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Some New Variational Properties of Hydromagnetic Equilibria

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Earlier variational formulations of problems in hydromagnetic equilibrium are extended by partially relaxing some of the boundary conditions. The resulting natural boundary conditions reflect situations of physical interest. Part of the analysis is done in the large and yields a simple intuitive condition which must be satisfied by an equilibrium which is free to move at its ends (the magnetic lines are not "tied"). Applications are made to stability theory.

1. INTRODUCTION

The distinction between systems in which magnetic lines are tied at boundaries or free to move is frequently discussed in connection with stability theory.[1] But it has apparently gone unnoticed that the class of equilibria (regardless of stability) is different in the two cases. Not every solution of the equilibrium equations

$$\nabla p = \mathbf{J}\times\mathbf{B}, \quad \mu_0\mathbf{J} = \operatorname{curl}\mathbf{B}, \quad \operatorname{div}\mathbf{B} = 0 \qquad (1.1)$$

satisfying boundary conditions including tied ends remains a legitimate equilibrium state if the ends of the magnetic lines are free to move (holding all other boundary conditions fixed).

This distinction occurs most naturally in a variational formulation. Let us take as the definition of an equilibrium state a stationary solution of an appropriate potential[2]

$$F[p(\mathbf{x}), \mathbf{B}(\mathbf{x})] = \int (B^2/2\mu_0 - p)\, d\mathbf{x}. \qquad (1.2)$$

An interior variation yields the equilibrium differential equations as "Euler" equations. But the potential must also be stationary with respect to variations at the boundary. If the class of admissible functions $\mathbf{B}$ and p is sufficiently restricted by the externally imposed boundary conditions, the boundary variation will vanish automatically. If not, a natural boundary condition will impose its will. If instead of just stationary states we insist on a minimum, then the minimum value under the natural boundary condition is clearly lower than a minimum attained under any imposed boundary condition. This has significance with respect to stability.

Occasionally the physical problem will suggest a certain boundary condition as being desirable; but only the equations can say whether it is permissible. For example, in a problem in plasma stability,[3] Taylor suggests that it would be pleasant to have the current vanish at both ends of a mirror-type equilibrium. Previous known (macroscopic) equilibrium theory[2] had only described equilibria in which the current can be specified at one end (e.g., zero); the value at the other end would only be determined after solution of the problem. Taylor finds the integral condition

$$\int \frac{\nabla B\cdot(\nabla p\times\mathbf{B})}{B^4}\, ds = 0 \qquad (1.3)$$

which must hold on each magnetic line in order for the solution to be compatible with the desired property of vanishing current at the ends. This condition seems very awkward and in order to easily satisfy it Taylor is led to consider much more special equilibria in which the integrand in (1.3) vanishes identically.[4]

We find, however, that the condition $J_n = 0$ results as a natural boundary condition obtained by partially relaxing an imposed boundary condition involving the pressure. This suggests that (1.3) is not awkward but, when interpreted properly, is even natural to the problem. Of course the pressure distribution can no longer be specified arbitrarily if (1.3) is to hold. Our main result is an explicit prescription of how the pressure should be distributed among the magnetic lines in order to satisfy the physically suggested boundary condition. We also describe a plausible method of computing these equilibria as a modification of the previous theory with tied ends.[2]

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[1] The earliest quantitative results are those of H. Rubin in J. Berkowitz, H. Grad, and H. Rubin, *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 31, p. 177.

[2] H. Grad and H. Rubin, in *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 31, p. 190.

[3] J. B. Taylor, Phys. Fluids **6**, 1529 (1963).

[4] Taylor's problem is a microscopic rather than our macroscopic one, but the principle is exactly the same (and we shall apply these results to Taylor's problem in a paper to follow).

By use of these techniques we find a simple *necessary* condition for the stability of a conducting fluid with free ends (this is necessary and sufficient for stability with respect to interchanges). One feature of this analysis is that we are able to exhibit a very large variety of equilibria which, although stationary, are not minima; they are, indeed, far more common than the minimum (stable) solutions. Another point of interest is that, with our techniques, the special (interchange) variations on which we concentrate can be easily analyzed with respect to finite displacements.

Analogous results for the plasma problem (microscopic guiding-center theory) will be presented in a paper to follow.

2. VARIATIONAL ANALYSIS

First we recall some of the results described in Ref. 2. Consider solution of the system (1.1) in a

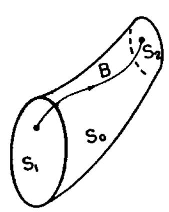

FIG. 1. Tubular domain.

tubular domain (Fig. 1). As boundary conditions we take

Problem I:

(a) B_n given on S [$B_n = 0$ on S_0, $B_n > 0$ on S_1 and $B_n < 0$ on S_2 with $\int_{S_1} B_n\,dS + \int_{S_2} B_n\,dS = 0$]
(b) p given on S_1
(c) J_n given on S_1 (or on S_2).

This formulation was suggested on the basis of plausible arguments involving the characteristics of the system (1.1). In addition, the following plausible iteration scheme for the solution of problem I was presented in Ref. 2. To start, take any magnetic field satisfying div $\mathbf{B} = 0$ and boundary condition (a) (e.g., the vacuum field if we take as our first iterate $\mathbf{J} = 0$). Carry the values of p given in (b) along the magnetic lines; p is now a known function of $\mathbf{x}$. Evaluate the perpendicular component of $\mathbf{J}$ from the first line of (1.1),

$$\mathbf{J}_\perp = \mathbf{B} \times \nabla p / B^2.$$

Writing

$$\mathbf{J} = \mathbf{J}_\perp + \sigma \mathbf{B}$$

and using div $\mathbf{J}$ = div $\mathbf{B} = 0$, we compute

$$\operatorname{div} \mathbf{J}_\perp = -\mathbf{B}\cdot\nabla\sigma = -B\,\partial\sigma/\partial s.$$

Knowing $\partial\sigma/\partial s$ and the initial value of σ from the remaining boundary condition (c) [$\sigma = J_n/B_n$], we can compute σ and therefore $\mathbf{J}$ in the domain. Finally we solve the inhomogeneous potential equation

$$\operatorname{div} \mathbf{B} = 0, \qquad \operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J}$$

subject to the boundary condition (a) to obtain a new magnetic field, and continue.

This iteration can be expected to converge if the tubular domain is not too large and the boundary conditions are not too wild, but no general convergence proof has yet been given.[5]

We continue with results from Ref. 2 based on the variational function (1.2).[6] We allow into competition admissible functions $\mathbf{B}(\mathbf{x})$ and $p(\mathbf{x})$ which satisfy

$$\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{B}\cdot\nabla p = 0, \tag{2.1}$$

and boundary conditions to follow. In other words, we examine F for any solenoidal field $\mathbf{B}(\mathbf{x})$ to which are assigned arbitrary constant values of p on each line. The conditions (2.1) can be parametrized in the form

$$\mathbf{B} = \nabla p \times \nabla\omega \tag{2.2}$$

[a function $\omega(\mathbf{x})$, not necessarily single-valued, can always be found given $\mathbf{B}(\mathbf{x})$ and p; see Ref. 2]. In terms of p and ω we now write

$$F[p, \omega] = \int \left[\frac{1}{2\mu_0}\,|\nabla p \times \nabla\omega|^2 - p\right] d\mathbf{x} \tag{2.3}$$

and restrict the class of admissible pairs $p(\mathbf{x})$, $\omega(\mathbf{x})$ by the boundary conditions

Problem II:

(a) $\nabla p \times \nabla\omega \cdot n = 0$ on S_0,
(b) p given on S_1 and S_2,
(c) ω given on S_1 and S_2.

It is shown in Ref. 2 that F is stationary for any admissible $p(\mathbf{x})$, $\omega(\mathbf{x})$ satisfying these boundary conditions and also the "Euler" equations

$$\mathbf{J}\cdot\nabla\omega = 1, \qquad \mathbf{J}\cdot\nabla p = 0 \tag{2.4}$$

which, together, imply

$$\nabla p = \mathbf{J} \times \mathbf{B}. \tag{2.5}$$

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[5] Convergence in certain special cases (two-dimensional or axial symmetry) can be verified without difficulty (M. Schechter, unpublished).

[6] This is the most convenient variational formulation for the study of equilibrium since it contains only the variables (magnetic field and pressure) which are present in the equilibrium equations. The connection with the stability problem is discussed in Sec. 4.

To compare problems I and II, we note that the flux element on any surface is given by

$$B_n\, dS = \nabla p \times \nabla\omega \cdot d\mathbf{S} = dp\, d\omega. \qquad (2.6)$$

The boundary conditions in problem II determine B_n just as in problem I. Instead of p and J_n given at one end of the tube, p is specified at both ends in problem II. But the specification of p and ω on S_1 and S_2 does more than give B_n on these surfaces; it fixes the ends of each magnetic line (p and ω are coordinates for a line). Alternatively, on a given cylindrical pressure surface, the assignment of ω at each end specifies a certain amount of "twist" to the magnetic field. In a less obvious way the assignment of J_n accomplishes the same in problem I.

In performing the variation of $F[p, \omega]$, the interior variation is found to vanish as a consequence of (2.4), leaving the boundary variation [Ref. 2, Eq. (25)]

$$\int_{S_1+S_2} [\delta p(\mathbf{B}\times\nabla\omega) - \delta\omega(\mathbf{B}\times\nabla p)]\cdot d\mathbf{S}. \qquad (2.7)$$

This vanishes, of course, if p and ω are held fixed on S_1 and S_2 as in problem II; $\delta p = \delta\omega = 0$. We now consider the possibility of widening the class of admissible functions by relaxing some of the boundary conditions of problem II.

First, still quoting Ref. 2, we relax the boundary condition on ω, keeping p and B_n fixed. This allows the ends of the magnetic lines to move around a constant p contour in S_1 or S_2 and yields the natural boundary condition

$$\oint_p \mathbf{B}\cdot d\mathbf{x} = 0 \qquad (2.8)$$

on each p contour (this integral is independent of the path on any given p cylinder). In other words, relaxing the twist associated with the specification of ω yields zero twist as a natural boundary condition. A statement equivalent to (2.8) is

$$\int_{p_0<p<p_1} J_n\, dS = 0. \qquad (2.9)$$

The net current through any ring $p_0 < p < p_1$ on S_1 or S_2 is zero.

We now proceed to further relax the boundary condition, allowing p and ω to vary arbitrarily at one end of the tube. This increased latitude in the admissible class of functions will yield a stronger natural boundary condition than (2.9), viz. $J_n = 0$. Specifically, we state

Problem III:

(a) $B_n = 0$ on S_0,
(b) $B_n > 0$ given on S_1,
(c) p and ω given on S_2 such that

$$\textstyle\int_{S_2} dp\, d\omega + \int_{S_1} B_n\, dS = 0.$$

We implicitly assume that the topology of the field lines is fixed, i.e. the domain is covered simply by magnetic lines all of which intersect S_1 and S_2. This implies certain automatic restrictions on the values of p and ω on S_1. For any admissible pair $p(\mathbf{x})$, $\omega(\mathbf{x})$, a surface $p =$ const is a flux tube. Therefore a closed curve $p =$ const on S_1 will include the same amount of flux as on S_2. Since B_n is specified on S_2, this flux property restricts the p values that can be found on S_1. A continuous deformation of the boundary values of p and ω on S_1 is limited to one which is flux-preserving.

Specifically, if $p(S)$, $\omega(S)$ and $p'(S)$, $\omega'(S)$ are two admissible sets on S_1, the Jacobian of the transformation is unity

$$\partial(p', \omega')/\partial(p, \omega) = 1.$$

Setting $p' = p + \delta p$ and $\omega' = \omega + \delta\omega$, to first order

$$(\partial/\partial\omega)(\delta\omega) + (\partial/\partial p)(\delta p) = 0.$$

Thus there exists a stream function $\psi(p, \omega)$ such that

$$\delta p = \partial\psi/\partial\omega, \qquad \delta\omega = -\partial\psi/\partial p.$$

Substituting into (2.7), we obtain

$$\begin{aligned}\int \left[\frac{\partial\psi}{\partial\omega}(\mathbf{B}\times\nabla\omega) + \frac{\partial\psi}{\partial p}(\mathbf{B}\times\nabla p)\right]\cdot d\mathbf{S} &= -\int \left(\frac{\partial\psi}{\partial\omega}\nabla\omega + \frac{\partial\psi}{\partial p}\nabla p\right)\times\mathbf{B}\cdot d\mathbf{S} \\ &= -\int (\nabla\psi\times\mathbf{B}\cdot\mathbf{n})\, dS \\ &= \int \tau(\psi)\, d\psi\end{aligned}$$

where

$$\tau(\psi) = \int_\psi \mathbf{B}\cdot d\mathbf{x}$$

is the line integral of $\mathbf{B}$ on a contour $\psi =$ const. By taking flows $\psi(\mathbf{x})$ with closed streamlines, we conclude that $\tau = 0$ for every closed curve on S_1. This implies, as a natural boundary condition [supplementing (a), (b), and (c)]

$$\text{III (d)} \qquad J_n = 0 \quad \text{on} \quad S_1. \qquad (2.10)$$

Problem III is seen to be a special case of problem I.

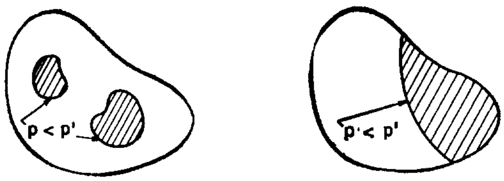

FIG. 2. Pressure contours.

It is convenient to introduce an alternative description of the allowable values that p and ω can take on S_1 when they are fixed on S_2. We introduce the flux function $\Phi(p)$[7] which is defined by the property

$$\Phi(p') = \int_{S'} B_n \, dS \quad \text{where} \quad p < p' \quad \text{on} \quad S'. \quad (2.11)$$

In other words, $\Phi(p')$ is the flux through that part of the surface (S_1 or S_2) where $p < p'$. The assigned boundary values on S_2 determine $\Phi(p)$; the admissible values on S_1 must be compatible with the same $\Phi(p)$. This is a more general admissibility concept than the previous one (and will be necessary later for the study of finite interchanges). For example (see Fig. 2), a set of boundary functions $p(S)$, $\omega(S)$ which is compatible with $\Phi(p)$ can be topologically complicated.

But even for infinitesimal variations, it is necessary to introduce the function $\Phi(p)$ as a constraint when we wish to allow p and ω to be free on both S_1 and S_2. Consider

Problem IV:

(a) B_n given on $S = S_0 + S_1 + S_2$ as before,
(b) $\Phi(p)$ given on S_1 and S_2.

By (b) we mean that a function $\Phi(p)$ is prescribed, and admissible sets $p(\mathbf{x})$, $\omega(\mathbf{x})$ are to be compatible with $\Phi(p)$ at both ends. The topology (e.g., whether lines $p =$ const are open or closed or disconnected sets, Fig. 2) is not specified; the reason for this will become clear later.

Evidently, the natural boundary condition for problem IV is

$$\text{IV (c)} \qquad J_n = 0 \quad \text{on} \quad S_1 \quad \text{and} \quad S_2. \quad (2.12)$$

We remark that in problems III and IV where natural boundary conditions appear, an equilibrium problem is well-posed in terms of solution of the differential equations (1.1) (rather than as a variational problem) only when subject to the combined boundary conditions, imposed and natural.

We recall from Ref. 2 that the boundary variation term (2.7) arises only from the magnetic energy, $\int (B^2/2\mu_0)\, d\mathbf{x}$; the term $\int p \, d\mathbf{x}$ contributes only to the interior variation which gives rise to the Euler equations (2.4). It will be illuminating to perform this magnetic variation *ab initio*, as a self-contained derivation of the formula (2.7). A flux-preserving, line-preserving perturbation of a solenoidal field, div $\mathbf{B} = 0$, can be described by the variational equation

$$\partial \mathbf{B}/\partial t = \text{curl}\, (\mathbf{u} \times \mathbf{B}). \quad (2.13)$$

The "time" t is merely the parameter in the variation; $\mathbf{u}$ is the first variation of the Lagrangian independent variable $\mathbf{x}$; and $\partial \mathbf{B}/\partial t$ is the first variation of $\mathbf{B}$ at a fixed $\mathbf{x}$. Varying the magnetic energy in a fixed domain,

$$\frac{d}{dt} \int \frac{1}{2\mu_0} B^2 \, d\mathbf{x} = \frac{1}{\mu_0} \int \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \, d\mathbf{x}$$
$$= \int \mathbf{u} \cdot \mathbf{B} \times \mathbf{J} \, d\mathbf{x} - \oint (\mathbf{B} \times \mathbf{u}) \times \mathbf{B} \cdot d\mathbf{S}. \quad (2.14)$$

To relate this boundary term (Poynting vector) to (2.7), we note that the values of p and ω are carried by the flow

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = 0, \qquad \frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0. \quad (2.15)$$

Interpreting $\partial p/\partial t$ as δp and $\partial \omega/\partial t$ as $\delta \omega$, (2.7) takes the form

$$\int [(\mathbf{u} \cdot \nabla \omega) \nabla p - (\mathbf{u} \cdot \nabla p) \nabla \omega] \times \mathbf{B} \cdot d\mathbf{S}$$
$$= -\int (\mathbf{B} \times \mathbf{u}) \times \mathbf{B} \cdot d\mathbf{S}$$

which agrees with (2.14).

The significance of the boundary condition $J_n = 0$ is further illuminated by a comparison of a force-free field, curl $\mathbf{B} \times \mathbf{B} = 0$, and a vacuum field, curl $\mathbf{B} = 0$ (see also Ref. 2). The variational function is the same in both cases, viz.

$$F = \frac{1}{2\mu_0} \int B^2 \, d\mathbf{x} = \frac{1}{2\mu_0} \int |\nabla p \times \nabla \omega|^2 \, d\mathbf{x}. \quad (2.16)$$

Here p is only a flux parameter describing the field. The two problems differ in the boundary data. The classical variation of $\int B^2 \, d\mathbf{x}$ with B_n fixed yields curl $\mathbf{B} = 0$ as the variational condition. The variation in terms of p and ω yields curl $\mathbf{B} \times \mathbf{B} = 0$ as the variational condition. Thus when p and ω are

[7] This is analogous to the procedure in a toroidal domain; see Ref. 2.

fixed (tied lines), we have a force free field. But if B_n alone is fixed on the boundary, relaxation of ω and p yields $J_n = 0$ as the natural boundary condition. Writing curl $\mathbf{B} \times \mathbf{B} = 0$ as curl $\mathbf{B} = \sigma \mathbf{B}$ where σ is constant on each line and using $J_n = 0$, we conclude that $\sigma = 0$ or curl $\mathbf{B} = 0$. The vacuum field is therefore obtained as a relaxation (dropping the end identification) of a force free field. The more usual procedure in the classical problem is to introduce a variation which entirely disregards line identification (e.g., by introducing the vector potential of $\delta\mathbf{B}$ and integrating by parts).

We may carry the process of relaxing boundary conditions one step farther and allow B_n as well as p and ω to vary, subject only to the total flux condition

$$\int_{S_1} B_n \, dS = \text{fixed}. \tag{2.17}$$

It is a classical result [and evident from (2.7) or (2.14)] that the natural boundary condition is now

$$\mathbf{B}_t = 0, \tag{2.18}$$

or $\mathbf{B}$ is normal to S_1. We shall not make use of this further relaxation because it makes it impossible to decouple the fluid domain of interest from the rest of the universe. So long as B_n is held fixed on S, the exterior field can be considered to be fixed and therefore irrelevant. But if B_n varies, we must consider as the magnetic potential $\int (B^2/2\mu_0) \, d\mathbf{x}$ summed over the outside and inside. In such a formulation, the natural boundary condition (2.18) is replaced by a vanishing jump in $\mathbf{B}_t$; the vector magnetic field is continuous.

3. INTERCHANGES

We define an interchange to be a perturbation which leaves $\mathbf{B}(\mathbf{x})$ unaltered and merely reassigns the constant values of p among the magnetic lines, keeping fixed the flux function $\Phi(p)$. We recall that this means that the flux which is associated with a given range of values of p, $p_0 < p < p_1$, is held fixed [and is equal to $\Phi(p_1) - \Phi(p_0)$]. This condition is compatible with a hypothetical perfectly conducting fluid flow in which the value of p is carried with a given field line (this could be termed an isobaric flow as distinguished from the more physical isentropic flow). Such an interchange is an admissible variation for problem IV in the last section. Of course, much more general variations are allowed in problem IV, varying p independently at both ends, S_1 and S_2, and varying B as well. By considering these special variations we can obtain *necessary* conditions, not only for stationary F but for a minimum.

It will now be convenient to distinguish the two roles played by p as pressure and as a magnetic coordinate. We write

$$\mathbf{B} = \nabla\alpha \times \nabla\beta, \tag{3.1}$$

where $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are fixed stream functions for the given field $\mathbf{B}(\mathbf{x})$, and we consider variations of the function $p(\alpha, \beta)$. The class of admissible functions $p(\alpha, \beta)$ can be described, for continuous variations, as an incompressible flow in (α, β) which carries the value p; and, in the large, as a class of functions with the property

$$\int_{p<p'} d\alpha \, d\beta = \Phi(p') \tag{3.2}$$

for every p'.

Since $\mathbf{B}$ is not varied, instead of $F[p, \omega]$ we consider the simpler variational function

$$P[p(\mathbf{x})] = -\int p \, d\mathbf{x}. \tag{3.3}$$

We introduce α, β, and the arc length s along a magnetic line as coordinates. Since $d\alpha \, d\beta = B \, dS$ for an element dS normal to $\mathbf{B}$, we have

$$d\mathbf{x} = (1/B) \, d\alpha \, d\beta \, ds. \tag{3.4}$$

Thus

$$P = -\int p(\alpha, \beta) \, d\alpha \, d\beta \, ds/B. \tag{3.5}$$

Or, defining

$$q(\alpha, \beta) = \int_{s_1}^{s_2} \frac{ds}{B(\alpha, \beta, s)} \tag{3.6}$$

integrated the full length of the line (α, β) from S_1 to S_2, we have

$$P = -\int p(\alpha, \beta) q(\alpha, \beta) \, d\alpha \, d\beta. \tag{3.7}$$

Here $q(\alpha, \beta)$ is fixed and $p(\alpha, \beta)$ is to be varied. The significance of an interchange as a special variation is that it reduces the integration from three dimensions to two.

It is intuitively clear how to minimize P.[8] For simplicity consider smooth functions $p(\alpha, \beta)$ and $q(\alpha, \beta)$. The values of p must be rearranged incompressibly such that the largest values of p are placed on the largest values of q. The maximum of $\int pq \, d\alpha \, d\beta$ is obtained when the contours $p =$ const coincide with $q =$ const and p is a monotone function of q. There is a unique function $\bar{p}(\alpha, \beta)$ which satisfies this condition. On a given contour $q(\alpha, \beta) = q_0$

[8] This type of analysis has been used in a similar context by C. S. Gardner [Phys. Fluids **6**, 839 (1963)].

which encloses a flux ϕ_0, the value of $\bar{p}(\alpha, \beta) = p_0$ is determined by $\Phi(p_0) = \phi_0$ (see Appendix).

Next we show that P is stationary whenever the contours p = constant and q = constant coincide. Let the variation of $p(\alpha, \beta)$ be described by an incompressible flow $\mathbf{u}(\alpha, \beta)$ which carries p

$$\operatorname{div} \mathbf{u} = 0, \qquad \frac{dp}{dt} = \frac{\partial p}{\partial t} + \mathbf{u}\cdot\nabla p = 0.$$

We have

$$\begin{aligned}\frac{dP}{dt} &= -\frac{d}{dt}\int pq\, d\alpha\, d\beta \\ &= -\int \frac{\partial p}{\partial t}\, q\, d\alpha\, d\beta \\ &= \int (\mathbf{u}\cdot\nabla p) q\, d\alpha\, d\beta.\end{aligned}$$

We recall the theorem that if $\int \mathbf{u}\cdot\mathbf{v} = 0$ for every incompressible $\mathbf{u}$ with $u_n = 0$ at the boundary, then curl $\mathbf{v} = 0$ (or we write $\mathbf{u} = \mathbf{n} \times \nabla\psi$ and integrate by parts). Thus

$$\nabla p \times \nabla q = 0,$$

which is what was to be proved.[9]

We shall refer briefly to the condition "p = const where q = const" as "p is a function of q", although the function may not be single-valued. For a minimum, p is in the usual sense a monotone function of q.

Note that for P either stationary or a minimum, the topology of the p surfaces is determined; it is the same as that of the given q surfaces. It is possible that the more general variation allowed in problem IV might yield more than one local minimum with different q topologies.

Summarizing, p a function of q is necessary and sufficient for $P = -\int pq\, d\alpha\, d\beta$ to be stationary, and p a monotone function of q is both necessary and sufficient for P to be an absolute minimum with regard to all finite perturbations. Since, as we have already remarked, interchanges are admissible variations for problem IV, we conclude that p a function of q is a necessary condition for $F = \int (B^2/2\mu_0 - p)\, d\mathbf{x}$ to be stationary and p a monotone function of q is a necessary condition for F to be a minimum. In other words, every hydromagnetic equilibrium with freedom to move at its ends must have p a function of q, and every absolute minimum equilibrium (related to stability) must have p a monotone function of q. These restrictions are in addition to the requirement that p and $\mathbf{B}$ satisfy the equilibrium differential equations.

It is clear that p a function of q, which makes the potential F stationary with respect to interchanges, must be related to $J_n = 0$ which makes F stationary with respect to more general variations. If we perform an interchange following the methods of the preceding section, restricting δp and $\delta\omega$ to be the same on S_1 and S_2, we find

$$j_n^1 + j_n^2 = 0, \tag{3.8}$$

where $j_n = J_n/B_n$ is the current density taken with respect to $d\alpha\, d\beta$ instead of dS

$$j_n\, d\alpha\, d\beta = J_n\, dS. \tag{3.9}$$

Instead of zero current at both ends, as in the case of arbitrary end variations, we have zero *difference* in current at the two ends of every magnetic line when we only admit interchanges. This condition on j_n is exactly equivalent to the condition that p be a function of q. The first criterion is perhaps more intuitive, but the second is more useful in constructing equilibria.

A direct proof of the equivalence of $j_n^1 + j_n^2 = 0$ with $p \sim q$ follows from the

Lemma: Given two vector fields $\mathbf{J}$ and $\mathbf{B}$ and a scalar p which satisfy div $\mathbf{J}$ = div $\mathbf{B} = 0$ and $\mathbf{J} \times \mathbf{B} = \nabla p$ (we do not insist on the relation $\mu_0 \mathbf{J}$ = curl $\mathbf{B}$); then $\mathbf{J}$ can be represented by a stream function ζ,

$$\mathbf{J} = \nabla\zeta \times \nabla p \tag{3.10}$$

where ζ differs from $q(\alpha, \beta, s) = \int_{s_1}^{s} ds/B$ by a constant on each magnetic line.

For the proof we compute

$$\mathbf{J} \times \mathbf{B} = (\nabla\zeta \times \nabla p) \times \mathbf{B} = \nabla p(\mathbf{B}\cdot\nabla\zeta)$$

and find that $\mathbf{B}\cdot\nabla\zeta = B\, \partial\zeta/\partial s = 1$ or

$$\zeta(s_2) - \zeta(s_1) = \int_{s_1}^{s_2} \frac{ds}{B} = q(s_2) - q(s_1) \tag{3.11}$$

with arbitrary endpoints s_1, s_2 on any line. This is equivalent to

$$\zeta - q = f(p, \omega), \tag{3.12}$$

where p and ω are coordinates of a line. Note that our present usage is that $q = \int ds/B$ is evaluated for arbitrary limits of integration; in the analysis of interchanges, q was always taken over the full length of the magnetic line. If a distinction is

[9] A similar criterion was obtained by B. B. Kadomtsev [*Plasma Physics and the Problem of Controlled Thermonuclear Reactions*, edited by M. A. Leontovich (Pergamon Press, New York, 1960), Vol. IV, p. 17] in a different problem, viz., a low pressure toroidal equilibrium with closed lines; cf. the discussion at the end of Sec. 4 on closed lines.

necessary, we shall make it by specifying the arguments $q(\alpha, \beta)$ or $q(\alpha, \beta, s)$. We can use q as the stream function for J if the end surface on which we assign $q = 0$ is assigned the boundary condition $J_n = 0$. More generally, q can be used instead of ζ to compute current *differences* between the two ends of a flux tube since $\nabla f \times \nabla p \cdot d\mathbf{S} = f_\omega \nabla \omega \times \nabla p \cdot d\mathbf{S} = -f_\omega \mathbf{B} \cdot d\mathbf{S}$ gives equal contributions at both ends. If p is a function of $q(\alpha, \beta)$, $\nabla p \times \nabla q = 0$, then the current density is the same at both ends (or $j_n^1 + j_n^2 = 0$ with the positive normal outward).

The condition (3.8) suggests

Problem V:

(a) B_n given on S,
(b) $\Phi(p)$ given,
(c) J_n given on S_1,
(d) $j_n^1 + j_n^2 = 0$.

Problem IV is a special case of problem V taking $J_n = 0$.

We now show how to construct solutions to problems IV and V by a modification of the iterations used in problem I. It suffices to consider problem V. We start with the vacuum magnetic field which takes the given boundary values B_n. Using $\mathbf{B}(\mathbf{x})$ we compute $\int ds/B$ and evaluate $q(\alpha, \beta)$. The given pressure function $\Phi(p)$ is then distributed so as to maximize $\int pq\, d\alpha\, d\beta$. Using $\mathbf{J} \times \mathbf{B} = \nabla p$ we evaluate $\mathbf{J}_\perp$ and then $\mathbf{J}_\parallel$ as in problem I, using the given boundary condition on J_n at one end. Using $\mathbf{J}$ we compute a new $\mathbf{B}$ and repeat the procedure. Note that there is an infinite variety of choices of $p(q)$ which will make F stationary as compared to a presumably unique minimum. In the iteration we use the value of J_n at one end only. But the assignment of p to be a function of q will insure that J_n takes the appropriate value at the other end.

In all the iteration procedures described, the computation of $\mathbf{J}$ from $\mathbf{J} \times \mathbf{B} = \nabla p$ and a boundary condition on J_n can be stated compactly in terms of the auxiliary function $q(\alpha, \beta, s)$. For example, if $J_n = 0$ on S_1, from a given $\mathbf{B}$ we compute $q(\alpha, \beta, s) = \int_{s_1}^{s} ds/B$ ($q = 0$ on S_1), and then evaluate $\mathbf{J} = \nabla q \times \nabla p$. If $J_n \neq 0$, we also compute the function $\zeta_1(p, \omega)$ on S_1 to satisfy $dp\, d\zeta = J_n dS$ ($\partial\zeta/\partial\omega = J_n/B_n$ defines ζ within an arbitrary and irrelevant added function of p), and then obtain $\mathbf{J} = \nabla(q + \zeta_1) \times \nabla p$. This is exactly equivalent to the previous computation of $\mathbf{J}_\perp$ followed by $\mathbf{J}_\parallel$.

Some of the previous results are particularly illuminating in the low pressure (low "β") limit. In problems I, II, and V, the limit is a force free field, whereas in III and IV it is a vacuum field. In problems I and V, the given boundary condition $J_n \neq 0$ precludes a vacuum field. In problem II, we define the limit by considering fixed functions $p_1(S_1)$ and $p_2(S_2)$ each multiplied by a small parameter. In the limit, the pressure is zero but the line identification is kept. Only by relaxing the line identification do we get $J_n = 0$ and a limiting field which is a vacuum.

When the limit $p \to 0$ is a vacuum field, the iterations previously described are expansions about $p = 0$. In problem III (or problem I with $J_n = 0$), we start with the vacuum field, assign the small pressure values imposed by the boundary condition and then compute the small perturbation current as a first order correction. In problem IV we start with the vacuum field, compute $q(\alpha, \beta) = \int ds/B$, assign the small pressure values accordingly, and and then compute the perturbation of the current. To lowest order, the equilibrium configuration is a vacuum field together with an assigned pressure function. In the first case p is arbitrarily assigned to the vacuum magnetic field lines; in the second case it is a function of q (which is computed from the given vacuum field). In the one case, the topology of the p-surfaces is arbitrary; in the other case it is determined by the vacuum field. It is evident in the second case, where p is a function of q, that there are infinitely more stationary equilibria than there are minima.

If p is assigned arbitrarily in problem III, a finite current (i.e., first order, comparable to p) is created at the end where it is not fixed to be zero. It is interesting to note that this residual current can be computed explicitly in terms of the given pressure assignment and the known vacuum value of q by the formula

$$J_n\, dS = dp\, dq = \nabla p \times \nabla q \cdot d\mathbf{S}.$$

4. STABILITY

The variational function $F = \int (B^2/2\mu_0 - p)\, d\mathbf{x}$ was originally proposed for the study of equilibrium.[10] In a special case (fluid and field separated at an interface) it was also successful for a study of stability.[10] But for more general stability analysis we must consider the energy

$$G = \int \left(\frac{1}{2\mu_0} B^2 + \rho e\right) d\mathbf{x} \tag{4.1}$$

[10] H. Grad in "Proceedings of Princeton Thermonuclear Conference, 1954." Published in U. S. Atomic Energy Commission report WASH-184, p. 144, 1955. Also in Refs. 1 and 2.

where e is the internal energy per mass and ρ is the mass density. We will show, however, that a minimum value for F is *sufficient* for stability in complete generality. We will also find simple necessary and sufficient conditions for interchange stability similar to the previous criteria on F involving p and q. Most stability analyses have been based on a linearized version (Rayleigh's principle) depending on the second variation of G.[11] A general variation of G can be defined by a displacement of the Lagrangian position $\mathbf{x}$. The magnetic energy variation is the same as for F. The internal energy variation is constrained by the conservation of mass in each flux tube and the constant entropy which is carried by a fluid particle. This is to be compared with the variation of F in which p (not entropy) is carried as a constant.

But we recall a basic thermodynamic inequality which states that in any domain (for example a flux tube), if the total mass is fixed and the entropy is constant following each displaced particle, then the energy is a minimum for a state with constant pressure over the domain. It therefore suffices, in investigating G for a minimum, to consider only special thermodynamic variations which keep p a constant on each line. Since the variation is adiabatic, the pressure value on a varied line will not be the same as on the original line, in contrast to the isobaric variation previously considered for F.

For interchanges which do not vary B and for the comparison between F and G in which the magnetic energy enters similarly, if suffices to consider

$$P = -\int p\, d\mathbf{x}, \qquad U = \int \rho e\, d\mathbf{x}. \tag{4.2}$$

We shall compare P and U by an elementary thermodynamic argument (cf. Ref. 1). The internal energy $e(\tau, \eta)$ is a convex function of its arguments ($\tau = 1/\rho$ is the specific volume, η is the entropy per mass). Actually we need only the property that it is convex in τ when η is fixed, and we recall that $\partial e/\partial \tau = -p$. The function

$$\phi(\tau) = e(\tau, \eta_0) + p_0\tau$$

is also convex in τ (p_0 is a given constant and η_0 is held fixed). It therefore assumes an absolute minimum where $\phi'(\tau) = 0$, viz., at the unique value τ_0 determined by $-\partial e/\partial \tau = p(\tau_0, \eta_0) = p_0$. We can write this minimum property as an inequality

$$e(\tau, \eta_0) + p_0\tau \geq e(\tau_0, \eta_0) + p_0\tau_0. \tag{4.3}$$

Now consider two domains D and D_0 in which the functions $\tau(\mathbf{x})$, $e(\mathbf{x})$, $\eta(\mathbf{x})$ and $\tau_0(\mathbf{x})$, $e_0(\mathbf{x})$, $\eta_0(\mathbf{x})$, respectively, are defined. We suppose that there is given a one-to-one mapping which identifies corresponding points $\mathbf{x} \leftrightarrow \mathbf{x}_0$. The functions η and η_0 are assumed to be equal at corresponding points, $\eta(\mathbf{x}) = \eta_0(\mathbf{x}_0)$, and the elements of mass are the same at image elements $\rho\, d\mathbf{x} = \rho_0\, d\mathbf{x}_0$. In other words, the state in D could have arisen by a mass-conserving adiabatic flow originating from D_0 (the two volumes are not necessarily the same). In particular we assume that p_0 is a constant in D_0 (but $\eta_0(\mathbf{x})$, $e_0(\mathbf{x})$, $\tau_0(\mathbf{x})$ are not necessarily constant). We integrate the inequality (4.3) with respect to mass, $dm = \rho\, d\mathbf{x} = \rho_0\, d\mathbf{x}_0$, and obtain

$$\int_D e(\tau, \eta_0)\rho\, d\mathbf{x} - \int_{D_0} e(\tau_0, \eta_0)\rho_0\, d\mathbf{x}_0 \geq p_0(V_0 - V), \tag{4.4}$$

where $V_0 = \int d\mathbf{x}_0$ and $V = \int d\mathbf{x}$ are the volumes of D_0 and D respectively. First, as a special case, taking $D = D_0$ and $V = V_0$, we conclude that the energy $\int e\rho\, d\mathbf{x}$ is a minimum with respect to any adiabatic variation in a given domain for the state where $p = p_0$ is constant in the domain (this is the result quoted above).

In our application we consider the domains D and D_0 to be two magnetic lines with $dx = ds/B$ and $V = \int ds/B = q$ (or, more intuitively, as flux tubes with $d\mathbf{x} = d\alpha\, d\beta\, ds/B$). The coordinates of D_0 are (α_0, β_0) and of D (α, β). Any variation (general as well as interchange) is a mapping from an original line (α_0, β_0) to a new position (α, β) together with an adiabatic mass-preserving variation of ρ, e, etc. along the line. But to investigate a minimum, it is only necessary to consider varied states in which $p =$ constant. We therefore now consider the inequality (4.4) under the restriction that p as well as p_0 is constant. Integrating (4.4) with respect to $d\alpha\, d\beta = d\alpha_0\, d\beta_0$ we obtain

$$U - U_0 \geq P - P_0. \tag{4.5}$$

The left side is the change in $U = \int \rho e\, d\mathbf{x}$ due to an arbitrary adiabatic variation. The right side is the change in P due to an isobaric variation which carries the value p_0 unaltered to the new location. This inequality holds for all admissible variations of the potentials P and U, not only interchanges, and not only variations of U in which p is constant on a line (the inequality is only strengthened otherwise).

This inequality contains one of our results: if a given equilibrium is a minimum for F considered with respect to isobaric variations, it is also a

[11] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. (London) **A244**, 17 (1958).

minimum for G with respect to adiabatic variations (i.e., stable).

For a more special equilibrium of a perfect gas in which the entropy η is a constant on each line (it may vary from line to line), we find a simple necessary and sufficient condition for U to be a minimum with respect to adiabatic interchanges. (This is therefore a necessary condition for absolute stability.) We have

$$U = \frac{1}{\gamma - 1} \int pq \, d\alpha \, d\beta. \qquad (4.6)$$

In displacing the gas adiabatically from one line to another, $pq^\gamma = $ const; thus $p = p_0 q_0^\gamma q^{-\gamma}$ and the varied U becomes

$$U = \frac{1}{\gamma - 1} \int p_0 q_0^\gamma q^{1-\gamma} \, d\alpha \, d\beta = \frac{1}{\gamma - 1} \int \sigma_0 q^{1-\gamma} \, d\alpha \, d\beta$$

where

$$p_0 q_0^\gamma = \sigma_0.$$

We wish to minimize U subject to arbitrary interchange of the values $\sigma_0(\alpha, \beta)$ carried as a constant. The minimum is attained when σ is a monotone decreasing function of $q^{1-\gamma}$. This is equivalent to σ a monotone increasing function of q $(\gamma > 1)$. We therefore state the theorem that pq^γ a monotone function of q is necessary and sufficient for U to be a minimum with respect to interchanges. We can compare this condition, pq^γ monotone in q, with the condition on P varied isobarically, viz., p monotone in q. Clearly the latter implies the former, but not conversely.

For an axially symmetric equilibrium at low pressure (**B** is a vacuum field), the following criterion has been given for stability with respect to interchanges[12]:

$$\int_{s_1}^{s_2} \frac{\kappa}{rB} \frac{ds}{B} > 0. \qquad (4.7)$$

Here r is the radius and κ is the curvature of the magnetic line (signed). It is easily verified that this inequality is equivalent to the statement that q decreases outward in this special case of a vacuum axially symmetric magnetic field provided that the domain terminates at end surfaces S_1 and S_2 which are orthogonal to the magnetic field. In this special case, (4.7) is a sufficient condition (equivalent to minimum P) but not a necessary condition (minimum U) for interchange stability. It is neither necessary nor sufficient for absolute stability and has no significance even with regard to interchange stability for more general end surfaces.

[12] M. N. Rosenbluth and C. L. Longmire, Ann. Phys. **1**, 120 (1957).

Given an arbitrary equilibrium configuration, we can choose new flux coordinates (α, β) such that $\alpha = $ const on the isobars; $p = p(\alpha)$ and $q = q(\alpha)$. The stability criterion $pq^\gamma \sim q$ now can be written

$$(p'/p + \gamma q'/q)q' \geq 0. \qquad (4.8)$$

In particular, for a general axially symmetric problem, the proper coordinate α is evident *a priori*. The condition (4.8) is necessary and sufficient for interchange stability in complete generality. A similar condition has been stated to be necessary and sufficient for absolute stability in the limit of low pressure in a periodic axially symmetric geometry,[11] but the proof given there is incomplete. The question whether interchange stability implies absolute stability for sufficiently low pressure is still open.

The stability problem in a toroidal system with closed magnetic lines is identical to that in the open-ended systems considered up to now. But the equilibrium problem is much more subtle. The reason is that the property of lines closing can be easily destroyed by a small perturbation. For example, the iterations previously described are useless in looking for perturbed closed-line equilibria in the neighborhood of a known vacuum field with closed lines.[13] But let us assume that we are somehow given a toroidal equilibrium with closed magnetic lines. First we remark tha the condition that p be a function of q is automatically satisfied, since it is equivalent to the statement that **J** is single-valued.[14] The criterion for a minimum of F with respect to interchanges is that p be monotone in q, and the criterion that G be a minimum with respect to interchanges is that $pq^\gamma \sim q$; the latter is therefore a necessary condition for absolute stability.

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APPENDIX: A VARIATIONAL PROBLEM

There does not seem to be any proof of the inequality that we desire in the literature,[15] although

[13] They are also useless in computing equilibria with closed (ergodic) flux surfaces contrary to an opinion expressed in J. M. Greene and J. L. Johnson, Phys. Fluids **4**, 875 (1961).

[14] See Eqs. (3.10) and (3.11); also Ref. 9.

[15] In one dimension, a similar theorem can be found in G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities* (Cambridge University Press, London, 1934), Chap. X.

it is intuitively clear and has been used previously for similar stability analyses.[8]

For simplicity of exposition it is convenient to consider two different differentiability classes for the functions $q(\alpha, \beta)$ and $p(\alpha, \beta)$. We consider a simply connected, bounded plane domain D. The given function $q(\alpha, \beta)$ has a continuous gradient which vanishes only at isolated points. This assures us of the existence of a family of curves $q =$ constant inside D. Consideration of the area A within which $q > q_0$ leads to the monotone continuous function $A(q)$ with inverse (also monotone and continuous) $q^*(A)$.

For $p(\alpha, \beta)$, we take a continuous bounded function in D and construct a similar function $p^*(A)$. The function p^* is also monotone and continuous but the inverse function, although monotone, is not necessarily continuous; e.g., there may be stretches of finite area where $p =$ const. Given a monotone continuous function $p^*(A)$ (defined on $0 < A < A_1$ where A_1 is the area of D), we consider the class of associated functions $\{p(\alpha, \beta)\}$ which are continuous in D and are equimeasurable with $p^*(A)$; i.e., for any function p the area A_0 of the subset of D on which $p > p_0$ is a monotone function of p_0 whose inverse is $p^*(A)$.

Our main theorem is that there is a unique representative $\bar{p}(\alpha, \beta)$ which is equimeasurable with $p^*(A)$ and which maximizes the integral

$$I = \int_D p(\alpha, \beta) q(\alpha, \beta) \, d\alpha \, d\beta.$$

We construct $\bar{p}$ explicitly as follows. For each value A_0 we assign the value $p^*(A_0)$ to the contour $q(\alpha, \beta) =$ const on which q takes the value $q^*(A_0)$. As A_0 ranges from 0 to A_1, every point of D is covered once and only once. Also it is easily seen that $\bar{p}(\alpha, \beta)$ is continuous.

We now show that any admissible $p(\alpha, \beta)$ which differs from $\bar{p}(\alpha, \beta)$ gives a smaller value to the integral I. For a given value q_0, we denote by S_0^- the set where $q < q_0$ and S_0^+ the set where $q > q_0$ (the common boundary is the contour $q = q_0$, and on it $\bar{p} = \bar{p}_0$). If every value taken by $p(\alpha, \beta)$ in S_0^- is less than every value taken in S_0^+, then by continuity, $p = \bar{p}_0$ on the boundary. If p in S_0^- is less than p in S_0^+ for every q_0, then $p(\alpha, \beta) = \bar{p}(\alpha, \beta)$. Since $p \neq \bar{p}$, there exists a pair of points (α', β') and (α'', β'') with the property that $q(\alpha', B') > q(\alpha'', \beta'')$ and $p(\alpha', \beta') < p(\alpha'', \beta'')$. By continuity, there exist two small domains of equal area surrounding (α', β') and (α'', β'') within which the same inequalities are satisfied. An isometric transposition of the values of p within these two areas increases the value of I.[16] Thus no function $p(\alpha, \beta)$ different from $\bar{p}(\alpha, \beta)$ yields a maximum.

This theorem can be extended to more general classes of functions, but at the expense of a certain degree of awkwardness. For example, if $q(\alpha, \beta)$ is allowed to be constant in a finite region, then I still has a maximum, but the maximizing function $\bar{p}(\alpha, \beta)$ will not be unique (any interchanges of $\bar{p}$ within a region where q is constant do not affect the value of I). The simplest way to include such generalizations is to consider measurable functions. This we shall not do because it is not necessary for our application.

[16] Strictly speaking the transposition is not allowable since it violates continuity. This difficulty can be sidestepped in many ways. Perhaps the simplest is to extend the class of admissible $p(\alpha, \beta)$ to include piecewise continuous functions which are compatible with the given (continuous) $p^*(A)$.