagnetic current is the sum of the currents produced by the diamagnetic currents due to flows in the various species of charged particles in the plasma: $\mathbf{J}_* = \sum_s n_s q_s \mathbf{V}_{*s}$. Like the species diamagnetic flows, it is called a "diamagnetic" current because it produces a magnetic field that reduces the magnetic field strength — in proportion to the plasma pressure P (see Problem 6.13). The electric field produces no perpendicular current in MHD because the $\mathbf{E}\times\mathbf{B}$ flows of all species are the same; hence, they produce no current: $\sum_s n_s q_s \mathbf{V}_{Es} = (\sum_s n_s q_s)\mathbf{V}_E = \rho_q \mathbf{V}_E \simeq 0$.

Like for the individual species diamagnetic flows [see (**??**) and Fig. **??**], the (fluid picture) diamagnetic current is equal to the (particle picture) current due to the combination of the particle guiding center drifts and the magnetization produced by the magnetic moments ($\boldsymbol{\mu}$) of all the charged particles gyrating in the $\mathbf{B}$ field:

$$\mathbf{J}_* = \mathbf{J}_D + \boldsymbol{\nabla}\times\mathbf{M}, \tag{6.10}$$

in which the particle drift (D) and the magnetization (M) currents are

$$\mathbf{J}_D \equiv \sum_s n_s q_s \bar{\mathbf{v}}_{Ds} = \frac{\mathbf{B}\times P(\boldsymbol{\nabla}\ln B + \boldsymbol{\kappa})}{B^2} + \frac{P}{B}\hat{\mathbf{b}}(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}), \tag{6.11}$$

$$\mathbf{J}_M \equiv \boldsymbol{\nabla}\times\mathbf{M}, \quad \mathbf{M} \equiv \sum_s \int d^3v\, \boldsymbol{\mu}_s f_{Ms} = -\frac{\hat{\mathbf{b}}}{B}\sum_s p_s = -\frac{P}{B}\hat{\mathbf{b}}. \tag{6.12}$$

Note that since the (dimensionless) magnetic susceptibility χ_M is defined by $\mathbf{M} = \chi_M \mathbf{B}/\mu_0$ [see (**??**)], in the MHD model of the plasma $\chi_M = -(\mu_0 P/B^2)$. The negative sign of χ_M indicates the diamagnetism effect of the magnetic moments of the gyrating particles in a magnetized plasma. As an illustration of the magnitude of this diamagnetism effect, when the plasma pressure P is equal to the magnetic energy density [see (**??**)] $B^2/2\mu_0$, the magnetic field strength is halved.

The polarization current is the current produced by the sum of the currents due to the polarization flows of the various species: $\mathbf{J}_p = \sum_s n_s q_s \mathbf{V}_p$. Since the ion mass is so much larger than the electron mass, the ion polarization flow dominates: $\mathbf{J}_p \simeq n_i Z_i e \mathbf{V}_{pi}$. There is no resistivity-driven current (i.e., no $\mathbf{J}_\eta$) because the classical diffusion induced by the plasma resistivity η is ambipolar [see (**??**)]. Also, there is no viscosity-induced current (i.e., no $\mathbf{J}_\pi$) in MHD because the stress tensor effects are neglected, assuming $\omega >> \nu_i$.

The total current in MHD is a combination of the parallel current, and the diamagnetic and polarization perpendicular currents:

$$\mathbf{J} = \mathbf{J}_\parallel + \mathbf{J}_* + \mathbf{J}_p = J_\parallel \frac{\mathbf{B}}{B} + \frac{\mathbf{B}\times\boldsymbol{\nabla} P}{B^2} + \frac{\mathbf{B}\times\rho_m d\mathbf{V}/dt}{B^2}. \tag{6.13}$$

The parallel component of the current density is defined by $J_\parallel \equiv \hat{\mathbf{b}}\cdot\mathbf{J} = (\mathbf{B}\cdot\mathbf{J})B$. Quasineutrality of the highly polarizable, magnetized plasma is ensured in MHD through

$$0 = \boldsymbol{\nabla}\cdot\mathbf{J} = (\mathbf{B}\cdot\boldsymbol{\nabla})(J_\parallel/B) + \boldsymbol{\nabla}\cdot\mathbf{J}_* + \boldsymbol{\nabla}\cdot\mathbf{J}_p, \quad \text{MHD charge continuity equation,} \tag{6.14}$$

which is a very important equation for analyzing MHD equilibria and instabilities. The derivative of the parallel current has been simplified here using the vector identity (**??**) and the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$:

$$\boldsymbol{\nabla}\cdot\mathbf{J}_\parallel = \boldsymbol{\nabla}\cdot(J_\parallel/B)\mathbf{B} = (\mathbf{B}\cdot\boldsymbol{\nabla})(J_\parallel/B) + (J_\parallel/B)\boldsymbol{\nabla}\cdot\mathbf{B} = (\mathbf{B}\cdot\boldsymbol{\nabla})(J_\parallel/B). \tag{6.15}$$

Taking the divergence of the diamagnetic current equation (6.20), we obtain (see Problem 6.3)

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{J}_* = \boldsymbol{\nabla}\cdot\mathbf{J}_D &= \frac{\mathbf{B}\times(\boldsymbol{\nabla}\ln B + \boldsymbol{\kappa})}{B^2}\cdot\boldsymbol{\nabla}P + \frac{1}{B}(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}P)(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}), \\ &= -\mathbf{J}_*\cdot(\boldsymbol{\nabla}\ln B + \boldsymbol{\kappa}) + (\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}P)(\mu_0 J_\parallel/B^2).\end{aligned} \tag{6.16}$$

Here, we have used vector identities (**??**) and (**??**) to evaluate the divergence of $\mathbf{J}_*$ and Ampere's law to write $\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}} = \mu_0\mathbf{J}\cdot\mathbf{B}/B^2 = \mu_0 J_\parallel/B$ — see discussion after (**??**). Thus, like for the individual species current contributions, the net (divergence of the) electrical current flow in or out of an infinitesimal volume can be computed from either the divergence of the diamagnetic current (fluid picture) or the divergence of the particle drift current (particle picture).

The important effects of the (mostly radial) pressure gradients in the MHD model of a magnetized plasma are manifested through the diamagnetic current $\mathbf{J}_*$ it induces and, for inhomogeneous magnetic fields, the net charge flows induced [see (6.16)]. For the MHD charge continuity equation (6.14) to be satisfied, compensating parallel ($J_\parallel$) or polarization ($\mathbf{J}_p$) currents must flow in the plasma. These electrical currents can lead, respectively, to modifications of the MHD equilibrium (Chapter 20) and pressure-gradient-driven MHD instabilities (Chapter 21).

Next, we obtain an Ohm's law for MHD. A one-fluid "generalized Ohm's law" is obtained by multiplying the electron and ion momentum equations by q_s/m_s and summing them to produce an equation for $\partial\mathbf{J}/\partial t$ — see Problem 6.4. However, we proceed more physically and directly from the electron momentum equation. Using $\mathbf{V}_e = \mathbf{V}_i - \mathbf{J}/n_e e \simeq \mathbf{V} - \mathbf{J}/n_e e$ and the anisotropic frictional force $\mathbf{R}$ in (**??**), and dividing the electron momentum equation (**??**) by $-n_e e$, we find it can be written (to lowest order in m_e/m_i) as

$$\frac{m_e}{e^2}\frac{d}{dt}\left(\frac{\mathbf{J}_e}{n_e}\right) = \mathbf{E} + \mathbf{V}\times\mathbf{B} - \left(\frac{\mathbf{J}_\parallel}{\sigma_\parallel} + \frac{\mathbf{J}_\perp}{\sigma_\perp}\right) - \frac{\mathbf{J}\times\mathbf{B} - \boldsymbol{\nabla} p_e - \boldsymbol{\nabla}\cdot\boldsymbol{\pi}_e}{n_e e}, \quad \text{generalized Ohm's law.} \tag{6.17}$$

Here, we have neglected an ion flow inertia term on the left because it is order $m_e/m_i \lesssim 1/1836$ smaller than the inertial flow contribution coming from the $\mathbf{J}_p\times\mathbf{B}$ term evaluated using the polarization current (6.9). While the first and third terms on the right indicate a simple Ohm's law $\mathbf{E} = \mathbf{J}/\sigma$, there are a number of additional terms. To understand the role and magnitude of these other contributions to the generalized Ohm's law and obtain an MHD Ohm's law, we need to explore separately their contributions along and perpendicular to the magnetic field direction.

The parallel component ($\hat{\mathbf{b}}\,\cdot$) of the generalized Ohm's law is:

$$(m_e/e^2)\,\hat{\mathbf{b}}\cdot d_e(\mathbf{J}/n_e)/dt = E_\parallel - J_\parallel/\sigma_\parallel + (\nabla_\parallel p_e + \hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_e)/n_e e. \tag{6.18}$$

The electron inertia term on the left is small compared to $E_\parallel$ for scale lengths longer than the electromagnetic skin depth (see Section 1.5): $|(c/\omega_{pe})\boldsymbol{\nabla}| \sim kc/\omega_{pe} << 1$ — see Problem 6.5. Since c/ω_{pe} is typically a very short distance ($c/\omega_{pe} \simeq 10^{-3}$ m $= 1$ mm for $n_e \simeq 3\times 10^{19}$ m^{-3}), this is usually a good approximation in MHD which seeks to provide a plasma description on macroscopic scale lengths. Also, since $1/\sigma_\parallel \sim m_e\nu_e/n_e e^2$, the electron inertia term is of of order ω/ν_e compared to the parallel friction force term $J_\parallel/\sigma_\parallel$. In resistive MHD it is assumed that $\omega << \nu_e$ so the electron inertia can be neglected in the parallel Ohm's law.

The parallel electron pressure gradient term is neglected in MHD because of a fundamental approximation in MHD that electric field effects are larger than pressure gradient effects:

$$|E_\parallel| >> |\nabla_\parallel P|/n_e e, \quad |\mathbf{E}_\perp| >> |\boldsymbol{\nabla}_\perp P|/n_e e, \quad \text{MHD approximations.} \tag{6.19}$$

Physically, the MHD model describes situations in which collective electric field effects are more important than the thermal motion (pressure) effects of both electrons and ions. Mathematically, this approximation is appropriate (both along and across magnetic field lines — see Problem 6.6) when the $\mathbf{E}\times\mathbf{B}$ flow velocity $\mathbf{V}_E$ is large compared to the diamagnetic flow velocities $\mathbf{V}_{*e}, \mathbf{V}_{*i}$ and hence for $\omega, \omega_E >> \omega_{*e}, \omega_{*i}$.

Finally, we consider the contribution due to the parallel component of the viscous stress. While this term is negligible compared to $J_\parallel/\sigma_\parallel$ in a collisional plasma [see (**??**)], it can be important in more collisionless plasmas where $\lambda_e\nabla_\parallel \gtrsim 1$ in which $\lambda_e = v_{Te}/\nu_e$ is the electron collision length. For low collisionality plasmas in axisymmetric toroidal magnetic systems these parallel electron viscosity effects (from $\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{\parallel e}$) represent the viscous drag on the parallel electron flow carried by untrapped electrons due to their collisions with the stationary trapped electrons and ions, and they are included in a model called neoclassical MHD; there they result in order unity modifications of the parallel Ohm's law (see Chapter 16) — reductions in the parallel electrical conductivity and a so-called "bootstrap current" parallel to $\mathbf{B}$ induced by the radial gradient of the plasma pressure. In ideal and resistive MHD the parallel electron inertia, pressure gradient and viscosity effects are all neglected and the parallel Ohm's law becomes simply $E_\parallel = J_\parallel/\sigma_\parallel$.

Next, we consider the perpendicular component of the generalized Ohm's law. It is obtained by operating on (6.17) with $-\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\;)$:

$$0 = \mathbf{E}_\perp + \mathbf{V}\times\mathbf{B} + \mathbf{J}_\perp/\sigma_\perp - [\mathbf{J}\times\mathbf{B} - \boldsymbol{\nabla}_\perp p_e - (\boldsymbol{\nabla\cdot\pi}_e)_\perp]/n_e e \tag{6.20}$$

in which the $\perp$ subscript indicates the component perpendicular to $\mathbf{B}$ [see (**??**)]. The perpendicular electron inertia term has been neglected here because it is a factor of at least $\omega/\omega_{ce} = (\omega_{ci}/\omega_{ce})(\omega/\omega_{ci}) \lesssim (1/1836)(\omega/\omega_{ci}) <<< 1$ smaller than the $\mathbf{E}_\perp$ term and hence negligible in MHD — see Problem 6.7. The first two terms on the right give the dominant part of the perpendicular Ohm's law and when set to zero yield a perpendicular plasma flow velocity $\mathbf{V}_\perp = \mathbf{V}_E = \mathbf{E}\times\mathbf{B}/B^2$. The $\mathbf{J}\times\mathbf{B}$ term on the right is known as the Hall term; it indicates a perpendicular electric field caused by current flowing transverse to a magnetic field. In MHD the perpendicular current is composed of the diamagnetic and polarization currents defined in (6.8) and (6.9). The diamagnetic Hall term component $\mathbf{J}_*\times\mathbf{B} = \boldsymbol{\nabla}_\perp P$, and the $\boldsymbol{\nabla}_\perp p_e$ and $(\boldsymbol{\nabla\cdot\pi}_{\wedge e})_\perp$ terms are comparable in magnitude; they are all neglected in MHD because of the perpendicular part of the MHD approximation (6.19). Finally, the ratio of the polarization current contribution in the Hall term to the electric field term is $|\mathbf{J}_p\times\mathbf{B}|/(n_e e|\mathbf{E}_\perp|) \sim (\rho_m/n_e e)|d\mathbf{V}_\perp/dt|/|\mathbf{E}_\perp| \sim (1/\omega_{ci})|d\mathbf{E}_\perp/dt|/|\mathbf{E}_\perp| \sim \omega/\omega_{ci}$, which is small in the small gyroradius expansion necessary for the validity of MHD. Thus, our perpendicular Ohm's law in MHD becomes simply $\mathbf{E}_\perp + \mathbf{V}\times\mathbf{B} = \mathbf{J}_\perp/\sigma_\perp$.

The perpendicular Ohm's law can be combined with the MHD parallel Ohm's law to yield

$$\mathbf{E} + \mathbf{V}\times\mathbf{B} = \mathbf{J}_\parallel/\sigma_\parallel + \mathbf{J}_\perp/\sigma_\perp, \quad \text{complete MHD Ohm's law.} \tag{6.21}$$

The parallel electrical conductivity $\sigma_\parallel$ is at most a factor [see (**??**)] of $1/\alpha_e \leq 32/3\pi \simeq 3.4$ greater than the perpendicular conductivity $\sigma_\perp = \sigma_0$. Thus, it is customary in resistive MHD to not distinguish the electrical conductivity along and transverse to the magnetic field, but instead to just use an isotropic electrical resistivity defined by $\eta \equiv 1/\sigma_0 = m_e\nu_e/n_e e^2$. Hence, the MHD Ohm's law is usually written as simply $\mathbf{E} + \mathbf{V}\times\mathbf{B} = \eta\mathbf{J}$.

In MHD the Ohm's law is used to write the electric field in terms of the flow velocity $\mathbf{V}$ and current $\mathbf{J}$. Taking the cross product of the Ohm's law with the magnetic field $\mathbf{B}$, we obtain the perpendicular MHD mass flow velocity $\mathbf{V}_\perp$:

$$\mathbf{V}_\perp = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{\mathbf{B}\times\eta\mathbf{J}}{B^2} = \mathbf{V}_E + \mathbf{V}_\eta. \tag{6.22}$$

Thus, the perpendicular MHD mass flow velocity is the sum of the $\mathbf{E}\times\mathbf{B}$ flow velocity (**??**) and the (ambipolar) classical transport flow velocity (**??**), which although small is kept because it is a consequence of including resistivity in the Ohm's law. [The diamagnetic flow velocity $\mathbf{V}_*$ does not appear in the perpendicular MHD mass flow velocity $\mathbf{V}_\perp$ because of the MHD approximation (6.19); the polarization flow $\mathbf{V}_p$ and viscosity-driven flow $\mathbf{V}_\pi$ are not included in the MHD $\mathbf{V}_\perp$ because they are higher order in the small gyroradius expansion.]

The parallel ($\hat{\mathbf{b}}\,\boldsymbol{\cdot}$) component of the MHD Ohm's law (**??**) yields

$$E_\parallel = \eta J_\parallel. \tag{6.23}$$

In the ideal MHD limit where $\eta \to 0$, this equation requires $E_\parallel = 0$, which for a general $\mathbf{E} = -\boldsymbol{\nabla}\phi - \partial\mathbf{A}/\partial t$ is satisfied in equilibrium by the equilibrium potential Φ being constant along the magnetic field, and in perturbations by the parallel gradient of the potential being balanced by a parallel inductive (vector potential) component: $\tilde{E}_\parallel = -\nabla_\parallel\tilde{\phi} - \partial\tilde{A}_\parallel/\partial t = 0$.

Finally, we need a one-fluid energy equation or equation of state to close the hierarchy of MHD equations. In MHD it is customary to use an isentropic equation of state $(d/dt)\ln(P/\rho_m^\Gamma) \simeq 0$. Using $P = p_e + p_e$, $3/2 \Longrightarrow 1/(\Gamma-1)$ and working out the time derivative in terms of the time derivatives of the electron and ion entropies given in (**??**), (**??**), (**??**) and (**??**), we obtain

$$\frac{d}{dt}\ln\frac{P}{\rho_m^\Gamma} = \frac{\Gamma-1}{P}\left(p_e\frac{ds_e}{dt} + p_i\frac{ds_i}{dt}\right) \simeq \frac{\Gamma-1}{P}\left(-\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{q}_e - \boldsymbol{\nabla}\mathbf{V}_i\,\boldsymbol{:}\,\boldsymbol{\pi}_i + \eta J^2\right). \tag{6.24}$$

The last, approximate form indicates the dominant contributions to the overall plasma entropy production rate. Its last term indicates entropy production by joule heating; while this rate is usually small [$\simeq \nu_e(|\mathbf{J}|/n_e e v_{Te})^2 << \nu_e$, of order one over the plasma confinement time], it should be kept in resistive MHD for consistency with the inclusion of resistivity in the Ohm's law. As discussed after (**??**), the ion viscous dissipation rate is at most of order the ion collision frequency ν_i for fluidlike ions; thus, like the ion viscous stress tensor effects in the plasma momentum equation, it is usually neglected assuming $d/dt \sim -i\omega >> \nu_i$.

Most problematic for an isentropic plasma equation of state is the electron heat conduction. In a collisional plasma, parallel electron heat conduction leads to a plasma entropy production rate of order $\nu_e(\lambda_e\nabla_\parallel)^2 << \nu_e$, which is often smaller than MHD wave frequencies and hence negligible. However, in low collisionality plasmas where $\lambda_e\nabla_\parallel \gtrsim 1$, parallel electron heat conduction can cause entropy production rates of order ν_e or perhaps larger [see disussion after (**??**)], which can be of order MHD wave frequencies. On the other hand, if the electron fluid responds totally collisionlessly, there is no entropy production from electron heat conduction (or any other collisionless electron process). In MHD it is customary to neglect the electron heat conduction contributions to entropy production on the basis that either: 1) $d/dt \sim -i\omega >> \nu_e$; 2) parallel electron temperature gradients are quite small because of parallel heat conduction and thus lead to a negligible entropy production rate [$\omega >> \nu_e\lambda_e^2(\nabla_\parallel^2 T)/T$]; or 3) the relevant electron response is totally collisionless and hence leads to no entropy production. However, there could be circumstances where entropy-producing parallel electron heat conduction effects are important on MHD wave time scales.

6.2 MHD Equations

The equations used to describe the MHD model of a magnetized plasma and the associated electric and magnetic fields are thus given by

MHD Plasma Description (Ideal, $\eta \to 0$; Resistive, $\eta \neq 0$):

$$\text{mass density:} \quad \frac{\partial \rho_m}{\partial t} + \boldsymbol{\nabla}\cdot \rho_m \mathbf{V} = 0, \tag{6.25}$$

$$\text{charge continuity:} \quad \boldsymbol{\nabla}\cdot \mathbf{J} = 0, \tag{6.26}$$

$$\text{momentum:} \quad \rho_m \frac{d\mathbf{V}}{dt} = \mathbf{J}\times\mathbf{B} - \boldsymbol{\nabla} P, \tag{6.27}$$

$$\text{Ohm's law:} \quad \mathbf{E} + \mathbf{V}\times\mathbf{B} = \eta \mathbf{J}, \tag{6.28}$$

$$\text{equation of state:} \quad \frac{d}{dt} \ln \frac{P}{\rho_m^\Gamma} = (\Gamma - 1)\frac{\eta J^2}{P} \simeq 0, \tag{6.29}$$

$$\text{total time derivative:} \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}. \tag{6.30}$$

Maxwell Equations for MHD:

$$\text{Faraday's law:} \quad \frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E}, \tag{6.31}$$

$$\text{no magnetic monopoles:} \quad \boldsymbol{\nabla}\cdot \mathbf{B} = 0, \tag{6.32}$$

$$\text{nonrelativistic Ampere's law:} \quad \mu_0 \mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B}. \tag{6.33}$$

Gauss' law ($\boldsymbol{\nabla}\cdot \mathbf{E} = \rho_q$) does not appear in the list of Maxwell equations because in the MHD model plasmas are highly polarizable, quasineutral ($\rho_q \simeq 0$) fluids in which the electric field is determined self-consistently from Ohm's law, Ampere's law and the charge continuity equation $\boldsymbol{\nabla}\cdot \mathbf{J} = 0$.

The MHD model describes a very wide range of phenomena in small gyroradius, magnetized plasmas — macroscopic plasma equilibrium and instabilities, Alfvèn waves, magnetic field diffusion. It is the fundamental, lowest order model used in analyzing magnetized plasmas.

The physics content of the MHD plasma description is briefly as follows. The equation for the mass density ($\rho_m \simeq m_i n_i$) is also called the continuity equation and can be written in the form $\partial \rho_m/\partial t = -\mathbf{V}\cdot\boldsymbol{\nabla}\rho_m - \rho_m \boldsymbol{\nabla}\cdot\mathbf{V}$. When written in the latter form, it describes changes in mass density due to advection ($\mathbf{V}\cdot\boldsymbol{\nabla}\rho_m$) and compressibility ($\boldsymbol{\nabla}\cdot\mathbf{V} \neq 0$) by the mass flow velocity $\mathbf{V}$ — see Fig. **??**. The charge continuity equation is the quasineutral ($\rho_q \simeq 0$) form of the general charge continuity equation $\partial \rho_q/\partial t + \boldsymbol{\nabla}\cdot \mathbf{J} = 0$ that results from adding equations for the charge densities of the electron and ion species in the plasma. [While $\boldsymbol{\nabla}\cdot \mathbf{J} = 0$ also results from taking the divergence of the nonrelativistic (i.e., without displacement current) Ampere's law, it is often better to think of it as the equation that ensures quasineutrality of the plasma in the MHD model — as indicated in (6.14).] The momentum equation, which is also known as the equation of motion, provides the force density balance for a fluid element (infinitesimal volume of fluid) that is analogous to $m\mathbf{a} = \mathbf{F}$ for a particle: the inertial force ($\rho_m d\mathbf{V}/dt$) is equal to the magnetic force ($\mathbf{J}\times\mathbf{B}$) plus the (expansive) pressure gradient force ($-\boldsymbol{\nabla} P$, where $P = p_e + p_i$ is the total plasma pressure) on a fluid element. The MHD Ohm's law, which is a simplified form of the electron momentum equation, is just the basic laboratory frame Ohm's law $\mathbf{E}' = \eta\mathbf{J}$ for a fluid moving with plasma mass flow velocity $\mathbf{V}$: $\mathbf{E}' = \mathbf{E} + \mathbf{V}\times\mathbf{B}$. The MHD equation of state is an isentropic (adiabatic in thermodynamics) equation of state except for the small entropy production rate by joule heating ($\sim \eta J^2/P \sim 1/\tau_E$), which is usually negligibly small but is retained for consistency with inclusion of resistivity in Ohm's law. The total time derivative in (6.30) indicates that time-differentiated quantities change both because of local (Eulerian) temporal changes ($\partial/\partial t|_{\mathbf{x}}$) and because of being carried along (advected) with the MHD fluid ($\mathbf{V}\cdot\boldsymbol{\nabla}$) at the velocity $\mathbf{V}$.

After some manipulations, it can be shown (see Problems 6.8–6.9) that the MHD equations yield the following conservative forms of total MHD system mass, momentum and energy relations:

$$\text{MHD system mass equation:} \qquad \frac{\partial \rho_m}{\partial t} + \boldsymbol{\nabla}\cdot\rho_m\mathbf{V} = 0, \tag{6.34}$$

$$\text{MHD system momentum equation:} \qquad \frac{\partial(\rho_m\mathbf{V})}{\partial t} + \boldsymbol{\nabla}\cdot\mathsf{T} = \mathbf{0}, \tag{6.35}$$

$$\text{MHD system energy equation:} \qquad \frac{\partial w}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} = 0, \tag{6.36}$$

in which

$$\text{MHD stress tensor:} \qquad \mathsf{T} \equiv \rho_m\mathbf{V}\mathbf{V} + \left(P + \frac{B^2}{2\mu_0}\right)\mathsf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0}, \tag{6.37}$$

$$\text{MHD energy density:} \qquad w \equiv \frac{\rho_m V^2}{2} + \frac{P}{\Gamma - 1} + \frac{B^2}{2\mu_0}, \tag{6.38}$$

$$\text{MHD energy flux:} \qquad \mathbf{S} \equiv \left(\frac{\rho_m V^2}{2} + \frac{\Gamma}{\Gamma - 1}P\right)\mathbf{V} + \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}. \tag{6.39}$$

Here, the contributions to the MHD system stress tensor are due to the flow ($\rho_m\mathbf{V}\mathbf{V}$, Reynolds stress), isotropic pressure ($P\mathsf{I}$) and both isotropic expansion [$(B^2/2\mu_0)\mathsf{I}$] and tension ($-\mathbf{B}\mathbf{B}/\mu_0$) stresses in the magnetic field — see (**??**). The Reynolds stress is only important in systems with large flow; it is negligible in MHD systems with strongly subsonic flows ($\rho_m V^2/2P \sim V^2/c_S^2 << 1$). The system energy density is composed of the densities of the kinetic (flow) energy ($\rho_m V^2/2$), internal energy ($3P/2$ for a three-dimensional system with $\Gamma = 5/3$) and the magnetic field energy density ($B^2/2\mu_0$). Joule heating (ηJ^2) does not appear in the MHD system energy density equation because energy lost from the electromagnetic fields by joule heating [see (**??**)] increases the internal energy in the plasma [see (6.29)]; thus, the total MHD energy density, which sums these energies, remains constant. The terms in the MHD energy flux represent the flow of kinetic ($\rho_m V^2/2$) and internal $[P/(\Gamma-1)]$ energies with the flow velocity $\mathbf{V}$, mechanical work done on or by the plasma as it moves ($P\mathbf{V}$), and energy flow by the electromagnetic fields ($\mathbf{E}\times\mathbf{B}/\mu_0$) [Poynting vector — see (**??**)].

To illutrate the usefulness of these MHD system conservation equations, consider the system energy equation (6.36). Integrating this equation over the volume V of an isolated plasma, the divergence term can be converted using Gauss' theorem (**??**) into a surface integral that vanishes if there is no flow of plasma or electromagnetic energy across the surface that bounds the volume. For such an isolated system the integral of the system energy over the volume must be independent of time:

$$\int_V d^3x \left(\frac{\rho_m V^2}{2} + \frac{P}{\Gamma-1} + \frac{B^2}{2\mu_0}\right) \equiv W_k + W_p \;=\; \text{constant}, \tag{6.40}$$

in which

$$W_k \;=\; \int_V d^3x \, \frac{\rho_m V^2}{2}\,, \qquad \text{plasma kinetic energy}, \tag{6.41}$$

$$W_p \;=\; \int_V d^3x \left(\frac{P}{\Gamma-1} + \frac{B^2}{2\mu_0}\right), \quad \text{MHD potential energy}. \tag{6.42}$$

Thus, in the MHD model while there can be exchanges of energy between the plasma kinetic, and internal and magnetic energies, their sum must be constant. For a plasma motion to grow monotonically (as in a collective instability), increases in plasma kinetic energy due to dynamical motion of the plasma must be balanced by reductions in the potential (plasma internal plus magnetic field) energy in the plasma volume. In Chapter 21 the constancy of the total system energy in MHD will be used as the basis for developing a variational ("energy") principle for plasma instability, which can occur for a plasma perturbation that reduces the system potential energy W_p.

6.3 MHD Equilibrium

In this section we discuss the equilibrium ($\partial/\partial t = 0$) consequences of the system conservation relations for MHD (6.34)–(6.36). In equilibrium the mass density equation yields $\boldsymbol{\nabla}\cdot\rho_m\mathbf{V} = 0$. In one dimension ($x$), this equilibrium continuity equation yields $\rho_m(x)V_x(x) = \text{constant}$. Thus, in a one-dimensional flow situation the mass density will be higher (lower) where the flow velocity $\mathbf{V}$ is lower (higher). Equilibrium flows are negligible in MHD for many plasma situations; then the equilibrium continuity equation is trivially satisfied for any mass density profile $\rho_m(\mathbf{x})$.

Next, consider the stress-induced forces which contribute to the system momentum conservation equation (6.35). Consider first the magnetic (subscript B) contribution that is represented by the $\mathbf{J}\times\mathbf{B}$ force density in the momentum equation (6.27) and the magnetic field part of the system stress tensor T in (6.37). The stress in the magnetic field exerts a force density $\mathbf{f}_B$ on a fluid

Figure 6.1: Schematic illustration of the stresses and force densities on a fluid element of plasma in the MHD model: a) isotropic expansive pressure stress $\mathsf{T}_P = P\mathsf{I}$, b) anisotropic magnetic stresses T_B, c) pressure gradient force density $\mathbf{f}_P = -\nabla P$, and d) magnetic force density $\mathbf{f}_B$ in the normal ($\hat{\mathsf{N}} \propto$ curvature) and binormal ($\hat{\mathsf{B}}$) directions.

element (infinitesimal volume of MHD plasma fluid) given by

$$\begin{aligned}
\mathbf{f}_B \;&\equiv\; \mathbf{J}\times\mathbf{B} = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} = -\frac{B}{\mu_0}\hat{\mathbf{b}}\times(\nabla\times B\hat{\mathbf{b}}) \\
&= -\frac{B}{\mu_0}\hat{\mathbf{b}}\times(\nabla B\times\hat{\mathbf{b}}) - \frac{B^2}{\mu_0}\hat{\mathbf{b}}\times(\nabla\times\hat{\mathbf{b}}) \\
&= -\nabla_\perp\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\boldsymbol{\kappa} \;=\; -\nabla\cdot\frac{B^2}{2\mu_0}\left(\mathsf{I} - \frac{\hat{\mathbf{b}}\hat{\mathbf{b}}}{2}\right) \;\equiv\; -\nabla\cdot\mathsf{T}_B,
\end{aligned} \tag{6.43}$$

in which we have used vector identities (??), (??), (??), (??), (??), (??) and (??). The corresponding force density $\mathbf{f}_P$ due to the plasma pressure is

$$\mathbf{f}_P \;\equiv\; -\nabla P \;=\; -\nabla\cdot P\,\mathsf{I} \;\equiv\; -\nabla\cdot\mathsf{T}_P. \tag{6.44}$$

These stresses and force densities are illustrated schematically in Fig. 6.1 and discussed in the next few paragraphs.

Consider first the stresses. Adopting $\hat{\mathbf{e}}_{\mathrm{x}}, \hat{\mathbf{e}}_{\mathrm{y}}, \hat{\mathbf{b}} \equiv \mathbf{B}/B$ as the base vectors for a local magnetic field coordinate system, the sum of the pressure and magnetic stress tensors can be written (in matrix notation) as

$$\begin{aligned}
\mathsf{T}_P + \mathsf{T}_B \;&\equiv\; \left(\hat{\mathbf{e}}_{\mathrm{x}}\,\hat{\mathbf{e}}_{\mathrm{y}}\,\hat{\mathbf{b}}\right)\begin{pmatrix} P + B^2/2\mu_0 & 0 & 0 \\ 0 & P + B^2/2\mu_0 & 0 \\ 0 & 0 & P - B^2/2\mu_0 \end{pmatrix}\begin{pmatrix} \hat{\mathbf{e}}_{\mathrm{x}} \\ \hat{\mathbf{e}}_{\mathrm{y}} \\ \hat{\mathbf{b}} \end{pmatrix} \\
&= \hat{\mathbf{e}}_{\mathrm{x}}T_{\perp\perp}\hat{\mathbf{e}}_{\mathrm{x}} + \hat{\mathbf{e}}_{\mathrm{y}}T_{\perp\perp}\hat{\mathbf{e}}_{\mathrm{y}} + \hat{\mathbf{b}}T_{\|\|}\hat{\mathbf{b}},
\end{aligned} \tag{6.45}$$

$$\text{with} \qquad T_{\perp\perp} \equiv P + B^2/2\mu_0, \quad T_{\|\|} \equiv P - B^2/2\mu_0.$$

(For simplicity of presentation, often the directional vectors are omitted and only the elements of the matrix of tensor coefficients are shown.) The plasma pressure produces an isotropic tensor (I) expansive (positive) stress, which represents the thermal motion of particles expanding uniformly in all directions. The magnetic stress is anisotropic. From T_B and (6.45), we see that the magnetic stress is expansive (positive) in directions $\hat{\mathbf{e}}_{\mathrm{x}}, \hat{\mathbf{e}}_{\mathrm{y}}$ perpendicular to the magnetic field $\mathbf{B} = B\hat{\mathbf{b}}$, but in tension (negative) along magnetic field lines. Physically, the magnetic field can be thought of as providing a magnetic "pressure" $B^2/2\mu_0$ perpendicular to magnetic field lines, and tension along field lines — as if the magnetic field lines are elastic cords with tension stress of B^2/μ_0 along $\mathbf{B}$ pressing against the plasma fluid, which is trying to expand perpendicular to the magnetic field lines due to the combination of the pressure and magnetic energy density expansive forces.

The force density on an MHD fluid element is given (for subsonic flows where the Reynolds stress tensor $\rho_m\mathbf{VV}$ is negligible) by the divergence of this stress tensor:

$$\begin{aligned} \mathbf{f}_P + \mathbf{f}_B \;&\equiv\; -\boldsymbol{\nabla} P + \mathbf{J}\times\mathbf{B} \;=\; -\boldsymbol{\nabla\cdot}(\mathsf{T}_P + \mathsf{T}_B) \\ &=\; -\boldsymbol{\nabla} P - \boldsymbol{\nabla}\left(\frac{B^2}{2\mu_0}\right) + \frac{(\mathbf{B}\boldsymbol{\cdot\nabla})\mathbf{B}}{\mu_0} \\ &=\; -\boldsymbol{\nabla} P - \boldsymbol{\nabla}_\perp\left(\frac{B^2}{2\mu_0}\right) + \frac{B^2}{\mu_0}\boldsymbol{\kappa}. \end{aligned} \tag{6.46}$$

In the last form, the $-\boldsymbol{\nabla} P$ term represents the isotropic, pressure gradient force, the next term represents the perpendicular (to $\mathbf{B}$) force due to the magnetic "pressure" $B^2/2\mu_0$ and the last term represents the force due to the parallel tension of magnetic field lines, as if each "magnetic cord" presses on the fluid with a force density of $(B^2/\mu_0)\boldsymbol{\kappa} = -(B^2/\mu_0)\mathbf{R}_C/R_C^2$ where $\mathbf{R}_C$ is the local radius of curvature vector [see (**??**)] of a magnetic field line.

An MHD fluid element will be in force balance equilibrium, which is usually just called "equilibrium" in MHD, if the force density $\mathbf{f}_P + \mathbf{f}_B$ vanishes. Then, there is no net force to drive an inertial force response via the MHD momentum equation (6.27) and the system momentum conservation equation (6.35) is satified in equilibrium $[\partial(\rho_m\mathbf{V})/\partial t = 0]$. When there is no gradient in the plasma pressure (an unconfined plasma), the force balance equilibrium becomes

$$\mathbf{f}_B = \mathbf{J}\times\mathbf{B} = \mathbf{0}, \quad \text{force-free equilibrium with } \boldsymbol{\nabla} P = \mathbf{0}. \tag{6.47}$$

In order for a magnetic field system to be able to support a pressure gradient in force balance equilibrium, the current and magnetic field must not be parallel to each other; rather, their cross product must satisfy

$$\mathbf{J}\times\mathbf{B} = \boldsymbol{\nabla} P, \quad \text{MHD force-balance equilibrium.} \tag{6.48}$$

Taking the cross product of $\mathbf{B}$ with this equation, we obtain the diamagnetic current $\mathbf{J}_* = (\mathbf{B}\times\boldsymbol{\nabla} P)/B^2$ in (6.8), which is the sum of the diamagnetic flows of all species of charged particles in the plasma given in (**??**). The perpendicular $[-\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\,)]$ component of the MHD force-balance equation can also be written [from the last form of (6.46)] as

$$\boldsymbol{\kappa} = \boldsymbol{\nabla}_\perp \ln B + \frac{\mu_0}{B^2}\boldsymbol{\nabla}_\perp P, \quad \text{perpendicular equilibrium in MHD.} \tag{6.49}$$

This formula is the same as (**??**) given previously in Chapter 3 for the magnetic field curvature if we use the MHD equilibrium condition $\mathbf{J}\times\mathbf{B} = \boldsymbol{\nabla}_\perp P$.

Because the force density on the plasma is different in different directions, it is of interest to explore its forms and implications in various relevant directions. Since the magnetic field direction and curvature are two obviously important directions, a convenient coordinate system is the Frenet coordinate system whose orthogonal base vectors for a vector field (**B** here) are (see Section D.6)

$$\hat{\mathsf{T}} \equiv \hat{\mathbf{b}} \equiv \mathbf{B}/B, \quad \hat{\mathsf{N}} \equiv \boldsymbol{\kappa}/\kappa, \quad \hat{\mathsf{B}} \equiv \hat{\mathsf{T}} \times \hat{\mathsf{N}} = \hat{\mathbf{b}} \times \boldsymbol{\kappa}/\kappa, \tag{6.50}$$

which are unit vectors in the tangent ($\hat{\mathsf{T}}$), normal ($\hat{\mathsf{N}}$, or curvature) and binormal ($\hat{\mathsf{B}}$) directions of the **B** field. Decomposing the MHD force density on a fluid element into its components in these orthogonal directions, we find

$$\mathbf{f}_P + \mathbf{f}_B = -\,\hat{\mathbf{b}}\,(\hat{\mathbf{b}}\cdot\nabla P) - \hat{\mathsf{N}}\left[(\hat{\mathsf{N}}\cdot\nabla)(P + \frac{B^2}{2\mu_0}) - \frac{B^2}{\mu_0}\boldsymbol{\kappa}\right] - \hat{\mathsf{B}}\,(\hat{\mathsf{B}}\cdot\nabla)\left(P + \frac{B^2}{2\mu_0}\right). \tag{6.51}$$

The conditions for MHD force-balance equilibrium are thus (see Fig. 6.1)

$$\text{along } \mathbf{B}: \qquad 0 = \hat{\mathbf{b}}\cdot\nabla P = \frac{\partial P}{\partial \ell}, \tag{6.52}$$

$$\text{curvature direction:} \qquad 0 = \hat{\mathsf{N}}\cdot\nabla\left(P + \frac{B^2}{2\mu_0}\right) - \frac{B^2}{\mu_0}\boldsymbol{\kappa}, \tag{6.53}$$

$$\text{binormal direction:} \qquad 0 = \hat{\mathsf{B}}\cdot\nabla\left(P + \frac{B^2}{2\mu_0}\right). \tag{6.54}$$

Since there is no magnetic force along the magnetic field ($\mathbf{B}\cdot\mathbf{f}_B = \mathbf{B}\cdot\mathbf{J}\times\mathbf{B} = 0$), in order to satisfy the first (parallel) MHD force balance condition the plasma pressure P must be constant along magnetic field lines. (The axial confinement of plasma in a magnetic mirror is achieved via anisotropic pressure — see Problem 6.11.) When nested magnetic flux surfaces exist (see end of Section 3.2), $\partial P/\partial \ell = 0$ requires that the pressure be a function only of the magnetic flux ψ:

$$P = P(\psi) \quad \Longrightarrow \quad \mathbf{B}\cdot\nabla P = (\mathbf{B}\cdot\nabla\psi)\frac{dP}{d\psi} = 0, \tag{6.55}$$

which vanishes (assuming finite $dP/d\psi$), by virtue of the condition for the existence of a magnetic flux function (**??**): $\mathbf{B}\cdot\nabla\psi = 0$. Further, from the dot product of the current **J** with the MHD equilibrium force-balance condition (6.48) we find

$$\mathbf{J}\cdot\nabla P = 0. \tag{6.56}$$

From these last two equations we see that the vector fields **J** and **B** both lie within, and do not penetrate, magnetic flux surfaces. Further, we see from (6.48) that in force balance equilibrium the cross product of these two vectors in the flux surface must equal the pressure gradient, which is perpendicular to the flux surface (see Fig. 6.2):

$$\mathbf{J}\times\mathbf{B} = \nabla P(\psi) = \nabla\psi\frac{dP}{d\psi}. \tag{6.57}$$

Figure 6.2: In ideal MHD equilibrium the cross product of the current density $\mathbf{J}$ and magnetic field $\mathbf{B}$ vectors within a flux surface is equal to $\boldsymbol{\nabla} P = (dP/d\psi)\boldsymbol{\nabla}\psi$, which is normal to the flux surface.

Figure 6.3: Pressure P and magnetic energy density $B^2/2\mu_0$ profiles for: a) $\beta << 1$, and b) $\beta \simeq 1$.

When there is no magnetic field curvature, the force balance equilibrium condition is the same in all directions perpendicular to the magnetic field:

$$\boldsymbol{\nabla}_\perp \left(P + \frac{B^2}{2\mu_0} \right) = \mathbf{0}, \quad \text{MHD equilibrium with no } \mathbf{B} \text{ field curvature.} \tag{6.58}$$

To illustrate the implications of this equation, consider the MHD equilibrium of a localized plasma placed in a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{b}} = B_0\hat{\mathbf{e}}_z$, For a given plasma pressure profile $P(\mathbf{x}_\perp)$ that varies in directions $(\mathbf{x}_\perp)$ perpendicular to the magnetic field but does not extend to infinite dimensions, (6.58) yields

$$\frac{\partial}{\partial \mathbf{x}_\perp} \left(\frac{B^2}{2\mu_0} \right) = - \frac{\partial P}{\partial \mathbf{x}_\perp} \quad \Longrightarrow \quad B(\mathbf{x}_\perp) = B_0 \sqrt{1 - \beta(\mathbf{x}_\perp)}\,. \tag{6.59}$$

Here, we have defined the very important MHD parameter β by

$$\beta(\mathbf{x}_\perp) \;\equiv\; \frac{P(\mathbf{x}_\perp)}{B_0^2/2\mu_0} \;=\; 4.0 \times 10^{-25} \left(\frac{n_e}{B^2} \right) \left[T_e(\text{eV}) + \frac{n_i}{n_e} T_i(\text{eV}) \right],$$
$$\text{ratio of plasma pressure to magnetic energy density.} \tag{6.60}$$

Thus, in an MHD equilibrium, for a situation where the magnetic field $\mathbf{B}$ has no curvature, the plasma digs a magnetic well (region of reduced magnetic energy density) that is just deep enough so that the sum of the plasma pressure P and magnetic field energy density $B^2/2\mu_0$ is constant (at $B_0^2/2\mu_0$) in all directions perpendicular to the magnetic field. This result is illustrated in Fig. 6.3 for a cylindrical plasma where the plasma pressure vanishes at $r = a$ for two cases: small β and near unity β. The cylindrical form of (6.59) can also be obtained directly (see Problem 6.14) from the radial force balance equation by calculating the radial variation of the magnetic field $B_z(r)$ using the azimuthal component of Ampere's law.

When the magnetic field has curvature, the force balance condition in the normal (curvature) direction is changed to condition (6.53). Then, the pressure gradient in the curvature direction can be supported in force balance equilibrium by either the curvature-induced force density ($\boldsymbol{\kappa}B^2/\mu_0$) or the gradient in the magnetic energy density, or by some combination thereof. When the plasma pressure is low ($\beta << 1$), the magnetic field curvature is equal to the gradient of the magnetic field energy density [the situation for a vacuum magnetic field — see (**??**)] plus a small correction due to the plasma pressure. In the limit where the magnetic field curvature is weak (radius of curvature R_C much greater than the presssure gradient scale length $L_P \equiv P/|\boldsymbol{\nabla}_\perp P|$), the curvature effects are small and the variation in magnetic field strength is still approximately as given in (6.59). [In an axisymmetric tokamak both of these small corrections to (6.59) are unfortunately comparable in magnitude — see Chapter 20.] In the binormal direction, (6.54) shows that in force balance equilibrium, even with curvature in the magnetic field $\mathbf{B}$, $P + B^2/2\mu_0$ is constant in the binormal direction — increases in the plasma pressure P in the binormal direction are balanced by decreases in magnetic energy density $B^2/2\mu_0$, like in (6.59).

From the preceding discussion is is clear that the parameter β characterizes the relative importance of the plasma pressure P versus the magnetic field $\mathbf{B}$. For $\beta << 1$ the plasma pressure has a small effect on the MHD equilibrium and the magnetic field structure is approximately that determined from a vacuum magnetic field representation (**??**). Also, the diamagnetic current is small ($\mathbf{J}_* \simeq \mathbf{B}\times\boldsymbol{\nabla}\beta/2\mu_0$), as is the (diamagnetic) magnetic susceptibility due to the plasma magnetization produced by the magnetic moments of all the charged particles in the plasma gyrating in the magnetic field [$\chi_M \simeq -\beta/2$ — see discussion after (6.12)]. Since the magnetic field is much stronger than the plasma pressure in this regime, it can be used to provide a "magnetic bottle" for plasma confinement. In the opposite limit ($\beta >> 1$) where the the plasma pressure in much larger than the magnetic energy density, in general the plasma "pushes the magnetic field around" and carries it along with its natural motions (pressure expansion plus flows). A key question for magnetic fusion confinement systems is the maximum β they can stably confine in equilibrium; the $\beta \sim$ 5–10% that is needed for economically viable deuterium-tritium fusion reactors is apparently accessible in many types of toroidal confinement systems — see Chapter 21.

It is often asked: can a finite pressure plasma support itself entirely with the diamagnetic current and the magnetic field it produces, without any externally imposed magnetic field? That is, can a plasma organize itself into a closed magnetic equilibrium that has no connection to the outside world? In order to examine this question, we consider the equilibrium [$\partial(\rho_m\mathbf{V})/\partial t = 0$] MHD sytem momentum (or force balance) equation obtained from (6.35): $\boldsymbol{\nabla}\cdot\mathsf{T} = \mathbf{0}$. Taking the dot product of this equation with the position vector $\mathbf{x}$ from the centroid of the plasma system (to obtain a measure of the MHD system potential energy density), we obtain the relation

$$0 = \mathbf{x}\cdot\boldsymbol{\nabla}\cdot\mathsf{T} = \boldsymbol{\nabla}\cdot(\mathbf{x}\cdot\mathsf{T}) - \boldsymbol{\nabla}\mathbf{x}:\mathsf{T} = \boldsymbol{\nabla}\cdot(\mathbf{x}\cdot\mathsf{T}) - tr\{\mathsf{T}\}, \qquad (6.61)$$

in which we have used vector identities (**??**), (**??**) and (**??**). Integrating this last form over a volume larger than the proposed isolated plasma, we obtain

$$\oiint_S d\mathbf{S}\cdot(\mathbf{x}\cdot\mathsf{T}) = \int d^3x\; tr\{\mathsf{T}\}, \tag{6.62}$$

in which we have used the tensor form of Gauss' divergence theorem (**??**) to convert the volume integral to a surface integral. We now examine the integrals on the left and right separately. For the integral on the left we assume negligible flows ($\mathbf{V}\to\mathbf{0}$) and use (6.37) for T. Then, the integral on the left can be written as

$$\oiint d\mathbf{S}\cdot(\mathbf{x}\cdot\mathsf{T}) = \oiint d\mathbf{S}\cdot\left[\mathbf{x}\left(P+\frac{B^2}{2\mu_0}\right)-\frac{(\mathbf{x}\cdot\mathbf{B})\mathbf{B}}{\mu_0}\right]\sim\frac{1}{r^3}\quad\overset{r\to\infty}{\Longrightarrow}\quad 0. \tag{6.63}$$

As indicated at the end, in the limit of large radial distances r from the isolated plasma this integral vaishes — because since there are apparently no magnetic monopoles in the universe, the magnetic field $\mathbf{B}$ must decrease like that for a dipole field does ($|\mathbf{B}|\sim 1/r^3$) so the integrand scales as $1/r^5$ and when integrated over the surface ($|d\mathbf{S}|\to 4\pi r^2$) one finds that the integral decreases at least as fast as $1/r^3$. Next, we consider the integral on the right. Using the matrix definition of the stress tensor T given in (6.45), we find (for an isolated plasma within a finite volume V)

$$\int_V d^3x\; tr\{\mathsf{T}\} = \int_V d^3x\left(3P+\frac{B^2}{2\mu_0}\right)\quad\Longrightarrow\quad \text{constant}. \tag{6.64}$$

The only way this last integral can vanish, as is required by the combination of (6.62) and (6.63), is if the plasma pressure P and magnetic energy density $B^2/2\mu_0$ (both of which are intrinsically positive quantities) vanish. Thus, we have found a contradiction: no isolated finite-pressure plasma can by itself develop a self-confining magnetic field in force balance equilibrium. This proof and analysis is sometimes called a virial theorem (because it results from $\int d^3x\,\mathbf{x}\cdot\mathbf{f} = \int d^3x\,\mathbf{x}\cdot\boldsymbol{\nabla}\cdot\mathsf{T} = 0$) and was first derived by V.D. Shafranov.[1]

6.4 Boundary Conditions and Shock Relations

The basic subject to be discussed here are the jump conditions at a discontinuity in a plasma or at a plasma-vacuum interface, and then the corresponding bounday conditions at a vacuum wall or around coils for a "free-boundary" equilibrium. See Section 3.2 of the Freidberg book. These same equations become the shock conditions in a plasma. This section will be written later.

6.5 MHD Dynamics

To explore the elementary dynamical (evolution in time) properties of a plasma in the MHD model, we first assume that the plasma fluid moves with a velocity

[1] V.D. Shafranov, in *Reviews of Plasma Physics*, edited by M.A. Leontovich (Consultants Bureau, New York, 1966), Vol. II.

$\mathbf{V}(\mathbf{x}, t)$ and determine the changes in the mass density ρ_m, pressure P and magnetic field $\mathbf{B}$ induced by $\mathbf{V}$. Then, these responses are used in the momentum equation (6.27) which is then solved self-consistently to determine the mass flow velocity $\mathbf{V}$.

We begin by considering the temporal evolution of the mass density in response to $\mathbf{V}$, which is governed by (6.25):

$$\partial\rho_m/\partial t|_{\mathbf{x}} = -\mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho_m + \rho_m\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V} \quad\Longleftrightarrow\quad d\rho_m/dt = -\rho_m\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V}, \tag{6.65}$$

in which we have used the vector identity (**??**) in obtaining the first form and the total time derivative definition in (6.30) in obtaining the second form. Here, as shown in Fig. **??**, in the Eulerian (fixed position) picture [first form of (6.65)], the flow causes changes in the mass density at a fixed point by advecting ($-\mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}\rho_m$) the mass flow at velocity $\mathbf{V}$ into a region of different mass density, or by compressibility ($\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V} \neq 0$) of the flow. In the Lagrangian (moving with fluid element) picture [second form of (6.65)], the mass density only changes due to the compressibility of the flow ($\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V} \neq 0$).

The pressure evolution can be determined from the isentropic form of the MHD equation of state [i.e., (6.29) neglecting the small entropy production due to joule heating]:

$$\frac{d}{dt}\ln\frac{P}{\rho_m^\Gamma} \;=\; \frac{1}{P}\frac{dP}{dt} - \frac{\Gamma}{\rho_m}\frac{d\rho_m}{dt} \;=\; \frac{1}{P}\frac{dP}{dt} + \Gamma\,\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V} \;=\; 0, \tag{6.66}$$

in which (6.65) has been used to obtain the last form. With the total time derivative definition (6.30), this yields

$$\frac{\partial P}{\partial t} \;=\; -\,\mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}P - \Gamma\,P\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V} \;=\; -\,\mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}P - c_S^2\,\rho_m\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V} \tag{6.67}$$

in which

$$c_S \equiv \sqrt{\Gamma P/\rho_m}, \quad \text{MHD sound speed (m/s).} \tag{6.68}$$

Thus, like the mass density, the plasma pressure changes in MHD are due to advection ($\mathbf{V}\boldsymbol{\cdot}\boldsymbol{\nabla}P$) and flow compression ($\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{V} \neq 0$). The presence of the sound speed in the last form of (6.67) shows that the compressiblity of the flow leads to pressure changes that move at the MHD sound speed through the plasma. Thus, the fluid motion at velocity $\mathbf{V}$ causes advection and compressibility changes in the mass density ρ_m and plasma pressure P, which are scalar quantities.

Note that the MHD sound speed is different from the ion acoustic speed (**??**) in Section 1.4 — because in a MHD description both the electrons and ions have fluidlike (inertial) responses whereas for ion acoustic waves while the ions have a fluidlike response the electrons respond adiabatically. Unfortunately,in plasma physics the same symbol is usually used for both wave speeds — which is meant is usually clear from the context. Also note that for most plasmas with comparable electron and ion temperatures these two speeds are close in magnitude.

The next question is: what is the effect of the fluid motion on the magnetic field $\mathbf{B}(\mathbf{x}, t)$, which is a vector field? Physically, we know that plasmas have a very high electrical conductivity (low resistivity). In the ideal MHD model we set the resistivity to zero and hence effectively assume infinite electrical conductivity; thus, the plasma is a "superconductor" in ideal MHD. From the properties of a superconducting wire of finite cross-section, we know that the magnetic field is "frozen" into it and moves with the wire as it is moved. Thus, we can intuitively anticipate that a fluid element in our superconducting ideal MHD plasma will carry the magnetic field (or at least the bundle of magnetic field lines penetrating it) with it wherever it moves — and will always contain the same amount of magnetic flux (number of field lines[2]). We can also anticipate that the addition of resistivity in the resistive MHD model will allow some slippage of the magnetic field lines relative to the fluid element.

We now develop mathematical representations of the idea that the magnetic field is mostly frozen into an MHD fluid element and moves with it. Consider the time derivative of the magnetic flux $\Psi \equiv \iint_S \mathbf{B}\cdot d\mathbf{S}$ [see (**??**)] though an open surface S in the fluid that moves with the fluid at velocity $\mathbf{V}$:

$$\frac{d\Psi}{dt} = \frac{d}{dt}\iint_S \mathbf{B}\cdot d\mathbf{S} = \iint_S \left[\frac{d\mathbf{B}}{dt}\cdot d\mathbf{S} + \mathbf{B}\cdot\frac{d}{dt}(d\mathbf{S})\right]. \tag{6.69}$$

The total time derivative is appropriate here because we are seeking the change in the magnetic flux penetrating a (changing) surface whose boundary is distorted in time as it moves with the fluid velocity $\mathbf{V}(\mathbf{x},t)$, which is in general nonuniform. The time derivative of the (vectorial) differential surface area $d\mathbf{S}$ represents changes due to changes in its constituent differential line elements induced by the nonuniform flow — see Section D.4. Using (**??**) for this time derivative and the definition of the total time derivative in (6.30), we find

$$\begin{aligned} \frac{d\Psi}{dt} &= \iint_S \left[\left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{V}\cdot\boldsymbol{\nabla}\mathbf{B}\right) + \mathbf{B}\,(\boldsymbol{\nabla}\cdot\mathbf{V}) - \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{V}\right]\cdot d\mathbf{S} \\ &= \iint_S \left[\frac{\partial \mathbf{B}}{\partial t} - \boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B})\right]\cdot d\mathbf{S}, \end{aligned} \tag{6.70}$$

in which we have used vector identity (**??**) and the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ in going from the first to the second line.

For the evolution of the magnetic field $\mathbf{B}$ we use Faraday's law (6.31) together with the MHD Ohm's law (6.28) to specify the electric field $\mathbf{E}$:

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E} = \boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B}) - \boldsymbol{\nabla}\times\eta\mathbf{J} \simeq \boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B}) + \frac{\eta}{\mu_0}\nabla^2\mathbf{B}$$

MHD magnetic field evolution. (6.71)

Here, in the last, approximate form we have used $\mathbf{J} = \boldsymbol{\nabla}\times\mathbf{B}/\mu_0$ (Ampere's law), neglected $\boldsymbol{\nabla}\eta$ for simplicity, and used the vector identity (**??**) and the Maxwell

[2]While magnetic field lines do not really exist since their properties cannot be measured, they are a useful concept for visualizing the behavior of the magnetic field $\mathbf{B}$.

equation $\nabla\cdot\mathbf{B} = 0$. Substituting this magnetic field evolution into (6.70), using Ampere's law for $\mathbf{J}$ again and Stokes' theorem (**??**), we finally obtain

$$\frac{d\Psi}{dt} \;=\; -\iint_S d\mathbf{S}\cdot\nabla\times\eta\mathbf{J} \;=\; -\oint_C d\boldsymbol{\ell}\cdot\frac{\eta}{\mu_0}\mathbf{B}. \tag{6.72}$$

In ideal MHD where $\eta \to 0$, this becomes

$$\frac{d\Psi}{dt} \;=\; 0, \quad \text{ideal MHD frozen flux theorem.}^3 \tag{6.73}$$

Thus, in the absence of resistivity the magnetic flux (number of field lines) through an open surface that moves with the fluid velocity $\mathbf{V}$ is "frozen" into the fluid and hence constant: the magnetic field moves with the superconducting ideal MHD fluid just as we wanted to prove! The key ingredient in this derivation is the $\mathbf{V}\times\mathbf{B}$ term in the MHD Ohm's law. It led to the $\nabla\times(\mathbf{V}\times\mathbf{B})$ term in the magnetic field evolution equation (6.71) and causes the magnetic field to be carried along with the ideal MHD fluid. Hence, this $\nabla\times(\mathbf{V}\times\mathbf{B})$ term represents the advection of the vector field $\mathbf{B}$ by the flow velocity $\mathbf{V}$; note that this vector field advection operator is different in structure from the advection operator for scalar quantities such as the mass density $(-\mathbf{V}\cdot\nabla\rho_m)$. Since the MHD Ohm's law is an approximation to the electron momentum balance equation, it is fundamentally the electron fluid into which magnetic field is frozen (despite the fact that the advection is induced by the overall plasma mass flow velocity $\mathbf{V}$).

By taking the limit of an infintesimally small surface S in the preceding derivation, one can show that an individual magnetic field line is carried along with the superconducting ideal MHD plasma. This can also be shown directly by examining the conditions under which the time derivative of the definitions of magnetic field lines vanish — see Problems 6.15 and 6.16. However, it is important to note that all these derivations have some ambiguity because the labeling of a magnetic field line is not unique [see discussion after (**??**)] and the properties of magnetic field lines cannot be measured. Thus, while we can mark infintesimal elements of a fluid (e.g., with radioactive nuclei or fluorescing partially ionized atoms), and know that the magnetic field is frozen into the ideal MHD fluid elements as they move, the association with a particular magnetic field line from one instant in time to the next is not unique. The "frozen flux" methodology provides a prescription for labeling field lines as they move. While it is not a unique prescription, it represents a very important tool for visualizing the motion of magnetic fields in a moving plasma in the MHD model.

The frozen flux theorem provides a very strong constraint on the motions of the magnetic field in an ideal MHD plasma. In particular, in this model adjacent magnetic field lines and flux bundles that are originally adjacent to each other will forever remain adjacent. Also, magnetic flux bundles and fluid

[3] This theorem is also known as the Alfvén frozen flux theorem. It is the magnetic field analogue of the Kelvin circulation theorem (**??**) for the constancy of the circulation or vorticity flux in a vortex in an inviscid neutral fluid.

Figure 6.4: Possible MHD evolution of a set of field lines in a sheared slab magnetic field model: a) initial sheared magnetic field equilibrium, b) sinusoidal perturbation in ideal MHD ($\eta = 0$), and c) resistive MHD ($\eta \neq 0$) with magnetic field reconnection into magnetic island structures.

elements are tied together, cannot break up or tear, and cannot interchange positions relative to each other. Thus, as illustrated in Figure 6.4, in the ideal MHD model the topology of magnetic field lines and flux surfaces is conserved — nested magnetic flux surfaces remain forever nested (even though their shape may become highly distorted), and plasma in regions "inside" (or "outside") a given magnetic flux surface remain inside (outside) forever. The inclusion of resistivity in the MHD model allows diffusion of the magnetic field relative to the plasma, and hence reconnection of the magnetic field lines and changes in the magnetic topology — for example by forming a magnetic island such as indicated in Figure 6.4c. In section 6.7 we discuss the relative importance of resistivity in MHD analyses of plasmas.

The most convenient form of the MHD momentum equation (6.27) for dynamical analyses uses the middle form of the force density $\mathbf{f}_B$ in (6.46) and is given by

$$\rho_m \frac{d\mathbf{V}}{dt} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B}}{\mu_0}. \tag{6.74}$$

Note that we have now reduced the full MHD equation set (6.25)–(6.44) to just three (or seven component) equations — the scalar pressure equation in (6.67), the vector magnetic field evolution equation in (6.71) and this last vector momentum equation (6.76). These equations are usually all we need to describe the linear and nonlinear dynamics of plasmas in the MHD model. [The mass density equation (6.65) is only needed when the equilibrium mass density is inhomogeneneous.] Note that for these MHD dynamical model equations the charge continuity equation $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$ is automatically satisfied by our having used Ampere's law to replace the current $\mathbf{J}$ with $\boldsymbol{\nabla}\times\mathbf{B}/\mu_0$, which is divergence free. Also, the electric field $\mathbf{E}$ does not appear because it was replaced by $-\mathbf{V}\times\mathbf{B} + \eta\mathbf{J}$ using the MHD Ohm's law.

6.6 Alfvén Waves

To illustrate the fundamental wave responses of plasmas in the MHD model (Alfvén waves — named after their discoverer), we consider plasma responses to small perturbations in the simplest possible plasma and magnetic field model. Namely, for the equilibrium we consider a uniform, nonflowing ($\mathbf{V}_0 = 0$) plasma in an infinite, homogeneous magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_z = B_0\hat{\mathbf{b}}$. This model trivially satisfies the MHD equilibrium force balance condition (6.48) since $\mu_0 \mathbf{J}_0 = \boldsymbol{\nabla}\times\mathbf{B}_0 = \mathbf{0}$ and $\boldsymbol{\nabla} P = \mathbf{0}$ because both the equilibrium magnetic field $\mathbf{B}_0$ and pressure P_0 are uniform in space. For perturbed responses we assume

$$\rho_m = \rho_{m0} + \tilde{\rho}_m, \quad P = P_0 + \tilde{P}, \quad \mathbf{V} = \tilde{\mathbf{V}}, \quad \mathbf{B} = \mathbf{B}_0 + \tilde{\mathbf{B}}, \tag{6.75}$$

in which the zero subscript indicates equilibrium quantities and the tilde over quantities indicates perturbed variables. Decomposing the perturbed magnetic field into its parallel $[\tilde{\mathbf{B}}_\parallel = \hat{\mathbf{b}}(\hat{\mathbf{b}}\boldsymbol{\cdot}\tilde{\mathbf{B}}) = \tilde{B}_\parallel \hat{\mathbf{b}}]$ and perpendicular $[\tilde{\mathbf{B}}_\perp \equiv -\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\tilde{\mathbf{B}})]$ components, we find the square of the magnetic field strength $\mathbf{B}$ is

$$B^2 \equiv (\mathbf{B}_0 + \tilde{\mathbf{B}})\boldsymbol{\cdot}(\mathbf{B}_0 + \tilde{\mathbf{B}}) = B_0^2 + 2B_0\tilde{B}_\parallel + \tilde{B}_\parallel^2 + |\tilde{\mathbf{B}}_\perp|^2 \simeq B_0^2 + 2B_0\tilde{B}_\parallel. \tag{6.76}$$

We will use the last expression, which is the linearized form (i.e., it neglects terms that are second order in the perturbation amplitudes).

Substituting the equilibrium plus perturbed quantities in (6.75) and (6.76) into the ideal MHD equations for the evolution of the pressure (6.67), flow velocity (6.74) and magnetic field [(6.71) with $\eta \to 0$] and linearizing (neglect second and higher order terms in the perturbation amplitudes), we obtain

$$\frac{\partial \tilde{P}}{\partial t} = -\Gamma P_0 \boldsymbol{\nabla}\boldsymbol{\cdot}\tilde{\mathbf{V}}, \tag{6.77}$$

$$\rho_{m0}\frac{\partial \tilde{\mathbf{V}}}{\partial t} = -\boldsymbol{\nabla}\left(\tilde{P} + \frac{B_0\tilde{B}_\parallel}{\mu_0}\right) + \frac{1}{\mu_0}(\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla})\tilde{\mathbf{B}}, \tag{6.78}$$

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla}\times(\tilde{\mathbf{V}}\times\mathbf{B}_0) \;=\; -\mathbf{B}_0(\boldsymbol{\nabla}\boldsymbol{\cdot}\tilde{\mathbf{V}}) + (\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla})\tilde{\mathbf{V}}. \tag{6.79}$$

In the last equation we used vector identity (**??**) and set to zero terms involving gradients of the homogeneous equilibrium magnetic field $\mathbf{B}_0$. Equations for the parallel and perpendicular components of the magnetic field are obtained from the corresponding projections of the magnetic field evolution equation:

$$\partial\tilde{B}_\parallel/\partial t = -B_0(\boldsymbol{\nabla}\boldsymbol{\cdot}\tilde{\mathbf{V}}) + (\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla})\tilde{V}_\parallel, \tag{6.80}$$

$$\partial\tilde{\mathbf{B}}_\perp/\partial t = (\mathbf{B}_0\boldsymbol{\cdot}\boldsymbol{\nabla})\tilde{\mathbf{V}}_\perp. \tag{6.81}$$

These equations can be combined into a single (vector) equation by taking the partial derivative of the perturbed momentum equation (6.78) and substituting in the needed partial derivatives from the other equations (see Problem 6.20):

$$\frac{\partial^2\tilde{\mathbf{V}}}{\partial t^2} = (c_S^2 + c_A^2)\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\cdot}\tilde{\mathbf{V}}) + c_A^2[\nabla_\parallel^2\tilde{\mathbf{V}}_\perp - \boldsymbol{\nabla}_\perp\nabla_\parallel\tilde{V}_\parallel - \hat{\mathbf{b}}\nabla_\parallel(\boldsymbol{\nabla}\boldsymbol{\cdot}\tilde{\mathbf{V}})] \tag{6.82}$$

in which

$$c_A \equiv \frac{B_0}{\sqrt{\mu_0\rho_{m0}}} \simeq 2.2\times10^{16}\frac{B_0}{\sqrt{n_i A_i}}\ \text{m/s}, \quad \text{Alfvén speed.} \tag{6.83}$$

Here, $A_i \equiv m_i/m_p$ is the atomic mass value of the ions, the perpendicular ($\perp$) and parallel ($\|$) subscripts indicate the respective components of the quantities as defined in (**??**)–(**??**). The magnitude of the Alfvén speed can be appreciated by noting its relationship to the sound speed defined in (6.68):

$$\frac{c_S^2}{c_A^2} = \frac{\Gamma P_0/\rho_{m0}}{B^2/\mu_0\rho_{m0}} = \frac{\Gamma}{2}\beta. \tag{6.84}$$

Thus, for $\beta < 1$ the Alfvén speed is a factor of about $1/\sqrt{\beta}$ greater than the MHD sound speed.

While (6.82) clearly has a wavelike structure, it is a quite complicated and anisotropic wave equation. We consider here only some special cases to illustrate the basic waves involved. (Section 7.6* provides a comprehensive analysis.)

First, consider waves propagating purely perpendicular to the magnetic field by setting $\nabla_\| = 0$. Then, taking the divergence of (6.82) we obtain

$$\left[\frac{\partial^2}{\partial t^2} - (c_A^2 + c_S^2)\,\nabla_\perp^2\right](\boldsymbol{\nabla}_\perp\boldsymbol{\cdot}\tilde{\mathbf{V}}_\perp) = 0 \quad\Longrightarrow\quad \omega^2 = k_\perp^2(c_A^2 + c_S^2),$$

$$\text{compressional Alfvén waves.} \tag{6.85}$$

This wave equation describes "fast" compressional Alfvén waves. In the last form we assumed a wave-like response $\tilde{\mathbf{V}}_\perp \sim \exp[i(\mathbf{k}\boldsymbol{\cdot}\mathbf{x}-\omega t)]$ to obtain the wave dispersion relation. Compressional Alfvén waves propagate perpendicular to the magnetic field with a wave phase speed given by $V_{\varphi\perp} \equiv \omega/k_\perp = \sqrt{c_A^2 + c_S^2}$, which is the fastest MHD wave phase speed. These waves propagate by perpendicular flow compression ($\boldsymbol{\nabla}_\perp\boldsymbol{\cdot}\tilde{\mathbf{V}}_\perp \neq 0$) and also involve magnetic field compression [$\tilde{B}_\| \neq 0$ — see (6.80)] and pressure perturbations [$\tilde{P} \neq 0$ — see (6.85)]. Adding the pressure perturbation (6.77) and B_0/μ_0 times the magnetic perturbation (6.80) with $\nabla_\| = 0$, one can show that

$$\frac{\partial^2}{\partial t^2}\left(\tilde{P} + \frac{B_0\tilde{B}_\|}{\mu_0}\right) = -(c_A^2 + c_S^2)\,\boldsymbol{\nabla}_\perp\boldsymbol{\cdot}\,\rho_{m0}\frac{\partial\tilde{\mathbf{V}}_\perp}{\partial t} = (c_A^2 + c_S^2)\,\nabla_\perp^2\left(\tilde{P} + \frac{B_0\tilde{B}_\|}{\mu_0}\right) \tag{6.86}$$

in which for the last form we have used (6.78) with $\nabla_\| = 0$. Thus, the compressibility in the perpendicular flow also causes the sum of the perturbed pressure and magnetic field energy density to satisfy a compressional Alfvén wave equation. Physically, as can be noted from the importance of the perpendicular component of (6.78) in these waves, the compressional Alfvén waves are the responses of the plasma to imbalances in the perpendicular (to $\mathbf{B}$) force balance in the plasma. Thus, on "equilibrium" time scales (after these wave responses have propagated away), MHD plasma responses will be in radial force balance equilibrium and not have any driving sources for compressional Alfvén waves:

$$\mathbf{J}_0\times\mathbf{B}_0 = \boldsymbol{\nabla}_\perp P_0, \quad \boldsymbol{\nabla}_\perp\boldsymbol{\cdot}\tilde{\mathbf{V}}_\perp = 0, \quad \tilde{P} + B_0\tilde{B}_\|/\mu_0 = 0. \tag{6.87}$$

These are the lowest order conditions for equilibria and perturbations in an MHD plasma (even in inhomogeneous magnetic fields — see Chapter 21); they

Figure 6.5: Perturbations $(\tilde{\mathbf{B}}, \tilde{\mathbf{V}}, \tilde{P})$ in the three fundamental types of MHD waves: a) compressional Alfvén, b) shear Alfvén, and c) sound.

obtain for time scales long compared to the fast compressional Alfvén wave period: $t >> 1/k_\perp\sqrt{c_A^2 + c_S^2}$.

Next, consider incompressible ($\boldsymbol{\nabla\cdot}\tilde{\mathbf{V}} = 0$) MHD waves propagating purely along the magnetic field ($\boldsymbol{\nabla}_\perp = \mathbf{0}$). Then, the perpendicular component of the general MHD wave equation (6.82) becomes

$$\left(\frac{\partial^2}{\partial t^2} - c_A^2 \nabla_\parallel^2\right)\tilde{\mathbf{V}}_\perp = \mathbf{0} \quad \Longrightarrow \quad \omega^2 = k_\parallel^2 c_A^2, \quad \text{shear Alfvén waves.} \qquad (6.88)$$

These are called "slow" Alfvén waves because their (parallel) phase speed $V_{\varphi\parallel} \equiv \omega/k_\parallel = c_A$ is less than the phase speed for the compressional Alfvén waves. They are called shear (or torsional) Alfvén waves because their $\tilde{\mathbf{V}}_\perp$ induces a perpendicular magnetic field perturbation $\tilde{\mathbf{B}}_\perp$ that shears or twists the magnetic field — see (6.81). In the MHD model, instabilities often arise that indirectly excite shear Alfvén waves; such instabilities must have exponential growth rates $\mathcal{I}m\{\omega\} > k_\parallel c_A$ so they are not be stabilized by the energy required to excite these shear Alfvén waves.

Finally, consider compressible waves in the parallel flow ($\tilde{\mathbf{V}} = \tilde{V}_\parallel \hat{\mathbf{b}}$) propagating along the magnetic field ($\boldsymbol{\nabla}_\perp = \mathbf{0}$). Then, the parallel component of the general MHD wave equation (6.82) becomes

$$\left(\frac{\partial^2}{\partial t^2} - c_S^2 \nabla_\parallel^2\right)\tilde{V}_\parallel = 0 \quad \Longrightarrow \quad \omega^2 = k_\parallel^2 c_S^2, \quad \text{parallel sound waves.} \qquad (6.89)$$

These are neutral-fluid-type sound waves (see A.6) that propagate along the magnetic field by parallel compression of the flow ($\nabla_\parallel \tilde{V}_\parallel \neq 0$). They are electrostatic waves since, as can be seen from (6.80) and (6.81), they produce no magnetic perturbations (i.e., $\tilde{\mathbf{B}} = \mathbf{0}$ for these waves). MHD instabilities often indirectly excite parallel sound waves; such instabilities must have exponential growth rates $\mathcal{I}m\{\omega\} > k_\parallel c_S$ so they are not be stabilized by the energy required to excite the sound waves.

The properties of the perturbations in these three fundamental types of MHD waves are illustrated in Fig. 6.5. As shown in Fig. 6.5a, (fast) compressional Alfvén waves have: oscillatory parallel magnetic field perturbations $\tilde{B}_\parallel$ that increase or decrease the local magnetic field strength (density of field lines), compressible pependicular flows, and corresponding oscillatory pressure perturbations, all in the direction perpendicular to the equilibrium magnetic field direction $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_z$, which is horizontal in the figure. In contrast, the (slow) shear Alfvén waves (Fig. 6.5b) have: oscillatory perpendicular magnetic fields $\tilde{\mathbf{B}}_\perp$ and oscillatory perpendicular flows $\tilde{\mathbf{V}}_\perp$ along the magnetic field, but no pressure perturbation (because these perurbed flows are incompressible). Finally, as shown in Fig. 6.5c, the parallel sound waves have: no magnetic field perturbation (because they are electrostatic), an oscillatory compressible parallel flow $\tilde{V}_\|$ and corresponding pressure $\tilde{P}$ perturbations along the magnetic field direction.

In the more general case of propagation of MHD waves at arbitrary angles to the magnetic field direction, these three types of waves become coupled (see Section 7.6). These waves also become coupled in inhomogeneous magnetic fields — because the parallel and perpendicular directions vary spatially. Nonetheless, the basic wave characteristics we have discussed are usually still evident in these more complicated situations.

6.7 Magnetic Field Diffusion in MHD

In order to examine the effect of electrical resistivity on a plasma in the MHD model, consider first the evolution of the magnetic field in (6.71) without the advection term:

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}, \quad \text{magnetic field diffusion equation.} \tag{6.90}$$

This equation describes the diffusion (see Section A.5) of the magnetic field (both its magnitude and directional components) that is caused by the electrical resistivity of a plasma. The diffusion coefficient is

$$D_\eta = \frac{\eta}{\mu_0} = \frac{m_e \nu_e}{\mu_0 n_e e^2} \simeq 1.4 \times 10^3 \left(\frac{Z_i}{T_e(\text{eV})]^{3/2}} \right) \left(\frac{\ln \Lambda}{17} \right) \ \text{m}^2/\text{s} \quad \text{magnetic field diffusivity.} \tag{6.91}$$

Phenomenologically, since we can write $D_\eta = \nu_e (c/\omega_{pe})^2$, magnetic field diffusion can be thought of [via $D \sim (\Delta x)^2/\Delta t$ — see (**??**)] as emanating from a random walk process in which magnetic field lines step a collisionless skin depth ($\Delta x \sim c/\omega_{pe}$) in an electron collision time ($\Delta t \sim 1/\nu_e$). The relative magnitude of the magnetic field diffusivity can be ascertained from its relationship to the classical diffusivity $D_\perp$ defined in (**??**):

$$\frac{D_\perp}{\eta/\mu_0} = \frac{\nu_e \varrho_e^2}{\nu_e (c/\omega_{pe})^2} \left(\frac{T_e + T_i}{2T_e} \right) = \frac{n_e(T_e + T_i)}{c^2 \epsilon_0 B^2} = \frac{\beta}{2}. \tag{6.92}$$

Thus, for a plasma with $\beta < 1$ particles diffuse classically across magnetic field lines slower than the magnetic field lines themselves diffuse relative to the plasma! However, in most plasmas of interest microscopic turbulence in plasmas causes an anomalous perpendicular transport that is rapid compared to the magnetic field diffusion; hence one can usually consider the magnetic field to be stationary for calculations of anomalous transport.

To illustrate the spatial and temporal scale lengths involved in magnetic diffision, consider the distance an electromagnetic wave can penetrate (see Section 1.5) into a resistive medium in which the magnetic field behavior is governed by (6.90). For wavelike perturbations $\tilde{\mathbf{B}} \sim \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$, the diffusion equation becomes

$$-i\omega\tilde{\mathbf{B}} = -k^2(\eta/\mu_0)\tilde{\mathbf{B}} \quad \Longrightarrow \quad k = \sqrt{i\omega\mu_0/\eta} = (1+i)\sqrt{(\omega/2)(\mu_0/\eta)}. \quad (6.93)$$

To use the analysis of Section 1.5, we identify this complex wavenumber k as the transmitted wavenumber k_T in (**??**). Thus, an electromagnetic wave will be dissipated and damped exponentially, as it oscillates spatially (due to $\mathcal{R}e\{k_T\}$) and propagates into a resistive medium, with a characteristic decay length of

$$\delta_\eta \equiv \frac{1}{\mathcal{I}m\{(k_T\}} = \sqrt{\frac{2}{\omega}\frac{\eta}{\mu_0}}, \quad \text{resistive skin depth.} \quad (6.94)$$

It is called a "skin" depth because of its analogy with the problem of determining how far an oscillating magnetic field (e.g., due to 60 Hz AC electricity) penetrates into a cylindrical wire of finite radius. This skin depth formula is appropriate for radian frequencies $\omega < \nu_e$, while the collisionless skin depth formula (**??**) is appropriate for higher frequencies — see Problem 6.25. For $T_e =$ 2000 eV, which gives $\eta/\mu_0 \simeq 0.016$ m^2/s (close to the resistivity of copper at room temperature of $\eta/\mu_0 \simeq 0.135$ m^2/s), the resistive skin depth ranges from 0.07 mm for $f = \omega/2\pi = 10^4$ Hz ($\omega = 2\pi \times 10^4$) to about 1 cm for 60 Hz.

Another way of illustrating the temporal behavior of magnetic field diffusion in a magnetized plasma is to ask: on what time scale τ will a magnetic field component diffuse away from being localized to a region of width $L_\perp$? Because for diffusive processes the diffusion coefficient scales with spatial and temporal steps as $D \sim (\Delta x)^2/\Delta t \sim L_\perp^2/\tau$ (see Appendix A.5), we can estimate phenomenologically that $\tau \sim L_\perp^2/(\eta/\mu_0)$. One often considers a cylindrical model consisting of a column of magnetized plasma with radius a that initially carries an axial current. For such a cylindrical model the resistivity-induced decay time of the current (and induced azimuthal magnetic field) is (see Section A.5)

$$\tau_\eta \simeq \frac{a^2}{6\,\eta/\mu_0}, \quad \text{resistive skin diffusion time.} \quad (6.95)$$

Here, the numerical factor of 6 is a cylindrical geometry factor which more precisely is the square of the first zero of the J_0 Bessel function: $j_{0,0}^2 \simeq 2.405^2 \simeq 5.78$ — see Appendix A.5 and (**??**). However, the additional accuracy is unwarranted both because of the approximations involved in the simple model used to derive τ_η and because of the intrinsic accuracy of the electrical resistivity ($\simeq 1/\ln\Lambda \sim$ 5–10%). For a plasma of radius $a = 0.3$ m with $T_e = 2000$ eV, which gives $\eta/\mu_0 \simeq 0.016$ m^2/s, the skin time is $\tau_\eta \sim 1$ s.

Finally, we discuss the relative importance of the two contributions to magnetic field evolution (6.71) in the MHD model: advection of the magnetic field by $\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B})$, and resistive diffusion by $(\eta/\mu_0)\nabla^2\mathbf{B}$. The relative importance of these two terms is indicated by the scaling properties of their ratio:

$$S = \frac{|\nabla\times(\mathbf{V}\times\mathbf{B})|}{|(\eta/\mu_0)\nabla^2\mathbf{B}|} \sim \frac{c_A/L_\parallel}{(\eta/\mu_0)/a^2} \simeq 1.6\times 10^{13}\,\frac{a^2 B[T_e(\mathrm{eV})]^{3/2}}{L_\parallel Z_i\sqrt{n_i A_i}}\left(\frac{17}{\ln\Lambda}\right), \quad \text{Lundquist number.}^4 \qquad (6.96)$$

Here, we have taken the typical velocity to be the Alfvén speed c_A and assumed scale lengths $L_\parallel$ [e.g., periodicity scale length along $\mathbf{B}$ — see (6.81)] for the advection process and a (e.g., plasma radius) for the magnetic diffusion. Typical Lundquist numbers range from 10^2 for cold, resistive plasmas, to 10^5–10^{10} for the earth's magnetosphere and magnetic fusion experiments, to 10^{10}–10^{14} for the sun's corona and astrophysical plasmas.

Because the Lundquist number is large for almost all magnetized plasmas of interest (and extremely large for high temperature plasmas), one might be tempted to just set the resistivity to zero ($S\to\infty$) and always use the ideal MHD model. Indeed, throughout most of a plasma the magnetic field is frozen into and moves with the plasma fluid. However, a small resistivity can be very important in resistive boundary layers. The boundary layers occur in the vicinity of magnetic field lines where the parallel derivative $(\mathbf{B}_0\cdot\nabla)\tilde{\mathbf{V}}_\perp$ in (6.81) vanishes so the $\tilde{\mathbf{B}}_\perp$ evolution becomes dominated by resistive evolution of $\tilde{\mathbf{B}}_\perp$, rather than by advection. The width of these resistive boundary layers scales inversely with a fractional power of the Lundquist number ($S^{-1/3}$ or $S^{-2/5}$) and hence is not negligible — see Chapter 22. Since resistivity allows the magnetic field lines to slip relative to the plasma fluid, they relax (in the resistive layers) the frozen flux constraint and thereby allow new types of instabilities — resistive MHD instabilities, which are described in Chapter 22. Since the resistivity only relaxes the frozen flux constraint in thin layers, resistive MHD instabilities grow much slower (by factors of $S^{-1/3}$ or $S^{-3/5}$) than ideal MHD instabilities. However, resistive MHD instabilities are quite important, because they can lead to turbulent plasma transport (see Section 25.3) and because in these narrow resistive boundary layers the magnetic field lines can tear or reconnect and thereby lead to changes in the magnetic topology (see Section 22.3). For example, they can nonlinearly evolve into a magnetic island structure like that shown in Fig. 6.4c.

6.8 Which Plasma Description To Use When?

In this section we discuss which types of plasma descriptions are used for describing various types of plasma processes in magnetized plasmas. This discussion also serves as an introduction to most of the subjects that will be covered in the remainder of the book. The basic logic is that the fastest, finest scale processes

[4] Many plasma physics textbooks refer to this as the "magnetic Reynolds number." However, S is the ratio of *linear* advection to a dissipative process rather than the ratio of *nonlinear* advection to a dissipative process, as the neutral fluid Reynolds number is — see (**??**). We will call S the Lundquist number to avoid the implication that this dimensionless number is indicative of nonlinear processes that always lead to turbulence when it is large.

require kinetic descriptions, but then over longer time and length scales more fluidlike, macroscopic models become appropriate. Also, the "equilibrium" of the faster time scale processes often provide constraint conditions for the longer time scale, more macroscopic processes.

In a magnetized plasma there are many more relevant parameters, and their relative magnitudes and consequences can vary from one application to another. Thus, to provide a table similar to Table **??** for magnetized plasmas, we need to specify the parameters for a particular application. We will choose parameters toward the edge ($r/a = 0.7$) of a typical 1990s "large-scale" tokamak plasma (e.g., the Tokamak Fusion Test Reactor: TFTR): $T_e = T_i = 1$ keV, $n_e = 3 \times 10^{19}$ m^{-3}, $B = 4$ T, deuterium ions, $Z_{\rm eff} = 2$, $L_\parallel = R_0 q \simeq 6$ m, $a = 0.8$ m, $L_p = 0.5$ m. In a magnetized plasma the unmagnetized phenomena listed in Table **??** still occur; however, their effects only influence responses along the magnetic field direction. Parameters for the gyromotion, bounce motion and drift motion of charged particles in this tokamak magnetic field structure are approximately the same as those indicated in (**??**) and (**??**).

Table 6.1 presents an outline of magnetized-plasma-specific plasma phenomena, and their relevant time scales, appropriate models and possible consequences for the tokamak plasma parameters indicated in the preceding paragraph. In it time scales are indicated in "half order of magnitudes" ($10^{0.5} = 3.16\cdots \sim 3$). As indicated, the fastest magnetic-specific process in magnetized plasmas is the gyromotion of particles about the magnetic field, for which the appropriate model is the Vlasov equation. The ion gyromotion leads to cyclotron (Bernstein) waves, finite ion gyroradius (FLR) effects and a perpendicular dielectric response (Sections 7.5, 7.6). There are of course also electron cyclotron motion and waves. The propagation of (electron and ion) cyclotron-type waves in plasmas and their use for wave heating of magnetized plasmas are discussed in Chapters 9 and 10. If the electron or ion distribution function is peaked at a nonzero energy (so $\partial f_0/\partial\varepsilon > 0$), it can lead to cyclotron instabilities (Chapter 18) whose nonlinear evolution to a steady state or bursting situation is often determined by collisions (Section 24.1).

The next fastest time scales are typically those associated with the the Alfvén wave and sound wave frequencies which are described by the ideal MHD model: (6.25)–(6.39) with $\eta \to 0$. As indicated in Table 6.1, in the usual situation where compressional Alfvén waves are stable, their effect is to impose radial ($\perp$ to $\mathbf{B}$) force balance equilbrium [(6.48) and Chapter 20] on the plasma and lower frequency perturbations in the plasma. The shear Alfvén and sound waves can lead to virulent macroscopic current-driven (kink) and pressure-gradient-driven (interchange) instabilities (Chapter 21). The nonlinear consequences of an ideal MHD instability is often dramatic movement or catastrophic loss of the plasma in a few to ten instability growth times; hence most magnetic confinement systems are designed to provide ideal MHD stability for the plasmas placed in them.

Next, we turn to the sequentially slower particle and plasma motions along ($\parallel$), across ($\wedge$) and perpendicular ($\perp$) to the magnetic field $\mathbf{B}$. The fastest motion along a magnetic field line is the electron bounce motion, which is de-

Table 6.1: Phenomena, Models For A Magnetized Plasma

Physical Process	Time Scales	Species, Plasma Model	Consequences
cyclotron waves	$1/\omega_{ci} \sim 10^{-8}$ s	Vlasov	dielectric resp.
cyclotron inst.	$1/\omega_{ci} \sim 10^{-8}$ s	Vlasov	NL, via collisions
Alfvén waves		ideal MHD	
compressional	$a/c_A \sim 10^{-7}$ s		$\boldsymbol{\nabla} P = \mathbf{J}\times\mathbf{B}$
shear	$L_\parallel/c_A \sim 10^{-6}$ s		$\mathbf{J}$-driven inst.
sound waves	$a/c_S \sim 10^{-5.5}$ s	ideal MHD	$\boldsymbol{\nabla} P$-driven inst.
parallel ($\parallel$) to $\mathbf{B}$		parallel kinetic	
electron bounce	$1/\omega_{be} \sim 10^{-6.5}$ s	$\parallel$ Vlasov	n_e, T_e const. $\parallel \mathbf{B}$
electron collisions	$1/\nu_e \sim 10^{-5}$ s	drift kinetic	η, $\mathbf{q}_{\parallel e}$, $\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{\parallel e}$
ion bounce	$1/\omega_{bi} \sim 10^{-4.5}$ s	$\parallel$ Vlasov	n_i, T_i const. $\parallel \mathbf{B}$
Ohm's law in MHD	$> 1/\nu_e \sim 10^{-5}$ s	resistive MHD	resistive inst.
cross ($\wedge$) to $\mathbf{B}$			
diamagnetic flow	$1/\omega_* \lesssim 10^{-5}$ s	gyrokinetic	drift wave inst.
cross flow equil.	$1/\nu_i \sim 10^{-3}$ s	drift kinetic	cross flow damp.
perp. ($\perp$) to $\mathbf{B}$			
plasma transport	$\tau_E \sim a^2/4\chi_\perp$	two-fluid	loss of plasma
$\mathbf{B}$ field evolution	$\tau_\eta \sim a^2\mu_0/6\eta$	res./neo. MHD	$\mathbf{B}$ field diffusion, magnetic islands

scribed by a parallel motion version of the Vlasov equation [the drift kinetic equation (**??**) without the collision operator and drift velocity $\mathbf{v}_D$]. On time scales longer than the electron bounce time ($1/\omega_{be}$), the lowest order distribution function becomes constant along field lines ($\nabla_\parallel f_{0e} = 0$ and hence density and temperature become constant along $\mathbf{B}$), and distinctions between trapped and untrapped electrons and their differing particle orbits become evident. For the parameters chosen, we have an electron collision length $\lambda_e = v_{Te}/\nu_e \simeq 200$ m $\sim 33L_\parallel$ and hence $\lambda_e\nabla_\parallel \sim 33 >> 1$. This is a typical toroidal plasma which is often (confusingly) called "collisionless" — because the collision length is long compared to the parallel periodicity length. Since the electron gyroradius is negligibly small, the collisional evolution of the electron species on the collision time scale ($1/\nu_e$) is governed by the (electron) drift kinetic equation (**??**). Its solution for axisymmetric toroidal plasmas is discussed in Section 16.2*. For times long compared to the electron collision time the plasma acquires its electrical resistivity η and the collisions of untrapped electrons produce entropy through "neoclassical" heat conduction ($\mathbf{q}_{\|e}$) and parallel viscosity ($\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{\|e}$) — see Chapter 16. Similarly, the lowest order ion distribution function, density and temperature become constant along magnetic field lines for time scales longer than the ion bounce time ($1/\omega_{bi}$) and their collisional effects (in relaxing cross flows within a magnetic flux surface) become evident on the ion collision time scale ($1/\nu_i$).

The plasma exhibits an electrical resistivity for time scales longer than the electron collision time ($1/\nu_e$). Its introduction into MHD leads to the resistive MHD model: (6.25)–(6.39). Since the introduction of resistivity relaxes the ideal MHD frozen flux constraint (in narrow layers), it can lead to resistive MHD instabilities related to their ideal MHD counterparts (kink $\rightarrow$ tearing, $\boldsymbol{\nabla}P$-driven "interchange" $\rightarrow$ resistive interchange), which, however, grow more slowly and hence are less virulent — see Chapter 22.

The next set of phenomena concern the effects of particle drifts and plasma species flows in the cross direction ($\wedge$ — perpendicular to $\mathbf{B}$ and within a flux surface if it exists). On this time scale a global (as opposed to local) description of the magnetic field is usually required. The diamagnetic flows of electrons and ions lead to drift-wave-type oscillations (Sections 7.4* and 8.6*) and instabilities (Section 23.3*). Since these "universal" instabilities involve modes with significant ion gyroradius (FLR) effects ($\varrho_i\boldsymbol{\nabla}_\perp \sim \mathbf{k}_\perp\varrho_i \sim 1$), the gyrokinetic equation is used to describe their nonlinear evolution into microsopic plasma turbulence (Chapter 25) that leads to anomalous radial transport (Chapter 26) of the plasma. On the same time scales the combination of the $\mathbf{E}\times\mathbf{B}$ and diamagnetic flows come into "equilibrium" (a steady state saturation or bounded cyclic behavior); flow components within a magnetic flux surface in directions in which the magnetic field is inhomogeneous (e.g., the poloidal direction in an axisymmetric tokamak) are damped on the ion collision time scale ($1/\nu_i$) — see Section 16.3*. Steady-state net radial transport fluxes can only be properly calculated after the flows within magnetic flux surfaces are determined and relaxed to their equilibium values. Also, in determining transport fluxes it is implicitly assumed that nested magnetic flux surfaces exist and that "radial" transport is to be calculated relative to them.

Finally, we reach the transport time scales on which the plasma and magnetic field diffuse radially out of the plasma confinement region, and radiation (Chapter 14) can be significant. Plasma transport (relative to the magnetic field) is usually modeled with two-fluid equations averaged over magnetic flux surfaces to yield equations that govern the transport of plasmas perpendicular to magnetic flux surfaces — see Chapter 17. However, the radial particle and heat diffusion coefficients $D_\perp$, $\chi_\perp$ are usually assumed to be the sum of those produced by anomalous transport (Chapter 26) and those due to classical [(**??**), (**??**) and Chapter 15] and neoclassical (Chapter 16) transport processes. For a cylindrical-type plasma model the characteristic time scale for the usally dominant plasma energy loss is (see Section 17.3) approximately $\tau_E \sim a^2/4\chi_\perp$ in which a is the plasma radius; for the plasma parameters we are considering it is of order 0.1 s. Simultaneously, the magnetic field is diffusing. The characteristic time scale for diffusive transport of magnetic field lines out of a cylindrical plasma is $\tau_\eta \sim a^2/(6\eta/\mu_0)$; for the plasma parameters we are considering it is of order 1 s. If resistive or neoclassical MHD tearing-type instabilities are present, they can reconnect magnetic field lines on rational magnetic flux surfaces and evolve nonlinearly by forming magnetic islands which grow (to saturation or total plasma loss) on a fraction (~ 0.1) of the magnetic field diffusion time scale τ_η. Since the magnetic field typically diffuses more slowly than energy is lost via anomalous transport (i.e., $\tau_\eta >> \tau_E$ or $\eta/\mu_0 << \chi_\perp$), it is usually reasonable to assume that the magnetic field is stationary and the plasma moves relative to it via Coulomb-collision-induced or anomalous plasma transport processes.

REFERENCES AND SUGGESTED READING

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Krall and Trivelpiece, *Principles of Plasma Physics* (1973), Chapts. 2.3 [?]

Freidberg, *Ideal Magnetoydrodynamics* (1987) [?]

Sturrock, *Plasma Physics, An Introduction to the Theory of Astrophysical, Geophysical & Laboratory Plasmas* (1994), Chapts. 11,12 [?]

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Bateman, *MHD Instabilities* (1980) [?]

Biskamp, *Nonlinear Magnetohydodynamics* (1993) [?]

The neoclassical MHD model for axisymmetric toroidal plasmas is described in

J.D. Callen, W.X. Qu, K.D. Siebert, B.A. Carreras, K.C. Shaing and D.A. Spong, "Neoclassical MHD Equations, Instabilities and Transport in Tokamaks," in *Plasma Physics and Controlled Nuclear Fusion Research 1986* (IAEA, Vienna, 1987), Vol. II, p. 157 [?]

PROBLEMS

6.1 Use the definition of the pressure in (**??**) with $\mathbf{v}_r \equiv \mathbf{v} - \mathbf{V}$ to show that the isotropic pressure of a species in the center-of-mass frame ($\mathbf{V}$) of an MHD plasma is

$$p_s^{\mathsf{CM}} = p_s + (n_s m_s/3)|\mathbf{V}_s - \mathbf{V}|^2. \ /*$$

6.2 Show that the plasma momentum equation (6.6) obtained by adding the electron and ion momentum equations is exact (i.e., it does not involve an $m_e/m_i << 1$ approximation). [Hint: To obtain the inertia term on the left it is easiest to use (**??**) for the electron and ion momentum equations. Also, first show that

$$\sum_s m_s n_s \mathbf{V}_s \mathbf{V}_s = \rho_m \mathbf{V}\mathbf{V} + \sum_s m_s n_s (\mathbf{V}_s - \mathbf{V})(\mathbf{V}_s - \mathbf{V})$$

in which $\mathbf{V}$ is the MHD mass flow velocity defined in (6.19).] //*

6.3 Evaluate $\boldsymbol{\nabla}\cdot\mathbf{J}_D$ and show that it is equal to $\boldsymbol{\nabla}\cdot\mathbf{J}_*$, and to the terms on the right of (6.16). Explain the physical significance of the equality of these two quantities. //*

6.4 Multiply the electron and ion momentum balance equations (**??**) by q_e/m_e and q_i/m_i and add them to obtain the exact generalized Ohm's law

$$\frac{\partial \mathbf{J}}{\partial t} + \boldsymbol{\nabla}\cdot(\mathbf{JV} + \mathbf{VJ} - \rho_q \mathbf{VV}) = \epsilon_0 \omega_p^2 \left\{ \mathbf{E} + \mathbf{V}\times\mathbf{B} - \left(\frac{\mathbf{J}_\parallel}{\sigma_\parallel} + \frac{\mathbf{J}_\perp}{\sigma_\perp} \right) \right.$$
$$\left. - \frac{(1 - Z_i m_e/m_i)\mathbf{J}\times\mathbf{B} - \boldsymbol{\nabla}\cdot[\mathsf{P}_e^{\mathsf{CM}} - Z_i(m_e/m_i)\mathsf{P}_i^{\mathsf{CM}}]}{(1 + Z_i m_e/m_i)n_e e} \right\}$$

in which $\mathsf{P}_s^{\mathsf{CM}} \equiv p_s \mathsf{I} + \boldsymbol{\pi}_s + n_s m_s(\mathbf{V}_s - \mathbf{V})(\mathbf{V}_s - \mathbf{V})$ is the pressure tensor of a species in the center-of-mass frame ($\mathbf{V}$) of the plasma. Show that this result simplifies to (6.17) for $m_e/m_i << 1$ and strongly subsonic relative species flows ($|\mathbf{V}_s - \mathbf{V}|/v_{Ts} << 1$). [Hint: Use $n_e/n_i = q_i/e = Z_i$ for this two species plasma and

$$\mathbf{V}_e = \mathbf{V} - \frac{m_i n_i(\mathbf{J}/n_e e)}{m_e n_e + m_i n_i}, \quad \mathbf{V}_i = \mathbf{V} + \frac{m_e n_e(\mathbf{J}/n_e e)}{m_e n_e + m_i n_i}.] \; ///*$$

6.5 Show that the electron inertia term is negligible compared to the electric field in the parallel generalized Ohm's law for $kc/\omega_{pe} << 1$. [Hint: Use the parallel component of the nonrelativistic Ampere's law: $\nabla^2 A_\parallel = -\mu_0 J_\parallel$ from (**??**).] //*

6.6 Show that for a wavelike perturbation in a sheared slab model magnetic field the perturbed electron pressure gradient is negligible in the parallel generalized Ohm's law when (6.19) is satisfied and $\omega >> \omega_{*e}$. [Hint: When the magnetic field is perturbed in MHD $\mathbf{B} \to \mathbf{B}_0 + \boldsymbol{\nabla}\times\tilde{\mathbf{A}} \simeq \mathbf{B}_0 + \boldsymbol{\nabla}\tilde{\mathbf{A}}_\parallel \times \hat{\mathbf{b}}$ and $\nabla_\parallel \equiv \hat{\mathbf{b}}\cdot\boldsymbol{\nabla}$ is changed accordingly.] //*

6.7 Show that the perpendicular electron inertia term is a factor of at least ω/ω_{ce} smaller than $\mathbf{E}_\perp$ in (6.20) and hence negligible in MHD. [Hint: Show that for the diamagnetic and polarization MHD currents the electron inertia term is smaller than that due to the electron polarization flow (**??**).] //*

6.8 Derive the MHD system momentum density equation (6.35). [Hint: Rewrite the momentum equation (6.27) using Ampere's law and vector identities (**??**), (**??**) and (**??**).] //

6.9 Derive the MHD system energy density equation (6.36). [Hint: Take the dot product of $\mathbf{V}$ with the MHD momemtum equation (6.45), and simplify the result using Ohm's law in the form $\mathbf{V}\times\mathbf{B} = \eta\mathbf{J} - \mathbf{E}$, vector identities (**??**) and (**??**), and

$$\mathbf{V}\cdot\boldsymbol{\nabla}P = \frac{1}{\Gamma - 1}\frac{\partial P}{\partial t} + \frac{\Gamma}{\Gamma - 1}\boldsymbol{\nabla}\cdot P\mathbf{V} - \eta J^2,$$

which is obtained from a combination of the equation of state (6.29) and the mass density equation (6.25).] //

6.10 Use the tensor form of Gauss' theorem (**??**) to calculate the force on a volume of MHD fluid in terms of a surface integral over the stress tensor. Use an infinitesimal volume form of your result to discuss the components of the force in the $\hat{\mathbf{e}}_{\mathrm{x}}, \hat{\mathbf{e}}_{\mathrm{y}}, \hat{\mathbf{b}}$ directions. //

6.11 The pressure tensor in an open-ended magnetic mirror is anisotropic because of the loss-cone. a) Show that for species distribution functions f_s which do not depend on the gyroangle φ the pressure tensor is in general of the form $\mathsf{P} = P_\perp(\mathsf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}) + P_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}}$ in which $P_\perp = P_\perp(\alpha, \beta, B)$ and $P_\parallel = P_\parallel(\alpha, \beta, B)$. b)

Work out $\boldsymbol{\nabla}\cdot\mathbf{P}$. c) Show that the condition for force balance along a magnetic field ($\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\cdot\mathbf{P}=0$) can be reduced to

$$\left.\frac{\partial P_\parallel}{\partial B}\right|_{\alpha,\beta} = \frac{P_\parallel - P_\perp}{B}.$$

d) Discuss how this result indicates confinement of plasma along the magnetic field in a magnetic mirror. ///

6.12 Obtain the angle between $\mathbf{J}$ and $\mathbf{B}$ in a screw pinch equilibrium as a function of a relevant plasma β. //*

6.13 Consider a pressure profile given by $P(x)/P(0)=\exp(-x^2/a^2)$ in a sheared slab magnetic field model with no curvature or shear. a) Calculate the diamagnetic current. b) Determine the $B_z(x)$ profile induced by this diamagnetic current. c) Show that the plasma pressure produces a diamagnetic effect. d) Show that your $B_z(x)$ agrees with (6.59). //

6.14 Consider the MHD radial force balance equilibrium of a cylindrical plasma with a pressure profile $P(r)$ that vanishes for $r\geq a$ which is placed in a uniform magnetic field $\mathbf{B}=B_0\hat{\mathbf{e}}_\mathbf{z}$. Use the azimuthal component of Ampere's law for J_θ and solve the resultant force balance equation for $B_z(r)$. Show that your result agrees with (6.59). /

6.15 One definition of a magnetic field line is $d\boldsymbol{\ell}\times\mathbf{B}=\mathbf{0}$. Show that its time derivative yields the magnetic evolution equation (6.71). How does this show that a magnetic field line is advected with the moving plasma in the ideal MHD limit? [Hint: Use vector identities (**??**) for $(d/dt)d\boldsymbol{\ell}$ and (**??**), (**??**).] //

6.16 Show that for a Clebsch magnetic field representation $\mathbf{B}=\boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\beta$ the ideal MHD evolution equation (6.71) is satisfied by $d\alpha/dt=d\beta/dt=0$. Why does this show that a magnetic field line is advected with an ideal MHD plasma? //

6.17 Derive the canonical flux invariant for an isentropic plasma species that is a combination of the magnetic flux and species vorticity flux which is deduced from the canonical momentum (**??**) $\mathbf{p}_s = m_s\mathbf{v}+q_s\mathbf{A}$ as follows. a) First, average the canonical momentum over a Maxwellian distribution to obtain $\bar{\mathbf{p}}_s = m_s\mathbf{V}_s+q_s\mathbf{A}$. b) Next, use this result to define a species canonical flux invariant

$$\psi_{\#s} \equiv \iint_S d\mathbf{S}\cdot\boldsymbol{\nabla}\times\left(\mathbf{A}+\frac{m_s}{q_s}\mathbf{V}_s\right) = \iint_S d\mathbf{S}\cdot\left(\mathbf{B}+\frac{m_s}{q_s}\boldsymbol{\nabla}\times\mathbf{V}_s\right).$$

c) Obtain $d\psi_{\#s}/dt$ and use the species momentum equation (**??**) to show that

$$\frac{d\psi_{\#s}}{dt} = -\iint_S d\mathbf{S}\cdot\boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla}p_s+\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_s-\mathbf{R}_s}{n_s q_s}\right).$$

d) Show that $d\psi_{\#s}/dt=0$ for an isentropic plasma species. e) Discuss how the canonical flux invariant $\psi_{\#s}$ combines the ideal MHD frozen flux theorem (6.73) and the Kelvin circulation theorem (**??**). f) Indicate the physical processes that can cause net transport of a plasma species relative to the canonical flux surfaces $\psi_{\#s}$. g) Why doesn't inertia contribute to transport relative to the $\psi_{\#s}$ surfaces? [Hint: Use vector identities (**??**), (**??**) and (**??**) in part c).] ///*

6.18 Show that for the MHD model the electron and ion canonical fluxes defined in the preceding problem are, to lowest order in $(m_e/m_i)^{1/2} << 1$,

$$\psi_{\#e} \simeq \iint_S d\mathbf{S}\cdot\left(1 - \frac{c^2}{\omega_{pe}^2}\nabla^2\right)\mathbf{B}, \quad \psi_{\#i} \simeq \iint_S d\mathbf{S}\cdot\left(\mathbf{B} + B\frac{c}{\omega_{pi}}\boldsymbol{\nabla}\times\frac{\mathbf{V}}{c_A}\right).$$

Use these two relations to discuss the degree to which the magnetic field is frozen into the electron and ion fluids in an ideal MHD plasma. //*

6.19 Show that the total mass M of an MHD plasma in a volume V that moves with the plasma flow velocity $\mathbf{V}$ will be conserved if the mass density satisfies the mass density equation (6.25). [Hint: Determine the condition for $dM/dt = 0$ and use vector identity (**??**) for $(d/dt)d^3x$.] //

6.20 Work out the terms on the right of (6.82). [Hint: Since $\mathbf{B} = B_0\hat{\mathbf{b}}$ is spatially uniform, it commutes with the $\nabla_\parallel \equiv \hat{\mathbf{b}}\cdot\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}_\perp = \boldsymbol{\nabla} - \hat{\mathbf{b}}\nabla_\parallel$ operators.] //

6.21 Work out formulas for the ratio of the electron and ion thermal speeds to the Alfvén speed in terms of $\beta_e \equiv 2\mu_0 p_e/B^2$ and $\beta_i \equiv 2\mu_0 p_i/B^2$. What are these ratios for a $\beta = 0.08$, $T_e = T_i$, electron-deuteron plasma? /

6.22 How large would the magnetic field strength B have to be for the Alfvén speed to be equal to the speed of light for $n_e = 10^{20}$ m^{-3} and $A_i = 2$? /

6.23 Show that for perturbations on the equilibrium time scale for compressional Alfvén waves

$$\tilde{B}_\parallel/B_0 = -(\beta/2)\,(\tilde{P}/P_0). \ /$$

6.24 Since to lowest order in $m_e/m_i << 1$ the MHD momentum equation results from the ion momentum equation, on the equilibrium time scale for compressional Alfvén waves the radial component of the ion momentum equation should be in equilibrium. Show that the equilibrium radial ion momentum (force) balance equation in a screw pinch plasma yields the following relation for the axial flow in terms of the radial electric field, pressure gradient and poloidal flow:

$$V_{iz} = -\frac{1}{B_\theta}\left(\frac{d\Phi_0}{dr} + \frac{1}{n_{0i}q_i}\frac{dp_{0i}}{dr} - V_{i\theta}B_z\right). \ /*$$

6.25 Determine the frequency ranges where an electromagnetic wave impinging on an unmagnetized plasma: a) propagates through it, b) is evanescent on a c/ω_{pe} length scale, and c) dissipatively decays in a resistive skin depth (6.94)? [Hint: review Section 1.5 and consider a time-dependent electrical conductivity.] //

6.26 Show that the Lundquist number can be written in terms of fundamental microscopic variables as

$$S = \frac{\omega_{ce}}{\nu_e}\,\frac{a}{c/\omega_{pi}}\,\frac{a}{L_\parallel}$$

Should S always be a large number for a magnetized MHD plasma? /