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M3DP USER'S GUIDE

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References:

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- A nonlinear two-fluid model for toroidal plasma, phys. plasma **7**, 4644(2000), L.E. Sugiyama, W. Park.
- Resistive MHD equations in toroidal geometry, X.Z.Tang, August, 2001
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1 Resistive MHD equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0; \tag{1}$$

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p + \mu \nabla^2 \mathbf{v} \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}; \tag{3}$$

$$\frac{dp}{dt} = -\gamma p \nabla \cdot \mathbf{v} + \rho \nabla \cdot \kappa \cdot \nabla \frac{p}{\rho} \tag{4}$$

$$\mathbf{J} = \nabla \times \mathbf{B}; \tag{5}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \tag{6}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7}$$

There are 8 variables $(\rho, \mathbf{v}, \mathbf{B}, p)$ and 8 equations (1), (2), (3), and (4). The number of variables can be reduced to 7 using (7). $\mathbf{J}$ and $\mathbf{E}$ can be derived from them using (5) and (6).

2 Resistive MHD equations in toroidal geometry

In (R, Z, φ) coordinates

2.1 Decompose v into (u, χ, v_φ), we have

$$\mathbf{v} = R^2 \epsilon \nabla u \times \nabla \varphi + \nabla_\perp \chi + v_\varphi \hat{\varphi}$$

2.2 Decompose B into (ψ, I), we have

$$\mathbf{B} = \nabla \psi \times \nabla \varphi + \frac{1}{R} \nabla_\perp F + R_0 I \nabla \varphi$$

2.3 F equation

$$\nabla_\perp^2 F = -\frac{a}{R} \tilde{I}', \ \ \text{with} F \equiv \frac{\partial f}{\partial \varphi}, I = \frac{R}{R_0} B_\varphi = 1 + \epsilon \tilde{I}. \tag{8}$$

2.4 Expression for J and C

$$\mathbf{J} = J_\varphi \hat{\varphi} + \nabla \tilde{I} \times \nabla \varphi - \frac{1}{R} \nabla_\perp F' \times \nabla \varphi + \frac{1}{R^2} \nabla_\perp \psi'$$

$$R J_\varphi \equiv -C = -\{\Delta^* \psi + \frac{1}{R} \frac{\partial F}{\partial z}\}.$$

2.5 Φ equation

From $\nabla \cdot \mathbf{B} = 0$, we have $\mathbf{B} = \nabla \times \mathbf{A}$ and

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} \times \nabla \Phi$$

Using $\nabla_\perp \cdot \mathbf{A} = 0$, we can derive the equation for Φ

$$\nabla_\perp^2 \Phi = \nabla_\perp \cdot \mathbf{E}, \ \ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}, \ \ , i.e.$$

$$\begin{aligned}
\nabla_\perp^2 \Phi \quad = \quad & \epsilon \nabla_\perp \tilde{I} \cdot \nabla_\perp U + I \nabla_\perp^2 U - \nabla_\perp (\frac{v_\varphi}{R}) \cdot \nabla_\perp \psi - \frac{v_\varphi}{R} \Delta^* \psi \\
& + \nabla_\perp \chi \times \nabla_\perp (\frac{\tilde{I}}{R}) \cdot \hat{\varphi} - \nabla_\perp F \times \nabla_\perp (\frac{v_\varphi}{R}) \cdot \hat{\varphi} \\
& + \frac{\eta}{R^2} [\frac{1}{R} (\frac{\partial F'}{\partial Z} - \frac{\partial \psi'}{\partial R}) - \frac{\partial \tilde{I}}{\partial z} + \frac{\partial C}{\partial \varphi}] \\
& + \frac{1}{R} \nabla_\perp \eta \times [\nabla_\perp \tilde{I} - \frac{1}{R^2} \nabla_\perp F'] \cdot \hat{\varphi} + \frac{1}{R^2} \nabla_\perp \eta \cdot \nabla_\perp \psi'
\end{aligned} \tag{9}$$

3 Scales and Variables using in M3DP

$\hat{()}$ ≡ code variables
$\mathbf{v} = v_0 \hat{\mathbf{v}}, v_o \equiv \epsilon B_0/\rho_0^{1/2} = \epsilon v_A$ (v_A is the poloidal Alfven velocity)
$\epsilon \equiv a/R_0$
$\mathbf{B} = B_0 \hat{\mathbf{B}}$
$l = L_0 \hat{l}, L_0 \equiv a$
$\rho = \rho_0 \hat{\rho}$, ρ_0 is simply ρ of where we choose v_A
$p = \epsilon B_0^2 \hat{p}$
$U = \epsilon v_0 \hat{U}$

writing $\rho = \rho_0 \frac{R_0^2}{R^2} d$, the only changed equation is (2), which now looks like

$$\frac{d\mathbf{v}}{dt} = \frac{R^2}{d} \mathbf{J} \times \mathbf{B} - \frac{\epsilon R^2}{d} \nabla p + \mu \frac{R^2}{d} \nabla^2 \mathbf{v} \quad \text{with} \quad \mu \equiv \mu^{emu} \frac{v_0}{a B_0^2} \tag{10}$$

rh : d rhi, rho: $\frac{1}{d}$
a,aold ,aa : ψ $chi, chiold : \chi$ simpf, bigf: F xphi: Φ
ei,eiold, si: $\tilde{I}$ bigi: I vphi: v_φ
xgram : γ rmu: μ etas: η
rpls1: R rneg1: $\frac{1}{R}$
lap_a: $\nabla_\perp^2 \psi$ lap_u: $\nabla_\perp^2 u$ lap_chi: $\nabla_\perp^2 \chi$ lap_phi: $\nabla_\perp^2 \phi$ lap_F: $\nabla_\perp^2 F$

4 The 7 time-dependent equations solved in M3DP

4.1 $R\hat{\varphi} \cdot \frac{\partial \mathbf{B}}{\partial t} \Rightarrow \frac{\partial \tilde{I}}{\partial t}$:

$$\begin{aligned}
\frac{\partial \tilde{I}}{\partial t} &= \epsilon R \nabla_\perp U \times \nabla_\perp \tilde{I} \cdot \hat{\varphi} - \nabla_\perp \chi \cdot \nabla_\perp \tilde{I} - \frac{v_\varphi}{R} \frac{\partial \tilde{I}}{\partial \varphi} \\
&+ R \nabla_\perp (\frac{v_\varphi}{R}) \times \nabla_\perp \psi \cdot \hat{\varphi} + R \nabla_\perp F \cdot \nabla_\perp (\frac{v_\varphi}{R}) \\
&-(\frac{1}{\epsilon} + \tilde{I}) \Delta^* \chi + \eta [\Delta^* \tilde{I} - \frac{1}{R} \nabla_\perp^2 F' + \frac{2}{R^2} (\frac{\partial F'}{\partial R} + \frac{\partial \psi'}{\partial Z})] \\
&+ \nabla_\perp \eta \cdot [\nabla_\perp \tilde{I} - \frac{1}{R} \nabla_\perp F' - \nabla_\perp \psi' \times \nabla \varphi] \qquad (11)
\end{aligned}$$

4.2 $R\hat{\varphi} \cdot \frac{\partial \mathbf{A}}{\partial t} \Rightarrow \frac{\partial \psi}{\partial t}$:

$$\begin{aligned}
\frac{\partial \psi}{\partial t} &= \epsilon R \nabla_\perp U \times \nabla_\perp \psi \cdot \hat{\varphi} + \epsilon R \nabla_\perp U \cdot \nabla_\perp F - \nabla_\perp \chi \cdot \nabla_\perp \psi \\
&+ \nabla_\perp \chi \times \nabla_\perp F \cdot \hat{\varphi} + \frac{\partial \Phi}{\partial \varphi} + \eta (\Delta^* \psi + \frac{1}{R} \frac{\partial F}{\partial Z}) \qquad (12)
\end{aligned}$$

Instead of finding $C = \Delta^* \psi + \frac{1}{R} \frac{\partial F}{\partial Z}$, we define a new variable

$$C_a \equiv \Delta^* \psi = \nabla_\perp^2 \psi - \frac{1}{R} \frac{\partial \psi}{\partial R}$$

and solve the follwoing equation

$$\begin{aligned}
\frac{\partial C_a}{\partial t} &= \epsilon R [\nabla_\perp (\Delta^* U) \times \nabla_\perp \psi + \nabla_\perp U \times \nabla_\perp C_a + 2 \nabla_\perp (\frac{\partial U}{\partial R}) \times \nabla_\perp (\frac{\partial \psi}{\partial R}) + 2 \nabla_\perp (\frac{\partial U}{\partial Z}) \times \nabla_\perp (\frac{\partial \psi}{\partial Z})] \cdot \hat{\varphi} \\
&\epsilon R [\nabla_\perp (\Delta^* U) \cdot \nabla_\perp F + \nabla_\perp U \cdot \nabla_\perp (\Delta^* F) + 2 \nabla_\perp (\frac{\partial U}{\partial R}) \cdot \nabla_\perp (\frac{\partial F}{\partial R}) + 2 \nabla_\perp (\frac{\partial U}{\partial Z}) \cdot \nabla_\perp (\frac{\partial F}{\partial Z})] \\
&- \nabla_\perp (\Delta^* \chi) \cdot \nabla_\perp \psi - \nabla_\perp \chi \cdot \nabla_\perp C_a - 2 \nabla_\perp (\frac{\partial \chi}{\partial R}) \cdot \nabla_\perp (\frac{\partial \psi}{\partial R}) - 2 \nabla_\perp (\frac{\partial \chi}{\partial Z}) \cdot \nabla_\perp (\frac{\partial \psi}{\partial Z}) \\
&[\nabla_\perp (\Delta^* \chi) \times \nabla_\perp F + \nabla_\perp \chi \times \nabla_\perp (\Delta^* F) + 2 \nabla_\perp (\frac{\partial \chi}{\partial R}) \times \nabla_\perp (\frac{\partial F}{\partial R}) + 2 \nabla_\perp (\frac{\partial \chi}{\partial Z}) \times \nabla_\perp (\frac{\partial F}{\partial Z})] \cdot \hat{\varphi} \\
&+ \frac{\partial}{\partial \varphi} \Delta^* \Phi \\
&+ \eta (\Delta^* C_a + \frac{1}{R} \frac{\partial}{\partial Z} \Delta^* F) \qquad (13)
\end{aligned}$$

4.3 expression for v

$\mathbf{v} = (\epsilon R \frac{\partial U}{\partial Z} + \frac{\partial \chi}{\partial R}) \hat{R} + (-\epsilon R \frac{\partial U}{\partial R} + \frac{\partial \chi}{\partial Z}) \hat{Z} + v_\varphi \hat{\varphi}$

4.4 $\frac{R_0}{R}\hat{\varphi}\cdot\nabla\times\frac{\partial \mathbf{v}}{\partial t} \Rightarrow \frac{\partial}{\partial t}\Delta^\dagger U$

$$
\begin{aligned}
\frac{\partial}{\partial t}\Delta^\dagger U \quad = \quad & \epsilon R\nabla_\perp U\times\nabla_\perp(\Delta^\dagger U)\cdot\hat{\varphi} - \nabla_\perp\chi\cdot\nabla_\perp(\Delta^\dagger U) - \Delta^\dagger U(2\epsilon\frac{\partial U}{\partial Z} + \Delta^\dagger\chi) \\
& -\frac{v_\varphi}{R}\frac{\partial}{\partial\varphi}\Delta^\dagger U - \nabla_\perp(\frac{v_\varphi}{R})\cdot\nabla_\perp(\frac{\partial U}{\partial\varphi}) \\
& +2R_0\frac{v_\varphi}{R}\frac{\partial}{\partial Z}\frac{v_\varphi}{R} + \frac{R_0}{R}\nabla_\perp(\frac{v_\varphi}{R})\times\nabla_\perp(\frac{\partial\chi}{\partial\varphi})\cdot\hat{\varphi} \\
& +R_0[\mathbf{B}\cdot\nabla(\frac{C}{d}) + \mathbf{J}\cdot\nabla(\frac{I}{d})] \\
& +\frac{2}{d}\frac{\partial p}{\partial Z} + R\nabla_\perp\frac{1}{d}\times\nabla_\perp p\cdot\hat{\varphi} \\
& -R_0\nabla\varphi\cdot\nabla\times(\mu\frac{R^2}{d}\nabla^2\mathbf{v}) \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\varphi\cdot\nabla\times R\nabla^2\mathbf{v} \quad &= \quad \nabla\cdot(\hat{\varphi}\times\nabla^2\mathbf{v}) \\
&= \quad \epsilon R\nabla^2(\Delta^\dagger U) - \frac{1}{R}\frac{\partial}{\partial Z}(\nabla^2\chi + \frac{1}{R}\frac{\partial\chi}{\partial R}) - \frac{1}{R^2}\frac{\partial v'_\varphi}{\partial Z} - \frac{\epsilon}{R}\frac{\partial^2 U}{\partial Z^2}
\end{aligned}
$$

4.5 $\hat{\varphi}\cdot\frac{\partial\mathbf{v}}{\partial t} \Rightarrow \frac{\partial v_\varphi}{\partial t}$

$$
\begin{aligned}
\frac{\partial v_\varphi}{\partial t} \quad = \quad & \epsilon R\nabla_\perp U\times\nabla_\perp v_\varphi\cdot\hat{\varphi} - \nabla_\perp\chi\cdot\nabla_\perp v_\varphi - \frac{v_\varphi}{R}[\epsilon R\frac{\partial U}{\partial Z} + \frac{\partial\chi}{\partial R} + \frac{\partial v_\varphi}{\partial\varphi}] \\
& +\frac{1}{d}[\nabla_\perp\tilde{I}\cdot\nabla_\perp F - \frac{1}{R}(\nabla_\perp F'\cdot\nabla_\perp F + \nabla_\perp\psi'\cdot\nabla_\perp\psi) \\
& +\nabla_\perp\tilde{I}\times\nabla_\perp\psi\cdot\hat{\varphi} + \frac{1}{R}\nabla_\perp\psi'\times\nabla_\perp F\cdot\hat{\varphi} + \frac{1}{R}\nabla_\perp F'\times\nabla_\perp\psi\cdot\hat{\varphi}] \\
& -\epsilon\frac{R}{d}\frac{\partial p}{\partial\varphi} + \hat{\varphi}\cdot(\mu\frac{R^2}{d}\nabla^2\mathbf{v}) \qquad (15)
\end{aligned}
$$

$$
\hat{\varphi}\cdot\nabla^2\mathbf{v} = \nabla^2 v_\varphi + \frac{2}{R^2}(\frac{\partial\chi'}{\partial R} + \epsilon R\frac{\partial U'}{\partial Z}) - \frac{v_\varphi}{R^2}
$$

4.6 $\nabla_\perp\cdot\frac{\partial\mathbf{v}}{\partial t} \Rightarrow \frac{\partial}{\partial t}(\frac{\partial\chi}{\partial R}),\ \ \frac{\partial}{\partial t}(\frac{\partial\chi}{\partial Z})$

$$
\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial\chi}{\partial R}) \quad = \quad & -\epsilon R\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial t}\right) - \mathbf{v}_\perp\cdot\nabla_\perp\left(\frac{\partial\chi}{\partial R} + \epsilon R\frac{\partial U}{\partial z}\right) \\
& -\epsilon v_\varphi\frac{\partial U'}{\partial z} - \frac{v_\varphi}{R}\frac{\partial\chi'}{\partial R} + \frac{v_\varphi^2}{R} - \frac{R^2}{d}\frac{\partial p}{\partial R} \\
& +\frac{1}{d}\left(\frac{1}{\epsilon} + \tilde{I}\right)\left[\frac{1}{R}\left(\frac{\partial F'}{\partial R} + \frac{\partial\psi'}{\partial z}\right) - \frac{\partial\tilde{I}}{\partial R}\right] + \frac{C}{d}\left(\frac{\partial F}{\partial z} - \frac{\partial\psi}{\partial R}\right) + \hat{R}\cdot\mu\frac{R^2}{d}\nabla^2\mathbf{v} \qquad (16)
\end{aligned}
$$

$$\hat{R}\cdot\nabla^2\mathbf{v} = \nabla^2(\frac{\partial\chi}{\partial R} + \epsilon R\frac{\partial U}{\partial Z}) - \frac{1}{R^2}(\frac{\partial\chi}{\partial R} + \epsilon R\frac{\partial U}{\partial Z}) - \frac{2}{R^2}\frac{\partial v_\varphi}{\partial\varphi}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\frac{\partial\chi}{\partial Z}) \quad = \quad & \epsilon R\frac{\partial}{\partial R}\left(\frac{\partial U}{\partial t}\right) - \mathbf{v}_\perp\cdot\nabla_\perp\left(\frac{\partial\chi}{\partial z} - \epsilon R\frac{\partial U}{\partial R}\right) \\ & +\epsilon v_\varphi\frac{\partial U'}{\partial R} - \frac{v_\varphi}{R}\frac{\partial\chi'}{\partial z} - \frac{R^2}{d}\frac{\partial p}{\partial z} \\ & +\frac{1}{d}\left(\frac{1}{\epsilon} + \tilde{I}\right)\left[\frac{1}{R}\left(\frac{\partial F'}{\partial z} - \frac{\partial\psi'}{\partial R}\right) - \frac{\partial\tilde{I}}{\partial z}\right] - \frac{C}{d}\left(\frac{\partial F}{\partial R} + \frac{\partial\psi}{\partial z}\right) + \hat{z}\cdot\mu\frac{R^2}{d}\nabla^2\mathbf{v}\end{aligned} \tag{17}$$

$$\hat{Z}\cdot\nabla^2\mathbf{v} = \nabla^2(\frac{\partial\chi}{\partial Z} - \epsilon R\frac{\partial U}{\partial R})$$

4.7 Density d

$$\frac{\partial d}{\partial t} \quad = \quad -\{d[\Delta^*\chi + \frac{1}{R}\frac{\partial v_\varphi}{\partial\varphi}] + \epsilon R\nabla_\perp d\times\nabla_\perp U\cdot\hat{\varphi} + \nabla_\perp\chi\cdot\nabla_\perp d + \frac{v_\varphi}{R}\frac{\partial d}{\partial\varphi} \tag{18}$$

4.8 Pressure p

$$\begin{aligned}\frac{\partial p}{\partial t} \quad = \quad & \epsilon R\nabla_\perp U\times\nabla_\perp p\cdot\hat{\varphi} - \nabla_\perp\chi\cdot\nabla_\perp p - \frac{v_\varphi}{R}\frac{\partial p}{\partial\varphi} \\ & -\gamma p[\Delta^\dagger\chi + 2\epsilon\frac{\partial U}{\partial Z} + \frac{1}{R}\frac{\partial v_\varphi}{\partial\varphi}] \\ & +d\nabla\cdot\kappa\cdot\nabla\left(\frac{p}{d}\right)\end{aligned} \tag{19}$$

$\mathbf{B}\cdot\nabla p\rightarrow 0$ is accelerated by solving

$$\left\{\begin{array}{l}\frac{\partial p}{\partial t} = \mathbf{B}\cdot\nabla v_a \\ \frac{\partial v_a}{\partial t} = \mathbf{B}\cdot\nabla p\end{array}\right. \tag{20}$$

for an artificial "sound" velocity v_a.

5 Quasi-implicit treatment of certain terms for stability

The time evolution equations that are of the form

$$\frac{\partial A}{\partial t} = g(R, Z, \varphi, t)\ \nabla_{\perp}^{2} A + \cdots \tag{21}$$

are treated quasi-implicitly.$g(R, Z, \varphi, t)$ is a multiplier of $\nabla_{\perp}^{2} A$ and has different forms in different equations.

Advancing time from t to $t + \Delta t$ time level

$$t^{n+1} = t^{n} + \Delta t$$

we have

$$\frac{\partial A}{\partial t}|^{n} = \frac{(A^{n+1} - A^{n})}{\Delta t}$$

and the solution of eqn.(21) becomes the solution of the following poission-type equation for A^{n+1}

$$A^{n+1} - \Delta t \cdot g(R, Z, \varphi, t)\ \nabla_{\perp}^{2} A^{n+1} = A^{n} + [\cdots]^{n}$$

i.e.

$$\left(I - \Delta t \cdot g(R, Z, \varphi, t) \nabla_{\perp}^{2}\right)\ \ A^{n+1} = A^{n} + [\cdots]^{n} \tag{22}$$

If we go one step further

$$\left(\frac{I}{\Delta t \cdot g(R, Z, \varphi, t)} - \nabla_{\perp}^{2}\right)\ \left(\Delta t \cdot g(R, Z, \varphi, t)\right)\ \ A^{n+1} = A^{n} + [\cdots]^{n}$$

i.e.,

$$\left(\nabla_{\perp}^{2} - \frac{I}{\Delta t \cdot g(R, Z, \varphi, t)}\right)\ \left(\Delta t \cdot g(R, Z, \varphi, t)\right)\ \ A^{n+1} = -(A^{n} + [\cdots]^{n}) \tag{23}$$

or

$$\left(\nabla_{\perp}^{2} - \frac{I}{\Delta t \cdot g(R, Z, \varphi, t)}\right)\ \left(g(R, Z, \varphi, t)\right)\ \ A^{n+1} = \frac{-(A^{n} + [\cdots]^{n})}{\Delta t} \tag{24}$$

The poission operator

$$\left(\nabla_{\perp}^{2} - \frac{I}{\Delta t \cdot g(R, Z, \varphi, t)}\right)$$

is inverted by the following subroutions, which vary since $g(R, Z, \varphi, t)$ varies.

- $poisvmu(aa, bb, isu, ibc, dtt, ss)$ solves

  isu=1 : (del.del.- 1/dtt/ss(r))(ss(r)*aa) = bb

  isu=0 : (del.del.- 1/dtt/ss(r))aa = bb

  b.c. ibc = 0 : a(ld,k) = 0.0

  ibc = 1 ; a(ld,k) = a(ldb,k)

  ibc = 2 : a(ld,k) from wa(ld,k)

  bb(l, k) = -bb(l, k) * (dtt * ss(l)); wss1(l, k) = ss(l)

  call poismc(aa, bb, wss1, dtt, ibc, 0)

  if(isu.eq.1) aa(l,k) = aa(l,k)/ss(l)

- $poisvmu0(aa, bb, isu, ibc, dtt, ss)$

  call $poisvmu(aa, bb, isu, ibc, dtt, ss)$

- $poisvmu3(aa, bb, isu, ibc, dtt, ss)$ solves

  isu=1 : (del.del.- 1/dtt/ss(r))(ss(r)*aa) = bb

  isu=0 : (del.del.- 1/dtt/ss(r))aa = bb

  b.c. ibc = 0 ; a(ld,k) = 0.0

  ibc = 1 ; a(ld,k) = a(ldb,k)

  ibc = 2 ; a(ld,k) from wa(ld,k)

  if(isu .eq. 1) bb(l, k) = -bb(l, k) * dtt;   else bb(l, k) = -bb(l, k) * (dtt * ss(l,k))

  call $poismc(aa, bb, ss, dtt, ibc, 0)$

  if(isu.eq.1) aa(l,k) = aa(l,k)/ss(l,k)

- $poisvmun(aa, bb, isu, ibc, dtt, ss)$ solves

  isu=1 : (del.del.- 1/dtt/ss(r,k)/hmt)(ss(r,k)*aa) = bb

  isu=0 : (del.del.- 1/dtt/ss(r,k)/hmt)aa = bb

  b.c. ibc = 0 ; a(ld,k) = 0.0

  ibc = 1 ; a(ld,k) = a(ldb,k)

  ibc = 2 ; a(ld,k) from wa(ld,k)

  call $poisvmu3(aa, bb, isu, ibc, dtt, ss)$

- $poisdmd(aa, bb, ibc, dtt, ss)$ solves

  (del.ss(r)del.- 1/dtt/hmt)aa = bb

  b.c. ibc = 0 ; a(ld,k) = 0.0

  ibc = 1 ; a(ld,k) = a(ldb,k)

  ibc = 2 ; a(ld,k) from wa(ld,k)

  ibc = 3 ; the second gradient at the wall approximatly 0

  call $poismc(aa, bb, wss1, dtt, ibc, 2)$

- $poisdmd0(aa, bb, ibc, dtt, ss)$ solves

  (del.ss(r)del.- 1/dtt)aa = bb

  b.c. ibc = 0 ; a(ld,k) = 0.0

  ibc = 1 ; a(ld,k) = a(ldb,k)

ibc = 2 ; a(ld,k) from wa(ld,k)

call $poisdmd(aa, bb, ibc, dtt, ss)$

- $poisdmd\_n(aa, bb, ibc, dtt, ss)$ solves

  (del.ss(r)del.- 1/dtt/hmt)aa = bb

  b.c. ibc = 0 ; a(ld,k) = 0.0

  ibc = 1 ; a(ld,k) = a(ldb,k)

  ibc = 2 ; a(ld,k) from wa(ld,k)

  ibc = 3 ; the second gradient at the wall approximatly 0

  bb(l, k) = bb(l, k) / ss(l)

  call $poisvmu(aa, bb, 0, ibc, dtt, ss)$

- $lopoismu(aa, bb, isu, ibc, dtt, ss)$ solves

  isu=1 : (del.(1/R)del.- 1/(dtt*R*ss))(ss(r)*aa) = bb/R

  isu=0 : (del.(1/R)del.- 1/(dtt*R*ss))aa = bb/R

  b.c. ibc = 0 ; a(ld,k) = 0.0

  ibc = 1 ; a(ld,k) = a(ldb,k)

  ibc = 2 ; a(ld,k) from wa(ld,k)

  bb(l, k) = -bb(l, k) * (dtt * ss(l)); wss1(l, k) = ss(l)

  call poismstc(aa, bb, wss1, dtt, ibc, 0)

  if(isu.eq.1) $aa(l, k) = aa(l, k)/ss(l)$

- $lopoisma(aa, bb, isu, ibc, dtt, ss)$ solves

  isu=1 : (del.(R)del.- R/(dtt*ss))(ss(r)*aa) = bb*R

  isu=0 : (del.(R)del.- R/(dtt*ss))aa = bb*R

  b.c. ibc = 0 ; a(ld,k) = 0.0

  ibc = 1 ; a(ld,k) = a(ldb,k)

  ibc = 2 ; a(ld,k) from wa(ld,k)

  ib0 = 3; if(ibc .eq. 1) ib0 = 4

  if(isu .eq. 1) bb(l, k) = -bb(l, k) * dtt; else bb(l, k) = -bb(l, k) * (dtt * ss(l))

  wss1(l, k) = ss(l)

  call $poismstc(aa, bb, wss1, dtt, ibc, 0)$

  if(isu.eq.1) $aa(l, k) = aa(l, k)/ss(l)$

- $poiss(aa, bb, ibc, ibc2)$

  Laplacian (wa) = wb, i.e., S * wa = M * wb

  ibc1 = ibc; if(ibc = 3) ibc1 = 1; if(ibc = 4) ibc1 = 1

  call $poissc(aa, bb, ibc1)$

- $lowpois(rneg0, aa, bb)$ solve for u iteratively.

  del squared(u) = known; u = 0. on the boundary

  The full del operator is used: bb = del+ aa = (1/R) (del . R del) aa (R = rmajor + xH(l))

  $callpoisdagc(aa, bb, 0)$

- $lowpoisa(rneg0, aa, bb)$ solves for a iteratively

  del star(a) = known current; a = given in main prog. on the boundary

  $callpoistarc(aa, bb, 0)$

6 Subroutines related to the 7 time-dependent equations

Vaiables to be solved

$rh; u, w, lap\_u, dudt; si, bigi, p, chi, chi\_x, chi\_y, lap\_chi; vphi; xphi, lap\_phi; c, c_a, a, lap\_a; bigf, lap\_F$

6.1 RHOEQN solving (18) for d (m3d/code/m1.F)

$$\frac{\partial d}{\partial t} = -RHS + pkkk \cdots$$

slove it explicitly

$$d^{n+1} = d^n - \Delta t \cdot RHS^n$$

where $RHS^n =$

$$d^n [(\Delta^* \chi)^n_{=((\nabla_\perp \chi)^n - \frac{1}{R}\frac{\partial \chi^n}{\partial R})} + \frac{1}{R}\frac{\partial v^n_\varphi}{\partial \varphi}] + \epsilon R \nabla_\perp d^n \times \nabla_\perp U^n \cdot \hat{\varphi} + \nabla_\perp \chi^n \cdot \nabla_\perp d^n + \frac{v^n_\varphi}{R}\frac{\partial d^n}{\partial \varphi}$$

Used variables: $\underline{chi, lap\_chi, vphi, rh, u}$ at nth time leve.

1. form $RHS^n \Rightarrow tally$
2. form $d^n - \Delta t \cdot RHS^n \Rightarrow rh$
3. if $pdissf$ is turned on
   form $\frac{-rh + R^2/R_0^2}{\Delta t \cdot pdissf} \Rightarrow wb$
   call $poisvmu(wa, wb, 0, 0, \Delta t, pdissf)$ in mpar1.F to solve

   $$d^{n+1} - \Delta t \cdot pdissf \nabla_\perp^2 d^{n+1} = rh$$

   $$(\nabla_\perp^2 - \frac{I}{\Delta t \cdot pdissf}) d^{n+1} = \frac{-rh}{\Delta t \cdot pdissf}$$

   for wa
   form new d^{n+1} : $rh = wa + R^2/R_0^2$
4. fix rh at origin by calling fixo(rh). rh now is the newly updated d^{n+1}
5. call defvar to define variable before each time step.

6.2 WEQN solving (14) for $w = \Delta^\dagger U$ (in m3d/code/m1.F)

call weqn(w,wo,sio,ao,po,doo,qo,chio,vphio,dt,1,dudt)
weqn (wnow,wold,einow,aold,pold,dnow,qold,chinow,vphinow,deltat,istep,dutt)

$$\frac{\partial w}{\partial t} = RHS + \mu \nabla_\perp^2 w$$

slove it explicitly

$$w^{n+1} = w^n + \Delta t \cdot RHS^n$$

where $RHS^n =$

$$R_0[\mathbf{B}^n \cdot \nabla(\frac{C^n}{d_i^n}) + R_0\mathbf{J}^n \cdot \nabla(\frac{I^n}{d_i^n})] + R\nabla_\perp \frac{1}{d_i^n} \times \nabla_\perp p^n \cdot \hat{\varphi}$$

$$+\epsilon R\nabla_\perp U^n \times \nabla_\perp w^n \cdot \hat{\varphi} + \frac{2}{d_i^n}\frac{\partial p^n}{\partial Z} - w^n(2\epsilon\frac{\partial U^n}{\partial Z} + (\Delta^\dagger \chi)^n_{=((\nabla_\perp \chi)^n + \frac{1}{R}\frac{\partial \chi^n}{\partial R}))} - \nabla_\perp \chi^n \cdot \nabla_\perp w^n$$

$$+2R_0\frac{v_\varphi^n}{R}\frac{\partial}{\partial Z}\frac{v_\varphi^n}{R} - \frac{v_\varphi^n}{R}\frac{\partial w^n}{\partial \varphi} - \nabla_\perp(\frac{v_\varphi^n}{R}) \cdot \nabla_\perp(\frac{\partial U^n}{\partial \varphi}) + \frac{R_0}{R}\nabla_\perp(\frac{v_\varphi^n}{R}) \times \nabla_\perp(\frac{\partial \chi^n}{\partial \varphi}) \cdot \hat{\varphi}$$

Here I am not sure d_i is $\frac{1}{d^n}$ or $\frac{1}{d^{n+1}}$
Used variables: $\underline{rhi, c, bigi, po, u, wo, lap\_chi, chio, vphio}$ at nth time leve.

1. form RHS^n
2. form $w^n + \Delta t \cdot RHS^n$ and save it in variable $wnow$
3. $fix(wnow)$. $wnow$ is the newly updated w^{n+1}.
4. if rmu and ibl3 (i.e., low β option) are turned on
   form $\Delta t \cdot rmu \cdot \frac{1}{R}\frac{\partial wnow}{\partial R}$ and add it to $wnow$. w^{n+1} is updated agian.
5. if rmu is turned on
   form $\frac{-wnow}{\Delta t \cdot hmt}$ as a variable wb
   call $poisvmu(wnow, wb, 1, ibx, \Delta t, rmus)$ in mpar1.F to solve

   $$w^{n+1} - \Delta t \cdot \mu \nabla_\perp^2 w^{n+1} = wnow$$

   $$(\nabla_\perp^2 - \frac{I}{\Delta t \cdot \mu})(\mu w^{n+1} = \frac{-wnow}{\Delta t}$$

   for $wnow$, i.e., w^{n+1} which is updated again

6. fixo(wnow). $wnow$ is the newly updated w^{n+1}, i.e., $(\Delta^\dagger U)^{n+1}$,
7. save $u^n \Rightarrow dutt$
8. call $lowpois(\frac{1}{R}, u, wnow)$ to solve

   $$\Delta^\dagger U = wnow$$

   for u^{n+1}
9. calculate $\frac{U^{n+1} - U^n}{\Delta t}$, i.e., $(\frac{\partial U}{\partial t})^{n+1} \Rightarrow dutt$, which is $dudt$ outside of $WEQN$
10. set $u^{n+1} = 0, w^{n+1} = 0$ on the boundary or ghost cells which are not clear right now
11. form lap_u, i.e., $(\nabla_\perp^2 U)^{n+1}$, from $wnow - \frac{1}{R}\frac{\partial u^{n+1}}{\partial R}$, i.e., $(\Delta^\dagger u)^{n+1} - \frac{1}{R}\frac{\partial u^{n+1}}{\partial R}$

6.3 CHIAIAP_3 solving (11), (16)-(17), (19) for $(\tilde{I}, \chi, p)$ (in m3d/code/m1.F)

call chiaiap_3(chio,sio,ao,po,peo,vphio,wo,dudt,chi,si,p,dt)
chiaiap_3(chit,sit,aa,pt,pet,vpht,ww,duut,chin,sinn,pn,sdt)

Rewriting the evolution equations as

$$\frac{\partial p}{\partial t} = \mathcal{S}_p - \gamma p \Delta^\dagger \chi \tag{25}$$

$$\frac{\partial \tilde{I}}{\partial t} = \mathcal{J}_2 - I\Delta^* \chi + \eta \Delta^* I \tag{26}$$

$$\frac{\partial}{\partial t}(\frac{\partial \chi}{\partial R}) = \mathcal{J}_4 - \frac{R_0^2}{d}\frac{\partial}{\partial R}(\frac{I^2}{2}) - \frac{R^2}{d}\frac{\partial p}{\partial R} + \mu \frac{R^2}{d}\nabla_\perp^2(\frac{\partial \chi}{\partial R}) \tag{27}$$

$$\frac{\partial}{\partial t}(\frac{\partial \chi}{\partial Z}) = \mathcal{J}_3 - \frac{R_0^2}{d}\frac{\partial}{\partial Z}(\frac{I^2}{2}) - \frac{R^2}{d}\frac{\partial p}{\partial Z} + \mu \frac{R^2}{d}\nabla_\perp^2(\frac{\partial \chi}{\partial Z}) \tag{28}$$

solve them quasi-implicitly

$$(\frac{\partial \chi}{\partial R})^{n+1} = (\frac{\partial \chi}{\partial R})^{n} + \Delta t(\mathcal{J}_4^n - \frac{R_0^2}{d}\frac{\partial}{\partial R}(\frac{I^2}{2})^{n+\frac{1}{2}} - \frac{R^2}{d}\frac{\partial p^{n+\frac{1}{2}}}{\partial R} + \mu \frac{R^2}{d}\nabla_\perp^2(\frac{\partial \chi}{\partial R})^{n+1}) \tag{29}$$

$$(\frac{\partial \chi}{\partial Z})^{n+1} = (\frac{\partial \chi}{\partial Z})^{n} + \Delta t(\mathcal{J}_3^n - \frac{R_0^2}{d}\frac{\partial}{\partial Z}(\frac{I^2}{2})^{n+\frac{1}{2}} - \frac{R^2}{d}\frac{\partial p^{n+\frac{1}{2}}}{\partial Z} + \mu \frac{R^2}{d}\nabla_\perp^2(\frac{\partial \chi}{\partial Z})^{n+1}) \tag{30}$$

1. $its = 0, rhdo = rhi$

2. $\Delta t \mathcal{J}_4^n$ is formed under "calculate xjay4" and saved as xjay4, where

$$\begin{aligned}
\mathcal{J}_4^n &= -\epsilon R \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial t}\right)^{n+1} - \epsilon v_\varphi^n \frac{\partial U^{n+1'}}{\partial z} - \frac{v_\varphi^n}{R}\frac{\partial \chi^{n'}}{\partial R} + \frac{v_\varphi^n 2}{R} \\
&+ \frac{1}{d_i^n}\left(\frac{1}{\epsilon} + \tilde{I}^n\right)\frac{1}{R}\left(\frac{\partial F^{n'}}{\partial R} + \frac{\partial \psi n'}{\partial z}\right) + \frac{C^n}{d_i^n}\left(\frac{\partial F^n}{\partial z} - \frac{\partial \psi^n}{\partial R}\right) \\
&- \mathbf{v}_\perp \cdot \nabla_\perp \left(\frac{\partial \chi}{\partial R} + \epsilon R \frac{\partial U}{\partial z}\right) - \frac{R^2}{d}\frac{\partial p}{\partial R} - \frac{R_0^2}{d}\frac{\partial}{\partial R}\frac{I^2}{2} + \hat{R}\cdot \mu \frac{R^2}{d}\nabla^2 \mathbf{v}
\end{aligned} \tag{31}$$

Used variables:$\underline{dudt, u, w, chio, c, vphio, bigf, bigi, ao, sio, pold, ri = 1 + \epsilon \cdot sio}$. $\frac{R^2}{d}\frac{\partial p}{\partial R}$ is formed as $\frac{R^2}{d_i}\frac{\partial p}{\partial R} - \frac{R^2}{d_o}\frac{\partial p}{\partial R}$, and therefore cancelled. similarly, $\frac{R_0^2}{d}\frac{\partial}{\partial R}\frac{I^2}{2}$.

3. $\Delta t \mathcal{J}_3^n$ is formed under "calculate xjay3" and saved as xjay3, where

$$\begin{aligned}
\mathcal{J}_3^n &= \epsilon R \frac{\partial}{\partial R}\left(\frac{\partial U}{\partial t}\right)^{n+1} + \epsilon v_\varphi^n \frac{\partial U^{n+1'}}{\partial R} - \frac{v_\varphi^n}{R}\frac{\partial \chi^{n'}}{\partial z} \\
&+ \frac{1}{d^n}\left(\frac{1}{\epsilon} + \tilde{I}^n\right)\frac{1}{R}\left(\frac{\partial F^{n'}}{\partial z} - \frac{\partial \psi^{n'}}{\partial R}\right) - \frac{C^n}{d^n}\left(\frac{\partial F^n}{\partial R} + \frac{\partial \psi^n}{\partial z}\right) \\
&- \mathbf{v}_\perp \cdot \nabla_\perp \left(\frac{\partial \chi}{\partial z} - \epsilon R \frac{\partial U}{\partial R}\right) - \frac{R^2}{d}\frac{\partial p}{\partial z} - \frac{R_0^2}{d}\frac{\partial}{\partial Z}\frac{I^2}{2} + \hat{z}\cdot \mu \frac{R^2}{d}\nabla^2 \mathbf{v}
\end{aligned} \tag{32}$$

Used variables:$\underline{dudt, u, w, chio, c, vphio, bigf, bigi, ao, sio, pold, ri = 1 + \epsilon \cdot sio}$
$\frac{R^2}{d}\frac{\partial p}{\partial Z}$ is formed as $\frac{R^2}{d_i}\frac{\partial p}{\partial Z} - \frac{R^2}{d_o}\frac{\partial p}{\partial Z}$, and therefore cancelled. similarly, $\frac{R_0^2}{d}\frac{\partial}{\partial Z}\frac{I^2}{2}$.

4. starting from "calculate xjay6"
   Because $\frac{R_0^2}{2}\frac{\partial I^2}{\partial t} = R_0^2 I \frac{\partial I}{\partial t} = R_0 I \frac{\partial \tilde{I}}{\partial t}$, we have

$$\frac{1}{2}\frac{\partial I^2}{\partial t} = I\epsilon\frac{\partial \tilde{I}}{\partial t} = I\epsilon(\mathcal{J}_2 - \frac{I}{\epsilon}\Delta^*\chi + \eta\Delta^* I)$$

   so we first solve

$$(\frac{I^2}{2})^{n+\frac{1}{2}} = (\frac{I^2}{2})^n + \Delta t \cdot I^n \epsilon \frac{\tilde{I}^{n+\frac{1}{2}} - \tilde{I}^n}{\Delta t}$$

5. term1 $(\frac{I^2}{2})^n = \frac{(1+\epsilon\tilde{I}^n)}{2} \Rightarrow tally$

6. term2 $\frac{1}{R}\frac{\partial}{\partial R}\chi^n \Rightarrow fun$

7. call $JAY2$ to form $\epsilon\mathcal{J}_2^n$ and save it to $xjay2$.
   Used variables:$\underline{chio, sio, ao, vphio, peo, u, bigf}$

8. if η is turned on, we solve $\tilde{I}$ equation

$$\tilde{I}^{n+\frac{1}{2}} - \Delta t \cdot \eta\nabla_\perp^2 \tilde{I}^{n+\frac{1}{2}} = \tilde{I}^n + \Delta t \mathcal{J}_2^n$$

   where

$$\begin{aligned} \mathcal{J}_2^n \quad = \quad & \epsilon R\nabla_\perp U^{n+1} \times \nabla_\perp \tilde{I}^n \cdot \hat{\varphi} - \nabla_\perp \chi^n \cdot \nabla_\perp \tilde{I}^n + R\nabla_\perp(\frac{v_\varphi^n}{R}) \times \nabla_\perp \psi^n \cdot \hat{\varphi} \\ & + R\nabla_\perp F^n \cdot \nabla_\perp(\frac{v_\varphi^n}{R}) - \frac{v_\varphi^n}{R}\frac{\partial \tilde{I}^n}{\partial \varphi} + \eta[\Delta^*(\tilde{I}^n - sy) + \frac{1}{R^2}\tilde{I}^{n''} + \frac{2}{R^2}(\frac{\partial F^{n'}}{\partial R} + \frac{\partial \psi^{n'}}{\partial Z})] \\ & + \nabla_\perp \eta \cdot [\nabla_\perp(\tilde{I}^n - sy) - \frac{1}{R}\nabla_\perp F^{n'}] - \frac{1}{R}\nabla_\perp \eta \times \nabla_\perp \psi^{n'} \cdot \hat{\varphi} \\ & - (\frac{1}{\epsilon} + \tilde{I})\Delta^*\chi \end{aligned} \tag{33}$$

   form $\tilde{I}^n + \Delta t \mathcal{J}_2^n - \Delta t[\eta\nabla_\perp^2(\tilde{I} - sy) + \nabla_\perp\eta \cdot \nabla_\perp(\tilde{I} - sy)] \Rightarrow fun8$.
   The last 2 terms cancell the same terms formed inside $\epsilon\mathcal{J}_2^n$
   fixo(fun8)
   $fun8 = \tilde{I}^n - sy$ on the boundary or ghost cells
   form $hpt\frac{-fun8+sy}{\Delta t} \Rightarrow wb$
   call $poisvmu(fun, wb, 1, ibx, \Delta t, \eta)$ to solve

$$(\nabla_\perp^2 - \frac{I}{\Delta t \cdot \eta})(\eta \tilde{I}^{n+\frac{1}{2}}) = \frac{-fun8 + sy}{\Delta t}$$

for fun
fixo(fun)
form $fun + sy \Rightarrow \tilde{I}^{n+\frac{1}{2}}$
form $\epsilon \frac{\tilde{I}^{n+\frac{1}{2}} - \tilde{I}^n}{\Delta t} \Rightarrow XJAY2$

9. form $I^n \cdot xjay2$
form $\Delta t I^n \cdot xjay2$
form $(\frac{I^2}{2})^n + \Delta t I^n \cdot xjay2$ and save in $\mathcal{J}_6$, which is newly updated $(\frac{I^2}{2})^{n+\frac{1}{2}}$
fixo(xjay6)

10. starting from "calculate xjay7"
call $PEQN$ to deal with

$$p^{n+\frac{1}{2}} = p^n + \Delta t(\mathcal{S}_p^n - \gamma p \Delta^\dagger \chi)$$

where

$$\begin{aligned} \mathcal{S}_p^n &= \epsilon R \nabla_\perp U^{n+1} \times \nabla_\perp p^n \cdot \hat{\varphi} - \gamma p^n [\Delta^\dagger \chi^n + 2\epsilon \frac{\partial U^{n+1}}{\partial Z} + \frac{1}{R} \frac{\partial v_\varphi^n}{\partial \varphi}] \\ &\quad - \frac{v_\varphi^n}{R} \frac{\partial p^n}{\partial \varphi} - \nabla_\perp \chi^n \cdot \nabla_\perp p^n + d \nabla \cdot \kappa \cdot \nabla \left(\frac{p}{d} \right) \end{aligned} \tag{34}$$

saved in pn. Used variables: $\underline{u, po, lap\_chi, vphio, chio}$

11. form $\Delta t \gamma p^n \cdot hmt \cdot (\nabla_\perp^2 \chi)^n$ and add it to pn to cancell the same term added in $PEQN$ save the reesult in $xjay7$.
fixo(xjay7). $xjay7$ is the newly updated $p^{n+\frac{1}{2}}$

12. under "calculate xjay8" as term2 and term3
form $-\Delta t \frac{R_0^2}{d_o^n} \frac{\partial}{\partial R} (\frac{I^2}{2})^{n+1}$ and add to xjay4
form $-\Delta t \frac{R_0^2}{d_o^n} \frac{\partial}{\partial Z} (\frac{I^2}{2})^{n+1}$ and add to xjay3
form $-\Delta t \epsilon \frac{R^2}{d_o^n} \frac{\partial \bar{p}}{\partial R}^{n+1}$ and add to $xjay4$
form $-\Delta t \epsilon \frac{R^2}{d_o^n} \frac{\partial \bar{p}}{\partial Z}^{n+1}$ and add to xjay3
$\bar{p}^{n+1} = p^{n+1} - \Delta t \gamma p^n (\Delta_\perp^2 \chi)^n$. Used variables:$\underline{rhdo, lap\_chi}$

13. from term31 to term33,
form $-\frac{1}{2} \Delta t \frac{\partial}{\partial R} [\nabla_\perp U^{n+1} \cdot \nabla_\perp U^{n+1} + \nabla_\perp \chi^n \cdot \nabla_\perp \chi^n + 2\epsilon R \nabla_\perp \chi^n \times \nabla_\perp U^{n+1} \cdot \hat{\varphi}]$ and add to $xjay4$
form $-\frac{1}{2} \Delta t \frac{\partial}{\partial Z} [\nabla_\perp U^{n+1} \cdot \nabla_\perp U^{n+1} + \nabla_\perp \chi^n \cdot \nabla_\perp \chi^n + 2\epsilon R \nabla_\perp \chi^n \times \nabla_\perp U^{n+1} \cdot \hat{\varphi}]$ and add to $xjay3$

14. adding $(\frac{\partial \chi}{\partial R})^n$ to xjay4; $(\frac{\partial \chi}{\partial Z})^n$ to xjay3
Now we finish the the R.H.S. of (31) and (32)

15. form η from $\frac{1}{d_o^n}[\Delta t R_0^2 (I^n)^2 + \Delta t \gamma R_0 \alpha p^n] \cdot hmt \cdot resc9$;
form $-\frac{xjay4}{\Delta t \cdot \eta} \Rightarrow fun7$, and solve
$poisvmu3(chi\_x, fun7, 0, 0, \Delta t, \eta)$ for chi_x, i.e., $(\frac{\partial \chi}{\partial R})^{n+1}$
$poisvmu3(chi\_y, fun7, 0, 0, \Delta t, \eta)$ for chi_y, i.e., $(\frac{\partial \chi}{\partial Z})^{n+1}$

$$(\frac{\partial \chi}{\partial R})^{n+1} - \Delta t \cdot \mu \frac{R^2}{d} \nabla_\perp^2 (\frac{\partial \chi}{\partial R})^{n+1} = xjay4$$

$$(\nabla_\perp^2 - \frac{I}{\Delta t \cdot \mu \frac{R^2}{d}})(\frac{\partial \chi}{\partial R})^{n+1} = \frac{-xjay4}{Deltat \cdot \mu \frac{R^2}{d}}$$

$$(\frac{\partial \chi}{\partial Z})^{n+1} - \Delta t \cdot \mu \frac{R^2}{d} \nabla_\perp^2 (\frac{\partial \chi}{\partial Z})^{n+1} = xjay3$$

$$(\nabla_\perp^2 - \frac{I}{\Delta t \cdot \mu \frac{R^2}{d}})(\frac{\partial \chi}{\partial Z})^{n+1} = \frac{-xjay3}{Deltat \cdot \mu \frac{R^2}{d}}$$

16. call $div(chi\_x, chi\_y, wb)$ to form lap_chi, i.e., $(\nabla_\perp^2 \chi)^{n+1}$ and save in wb

17. call $poiss(chin, wb, 1, 0)$ to solve equation

$$\nabla_\perp^2 \chi = wb$$

for $chin$, i.e., χ^{n+1}

18. call $wgrad(chin, chi\_x, chi\_y, 1)$ to calculate $(\frac{\partial \chi}{\partial R})^{n+1}$ and $(\frac{\partial \chi}{\partial Z})^{n+1}$
call $div(chi\_x, chi\_y, lap\_chi)$ to form lap_chi, i.e., $(\nabla_\perp^2 \chi)^{n+1}$
NOTE: I DON'T UNDERSTAND THIS PART BECAUSE IT'S JUST A REPEAT.

19. call $PEQN$ again to update p^{n+1} by

$$p^n + \Delta t (S_p^n - \gamma p^n (\Delta^\dagger \chi)^{n+1})$$

, where
$(\Delta^\dagger \chi)^{n+1} = (\nabla_\perp^2 \chi)^{n+1} + \frac{1}{R} \frac{\partial \chi^n}{\partial R}$

20. form $-\Delta t I^{n2} \cdot (\nabla_\perp^2 \chi)^{n+1}$, which calcells the same term in p^{n+1}
add to $\mathcal{J}_6(fun6)$, which is the newly updated $(\frac{I^2}{2})^{n+1}$.

21. solution of $\tilde{I}^{n+1}(sinn)$

$$\tilde{I}^{n+1} - \Delta t \cdot \eta \nabla_\perp^2 \tilde{I}^{n+1} = \tilde{I}^n + \Delta t \mathcal{J}_2^{n+\frac{1}{2}} - \Delta t \frac{I^n}{\epsilon} (\Delta^* \chi)^{n+1}$$

form $\tilde{I}^n - \Delta t R_0 I^n (\Delta^* \chi)^{n+1} \Rightarrow tally$, where $(\Delta^* \chi)^{n+1} = (\nabla_\perp^2 \chi)^{n+1} - \frac{1}{R} \frac{\partial \chi^{n+1}}{\partial R}$

22. call $jay2$ to form $\epsilon\mathcal{J}_2^{n+\frac{1}{2}}$
where $\mathcal{J}_2^{n+\frac{1}{2}} =$

$$\epsilon R\nabla_\perp U^{n+1} \times \nabla_\perp \tilde{I}^n \cdot \hat{\varphi} - \nabla_\perp \chi^{n+1} \cdot \nabla_\perp \tilde{I}^n + R\nabla_\perp(\frac{v_\varphi^n}{R}) \times \nabla_\perp \psi^n \cdot \hat{\varphi}$$
$$+R\nabla_\perp F^n \cdot \nabla_\perp(\frac{v_\varphi^n}{R}) - \frac{v_\varphi^n}{R}\frac{\partial \tilde{I}^n}{\partial \varphi} + \eta[\Delta^*(\tilde{I}^n - sy) + \frac{1}{R^2}\tilde{I}^{n''} + \frac{2}{R^2}(\frac{\partial F^{n'}}{\partial R} + \frac{\partial \psi^{n'}}{\partial Z})]$$
$$+\nabla_\perp \eta \cdot [\nabla_\perp(\tilde{I}^n - sy) - \frac{1}{R}\nabla_\perp F^{n'}] - \frac{1}{R}\nabla_\perp \eta \times \nabla_\perp \psi^{n'} \cdot \hat{\varphi}$$
$$-(\frac{1}{\epsilon} + \tilde{I})\Delta^* \chi \tag{35}$$

multiply it with $\Delta t R_0$ (because of ϵ) and add to *tally*

23. form $-\Delta t\eta\nabla_\perp^2(\tilde{I}^n - sy)$ and add it to *tally*
form $-\Delta t\nabla_\perp \eta \cdot \nabla_\perp(\tilde{I}^n - sy)$ and add it to *tally*, which is *sinn*, i.e., $\tilde{I}^{n+1}$, now
NOTE: THESE 2 TERMS CANCELL THE SAME TERMS FORMED IN $\epsilon\mathcal{J}_2^n$

24. fixo(tally)

25. on the boundary $tally = \tilde{I}^n$. So *tally* is newly updated $\tilde{I}^{n+1}$

26. if η is turned on
form wb by $hpt\frac{-tally+sy}{\Delta t}$
call $poisvmu(fun, wb, 1, ibx, \Delta t, \eta)$ to solve

$$(\nabla_\perp^2 - \frac{I}{\Delta t \cdot \eta})(\eta \tilde{I}^{n+1}) = \frac{-tally + sy}{\Delta t}$$

for fun
fixo(fun) form new $\tilde{I}^{n+1}$ by $fun + sy$ fixo(sinn)

27. call *sbigi* to form I^{n+1} from $1 + \epsilon\tilde{I}^{n+1}$
$(bigi = 1 + \epsilon si, siissinn, whichisnewlyupdated\tilde{I}^{n+1})$

6.4 VPHIEQ solving (15) for v_φ (in m3d/code/m1.F)

call vphieq(si,vphi,vphio,ao,p,chi,dt,1)
vphieq(einow,vphinow,vphiold,aold,pold,chinow,deltat,istep)

$$\frac{\partial v_\varphi}{\partial t} = RHS + \mu\nabla_\perp^2 v_\varphi$$

slove it explicitly

$$v_\varphi^{n+1} = v_\varphi^n + \Delta t \cdot RHS^n$$

where

$$
\begin{aligned}
RHS^n &= \frac{1}{d_i^n}[\frac{1}{R}(-\nabla_\perp F^{n'} \cdot \nabla_\perp F^n - \nabla_\perp \psi^{n'} \cdot \nabla_\perp \psi^n + \nabla_\perp \psi^{n'} \times \nabla_\perp F^n \cdot \hat{\varphi} + \nabla_\perp \psi^n \times \nabla_\perp F^{n'} \cdot \hat{\varphi}) \\
&\quad -\epsilon R \frac{\partial p^{n+1}}{\partial \varphi} + \nabla_\perp \tilde{I}^{n+1} \cdot \nabla_\perp F^n + \nabla_\perp \tilde{I}^{n+1} \times \nabla_\perp \psi^n \cdot \hat{\varphi}] \\
&\quad +\epsilon R \nabla_\perp U^{n+1} \times \nabla_\perp v_\varphi^n \cdot \hat{\varphi} - \nabla_\perp \chi^{n+1} \cdot \nabla_\perp v_\varphi^n - \frac{v_\varphi^n}{R}(\epsilon R \frac{\partial U^{n+1}}{\partial Z} + \frac{\partial \chi^{n+1}}{\partial R} + \frac{\partial v_\varphi^n}{\partial \varphi}) \qquad (36)
\end{aligned}
$$

Here $\frac{1}{d_i^n}$ uses rhi. I am not sure rhi is $\frac{1}{d^n}$ or $\frac{1}{d^{n+1}}$; Used variables: $\underline{rhi, bigf, ao, p, u, si, vphio, chi}$ at nth time level

1. form RHS^n
2. form $v_\varphi^n + \Delta t \cdot RHS^n \Rightarrow vphinow$, which is newly updated v_φ^{n+1}
3. if rmu is turned on
   form $\frac{-vphinow+qy}{\Delta t \cdot rmus}$ as a variable wb
   call $poisvmu(wd, wb, 0, ibx, \Delta t, rmus)$ in mpar1.F to solve

   $$v_\varphi^{n+1} - \Delta t \cdot \mu \nabla_\perp^2 v_\varphi^{n+1} = vphinow$$

   $$(\nabla_\perp^2 - \frac{I}{\Delta t \cdot \mu}) v_\varphi^{n+1} = \frac{-vphinow}{\Delta t \cdot \mu}$$

   for wd
   form new v_φ^{n+1} from $vphinow = wd + qy$
4. fix $vphinow$ at origin by calling fixo(vphinow). $vphinow$ is the newly updated v_φ^{n+1}

6.5 PHIEQN solving the Φ equation (9) for Φ and $\nabla_\perp^2 \Phi$ (in m3d/code/m1.F)

call phieqn(sio,vphi,ao,w,chi,peo)
phieqn (einow, vphinow, aold, wnow,chinow,peold)

$$\nabla_\perp^2 \Phi = RHS$$

$$
\begin{aligned}
RHS^n &= \epsilon \nabla_\perp \tilde{I}^n \cdot \nabla_\perp U^{n+1} + I^{n+1}(\nabla_\perp^2 U)^{n+1} - \nabla_\perp(\frac{v_\varphi^{n+1}}{R}) \cdot \nabla_\perp \psi^n - \frac{v_\varphi^{n+1}}{R}(\nabla_\perp^2 \psi)^n \\
&\quad +\nabla_\perp(\frac{v_\varphi^{n+1}}{R}) \times \nabla_\perp F^n \cdot \hat{\varphi} - \frac{\eta}{R^2}[\frac{1}{R}(-\frac{\partial F^{n'}}{\partial Z} + \frac{\partial \psi^{n'}}{\partial R}) + \frac{\partial \tilde{I}^n}{\partial z} - \frac{\partial C^n}{\partial \varphi}] \\
&\quad +\frac{1}{R^2}\nabla_\perp \eta \cdot \nabla_\perp \psi^{n'} - \frac{1}{R^2}\nabla_\perp \eta \times \nabla_\perp F^{n'} \cdot \hat{\varphi} + \frac{1}{R}\nabla_\perp \eta \times \nabla_\perp \tilde{I}^{n+1} \cdot \hat{\varphi} \\
&\quad +R_0 \nabla_\perp \chi^{n+1} \times \nabla_\perp(\frac{1+\epsilon \tilde{I}^n}{R}) \cdot \hat{\varphi} \qquad (37)
\end{aligned}
$$

calculate $RHS^n \Rightarrow wb$
fixo(wb)
form wb as $hpt \cdot wb$
call $poiss(xphi, wb, 0, 0)$ to solve

$$\nabla_\perp^2 \Phi = wb$$

for $xphi$, i.e., Φ^{n+1}.
fixo(xphi), $xphi$ is the newly updated Φ^{n+1}
save wb in lap_phi variable, i.e., $(\nabla_\perp^2 \Phi)^{n+1}$ Used variables: $\underline{sio, u, bigi, lap\_u, vphi, ao, lap\_a, bigf, c, a, si, chi}$

6.6 AEQN/CEQN solving (12)/(13) for ψ/C (in m3d/code/m1.F)

call ceqn(a,ao,chi,sio,peo,dt)
ceqn (anow,aold,chinow,einow,peold,deltat)

$$\frac{\partial C_a}{\partial t} = RHS + \eta \Delta^* C_a$$

slove it explicitly

$$C_a^{n+1} = C_a^n + \Delta t \cdot RHS^n$$

where $RHS^n =$

$$\begin{gathered}
\epsilon R[\nabla_\perp(\Delta^* U)^{n+1} \times \nabla_\perp \psi^n + \nabla_\perp U^{n+1} \times \nabla_\perp C_a^n + 2\nabla_\perp \tfrac{\partial U^{n+1}}{\partial R} \times \nabla_\perp \tfrac{\partial \psi^n}{\partial R} + 2\nabla_\perp \tfrac{\partial U^{n+1}}{\partial Z} \times \nabla_\perp \tfrac{\partial \psi^n}{\partial Z}] \cdot \hat{\varphi} \\
\epsilon R[\nabla_\perp(\Delta^* U)^{n+1} \cdot \nabla_\perp F^n + \nabla_\perp U^{n+1} \cdot \nabla_\perp (\Delta^* F)^n + 2\nabla_\perp(\tfrac{\partial U^{n+1}}{\partial R}) \cdot \nabla_\perp(\tfrac{\partial F^n}{\partial R}) + 2\nabla_\perp(\tfrac{\partial U^{n+1}}{\partial Z}) \cdot \nabla_\perp(\tfrac{\partial F^n}{\partial Z})] \\
-\nabla_\perp(\Delta^* \chi)^{n+1} \cdot \nabla_\perp \psi^n - \nabla_\perp \chi^n \cdot \nabla_\perp C_a^n - 2\nabla_\perp(\tfrac{\partial \chi}{\partial R})^{n+1} \cdot \nabla_\perp(\tfrac{\partial \psi^n}{\partial R}) - 2\nabla_\perp(\tfrac{\partial \chi}{\partial Z})^{n+1} \cdot \nabla_\perp(\tfrac{\partial \psi^n}{\partial Z}) \\
[\nabla_\perp(\Delta^* \chi)^{n+1} \times \nabla_\perp F^n + \nabla_\perp \chi^n \times \nabla_\perp(\Delta^* F)^n + 2\nabla_\perp(\tfrac{\partial \chi}{\partial R})^{n+1} \times \nabla_\perp(\tfrac{\partial F^n}{\partial R}) + 2\nabla_\perp(\tfrac{\partial \chi}{\partial Z})^{n+1} \times \nabla_\perp(\tfrac{\partial F^n}{\partial Z})] \cdot \hat{\varphi} \\
+\tfrac{\partial}{\partial \varphi}(\Delta^* \Phi)^{n+1}{}_{= (\nabla_\perp^2 \Phi)^{n+1} - \frac{1}{R}\frac{\partial \Phi^{n+1}}{\partial R}}
\end{gathered} \tag{38}$$

Used variables: $\underline{ao, u, bigf, w, c_a, lap\_chi, chio, chii\_x, chi\_y, xphi, lap\_phi}$

1. form
$$\frac{\partial \psi^n}{\partial R} = a_x, \frac{\partial U^{n+1}}{\partial R} = u_x, \frac{\partial F^n}{\partial R} = vr$$
$$\frac{\partial \psi^n}{\partial Z} = a_y, \frac{\partial U^{n+1}}{\partial Z} = u_y, \frac{\partial F^n}{\partial Z} = vt$$

2. form RHS^n

3. form $c_a^n + RHS^n$, which is newly updated C_a^{n+1}

4. if η is turned on
   form $etas = cinv \cdot eta \cdot simpf$
   form $C^{n+1} = C_a^{n+1} + \frac{1}{R}\frac{\partial F^n}{\partial Z}$, i.e. $(\Delta^* \psi)^{n+1} + \frac{1}{R}\frac{\partial F^n}{\partial Z}$

- if iuns=1
  form $\eta\frac{-c+cb}{\Delta t}$ as a variable wb
  call $lopoismu(wc, wb, 1, ibx, \Delta t, etas)$ in mpar1.F to solve

$$C_a^{n+1} - \Delta t \cdot \eta \Delta^* C_a^{n+1} = C_a$$

$$(\Delta^* - \frac{I}{\Delta t \cdot \eta}(\eta C_a^{n+1}) = \frac{-C_a + cb}{\Delta t}$$

  for wc
  form new C^{n+1} by $wc + cb$
  form new C_a^{n+1} by $C^{n+1} - \frac{1}{R}\frac{\partial F^n}{\partial Z}$

5. call $lowpoisa(\frac{1}{R}, anow, c_a)$ to solve

$$\Delta^* \psi = C_a$$

   for $anow$, i.e., ψ^{n+1}

6. form C^{n+1} by $C_a^{n+1} + \frac{\partial F^n}{\partial Z}$, i.e., $(\Delta^* \psi)^{n+1} + \frac{\partial F^n}{\partial Z}$

7. form lap_a, i.e., $(\nabla_\perp^2 \psi)^{n+1}$ by $(\Delta^* \psi)^{n+1} + \frac{1}{R}\frac{\partial \psi^{n+1}}{\partial R}$

6.7 FEQN solving the F equation (8) for F and $\nabla_\perp^2 F$ (in m3d/code/m1.F)

call feqn(si,bigf)
feqn (einow, bigft)

calculate the $RHS = -\frac{\tilde{I}^{n+1'}}{R} \Rightarrow wb$
form wb as $hpt \cdot wb$
save $wb \Rightarrow lap\_F$, i.e., $(\nabla_\perp^2 F)^{n+1}$
call $poiss(bigft, wb, 1, 0)$ to solve

$$\nabla_\perp^2 F = wb$$

for $bigft$, i.e., F^{n+1}. Used variables: $\underline{si}$.

7 operations

The fundamental operations in the time evolution equations are

- $+/- \Rightarrow$ ADD, ADDO
  ADD(b,c,bc,cc,ac): ac(l,k) = b*bc(l,k)+c*cc(l,k)
  ADDO(cc,bc,ac): ac(l,k)=cc(l,k) + bc(l,k)
  in m1.F

- multiplication/division of matix by scalars $\Rightarrow$ um02
  um02(b,bb,aa): aa(l,k) = b*bb(l,k)
  in m1.F

- multiplication/division of matix by vector $\Rightarrow$ um12, ud21
  um12(b,bb,aa): aa(l,k) = b(l)*bb(l,k)
  ud21(a,b,c): c(l,k) = a(l,k)/b(l)
  in m1.F

- multiplication of matrix by matrix , i.e., convolution $\Rightarrow$
  CONV, CONVS, WCONVO, WCONVON
  CONV(a,sym1,b,sym2,c), CONVS(a,sym1,b,sym2,c,kst):
  call wconvon(a,b,isy1,isy2,c,kst)
  isy1=1,isy2=1;if(sym1.lt.0.) isy1 = -1, if(sym2.lt.0.) isy2 = -1.
  in m1.F
  WCONVO(cac,dac,isy3,isy4,aac), WCONVON(cac,dac,isy3,isy4,aac,kstt):
  aac = cac d̂ac
  even symmetry:isy3=1 and/or isy4=1; odd symmetry:isy3=-1 and/or isy4=-1
  in mpar1.F

- $\nabla_\perp^2 A \Rightarrow$ DELSQ, DELSQ0
  DELSQ(aa,bb,isy), DELSQ0(aa,bb): $bb = \nabla_\perp^2 aa$, call delsqc(aa,bb)
  in mpar1.F

- $\nabla_\perp A \cdot \nabla_\perp B \Rightarrow$ GRADSQ, GRADSQ0
  GRADSQ(aa,bb,isy1,isy2,cc): $cc = (\nabla_\perp aa \cdot \nabla bb)$, call gradsqc(aa, bb, cc)
  GRADSQ0: (aa,bb,isy1,isy2,cc), call gradsq(aa,bb,isy1,isy2,cc)
  in mpar1.F

- $\nabla_\perp A \times \nabla_\perp B \cdot \varphi \Rightarrow$ GCRO, GCRO0
  GCRO(a,b,isy1,isy2,cc): $cc = \nabla_\perp A \times \nabla_\perp B \cdot \varphi$, call gcroc(a, b, cc)
  GCRO0(a,b,isy1,isy2,cc): call gcro(a,b,isy1,isy2,cc)
  in mpar1.F

- $\mathbf{B} \cdot \nabla g \Rightarrow$ BGRADG
- $\mathbf{J} \cdot \nabla g \Rightarrow$ JGRADG
- $\nabla \cdot \mathbf{A} \Rightarrow$ DIV
  DIV(ax,ay,cc): cc(l,k) = ax' + ay'
  in mpar1.f
- $\frac{R}{R_0} * A \Rightarrow$ ROVERR0 : $wa = \frac{R}{R_0} wb$
- $\frac{\partial}{\partial R}, \frac{\partial}{\partial Z}, \frac{\partial}{\partial \varphi}, \Rightarrow$ DXDR, DXDZ, DXDPHI
  DXDR(symr,xx,aa): aa = dxx/dR
  DXDZ(symr,xx,aa): aa = dxx/dZ
  DXDPHI(x,isym,xx): xx=dx/dphi
  in mpar1.f
- $\nabla_\perp \mathbf{A} \Rightarrow$ WGRAD
  WGRAD(xx,aa,bb,isym): call dxdrc(xx, aa), call dxdzc(xx, bb)
  in mpar1.F
- $(I - \Delta t \nabla_\perp^2) A = RHS \Rightarrow$ POIS...

8 Walkthrough of time advance

This starts from loop 100 and ends at 1555 in mh3d() function all of the 7 time-dependent and 2 poission equations are solved.

8.1 dtset is called to determine the time step at the present iteration

8.2 modifying the initial state

only go through at the first step,
e.g., give some initial perturbation for linear run,
change the strength of rigid ion beam.
call modini

8.3 define variables T, I, F at the beginning of each time step

call defvar

8.4 update resistivity for parallel and perpendicular to B

but we don't do here right now

8.5 fix origin

call fixo(cinv)

8.6 initialize variables which depend on other ones

call sbigi

8.7 advance to the next time level

ncy = ncy+1
tim = tim+dt

8.8 set kmin and set the temporary values at the reference time step

save all of the variables to their corresponding temporary variables
a to ao   u to uo   w to wo   v to vo
p to po   p to peo   rh to rhold
c_a to c_ao   si to sio   chi to chio

chi_x to chi_xo(l,k)   chi_y to chi_yo(l,k)   vphi to vphio

8.9 skip every thing else if iartp=1, do time advance

call timeadv from m1.F

1) copy rh to parameter $prcr1$, call ρ equation rhoeqn

2) copy w to $prcr1$, u to $prcr2$, lap_u to $prcr3$. call weqn and calculate $dwdt$ and $udth$.

3) copy chi to $prcr1$, p to $prcr2$, si to $prcr3$. call chiaiap_3 for $\tilde{I}, \chi, p$, and

inside chiaiap_3 call poisvmu3, poisvmu3, poiss, and peqn.

4) call pkkks

5) advance pressure using artificial sound term

6) copy $vphi$ to $prcr1$, and call vphieq.

7) calculate $dchidt$ and call phieqn

8) copy a to $prcr1$, c_a to $prcr2$, lap_psi to prcr3 and call ceqn.

9) copy $bigf$ to $prcr1$, lap_F to $prcr2$, and call feqn.

8.10

diagnose the results, print values only if ncy = n*nprnt or gamma small
decide whether the objective is reached so that the run can be stopped.
decide whether more harmonics are needed.
drop in the hole of the potential energy when deeded.
keep maximum kinetic energy within a given value when desired.
call diagnos which is in m1.F

8.11 write a restart/checkpoint file

call chekchek which is in mpar1.F

8.12 output printing and plotting