Kinetic Effects in ICW Mode Conversion in a Multi-ion Plasma

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Abstract. We discuss the role of kinetic effects on mode conversion near the ion-ion hybrid resonance in a warm plasma. A kinetic integral equation is derived for a plasma with gradients primarily along the magnetic field. We show that the kinetic kernel collapses to a delta function in the local limit reproducing the cold plasma result. The integral equation is solved using the method of finite elements. We examine wave scattering near the ion-ion hybrid resonance and obtain the absorption, reflection, and transmission coefficients for incoming waves on the ion cyclotron or magnetosonic branch. The results are compared with the cold plasma result. We find that in the limit of low plasma beta, the cold plasma result is recovered. On the other hand, for high plasma beta the kinetic results differ with significantly more wave absorption. In the warm plasma limit particles interact with the wave over a more extended region. These results suggest that it is adequate to use the wave absorption for the cold plasma result when considering mode conversion along field lines in regions of low plasma beta. In particular, the cold plasma result is suitable for looking at low energy ion conics in the topside ionosphere. In the plasma sheet, magentosheath, and magnetopause kinetic effects can be more important.

Introduction

Kinetic Integral Equation

As a first step to understanding kinetic effects on mode conversion at near the ion cyclotron frequency we derive an integral equation which captures the most important effects. Wave propagation is governed by Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}; \nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \tag{1}$$

which when combined gives the well known equation for wave propagation

$$\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{4\pi}{c}\frac{\partial \mathbf{J}}{\partial t} \tag{2}$$

The current is obtained from the response of the plasma to the field perturbations.

$$\mathbf{J}(\mathbf{x}, t) = \int d^3\mathbf{v}\mathbf{v}\delta f(\mathbf{x}, \mathbf{v}, t) \tag{3}$$

where the linearized particle distribution evolves according to the linearized Vlasov equation

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q_j}{m_j}\left[\mathbf{E}_0 + \frac{\mathbf{v}}{c}\times\mathbf{B}_0\cdot\frac{\partial}{\partial \mathbf{v}}\right]\right)\delta f(x, v, t) = \tag{4}$$

$$-\frac{q_j}{m_j}\left(\delta\mathbf{E} + \frac{\mathbf{v}}{c}\times\delta\mathbf{B}\cdot\frac{\partial}{\partial \mathbf{v}}\right) f_0(\{C_n\}) \tag{5}$$

where $\{C_n(\mathbf{x}, \mathbf{v})\}$ are constants of the unperturbed motion in the fields $\mathbf{E}_0$ and $\mathbf{B}_0$. This linearized equation is formally solved by the standard method of characteristics

$$\delta f(\mathbf{x}, \mathbf{v}, t) = -\frac{q_j}{m_j}\int_{-\infty}^{t} dt' \left(\delta\mathbf{E}(\mathbf{x}', t') + \frac{\mathbf{v}'}{c}\times\delta\mathbf{B}(\mathbf{x}', t')\right)\cdot\frac{\partial}{\partial \mathbf{v}'} f_0(\epsilon', \mu', \mathbf{x}_{gc}) \tag{6}$$

where we have now specified the (approximate) constants of the motion ϵ, μ and $\mathbf{x}_{gc}$ as the energy (exact), magnetic moment (adiabatic), and guiding center position (adiabatic). The characteristic equations for $\mathbf{x}'(t')$ and $\mathbf{v}'(t')$ are given by

$$\frac{d\mathbf{x}'}{dt'} = \mathbf{v}'; \frac{d\mathbf{v}'}{dt'} = \frac{q_j}{m_j}(\mathbf{E}_0 + \frac{\mathbf{v}'}{c}\times\mathbf{B}_0(\mathbf{x}')) \tag{7}$$

with the constraint, $\mathbf{x}'(t'=t)=\mathbf{x}$ and $\mathbf{v}'(t'=t)=\mathbf{v}$.

Our objective is to describe the current response near the cyclotron resonance location ($\omega \approx \Omega_c$) in a plasma with magnetic field gradients. Our expectaction is that magnetic field gradients along the particle trajectory will "detune" the particles from the resonance leading to a nonlocal spreading of the dissipation region about the ion cyclotron resonance location. As particles stream freely along the magnetic field, the gradients in the cyclotron frequency along the magnetic field direction are the most important. Therefore, for simplicity, we first consider the case $\mathbf{E}_0=0$ and $\mathbf{B}_0=B_0(z)\hat{\mathbf{z}}$. Although $\nabla\cdot\mathbf{B}_0$ is not strictly satisfied, corrections resulting from using a background field which does satisfy $\nabla\cdot\mathbf{B}_0=0$ do not significantly alter our results. The resonance condition for particles is $\omega - n\Omega(z) - k_\parallel v_\parallel = 0$. The term which we include provides spatial dependence of the second term. The term which is neglected would alter $v_\parallel$ of the particle through conservation of the magnetic moment. This correction is higher order in the background gradient.

For this field configuration we have

$$\ddot{x}' = -\dot{y}'\Omega(z'); \quad \ddot{y}' = \dot{x}'\Omega(z'); \quad \ddot{z}' = 0 \tag{8}$$

where $\Omega(z) \equiv q_j B_0(z)/m_j c$. Changing independent variable to $\tau \equiv t' - t$ the equations can be solved

$$\dot{x}'(\tau) = v_\perp \cos S \tag{9}$$

$$\dot{y}'(\tau) = v_\perp \sin S \tag{10}$$

$$x'(\tau) = x + v_\perp \int_0^\tau d\tau' \cos S(\tau') \tag{11}$$

$$y'(\tau) = y + v_\perp \int_0^\tau d\tau' \sin S(\tau') \tag{12}$$

$$z'(\tau) = z + v_z \tau \tag{13}$$

where

$$S(\tau) = \phi + \int_0^\tau d\tau' \Omega(z + v_z \tau') \tag{14}$$

is a rapidly varying phase and $\dot{x}'(\tau = 0) = v_\perp \cos(\phi)$ and $\dot{y}'(\tau = 0) = v_\perp \sin(\phi)$ where ϕ is the gyrophase at $\tau = 0$.

For a simplistic magnetic field profiles it is possible to obtain exact solutions of Equations (11–12). For example, if the magnetic field is constant the orbits become the standard circular orbits describing a homogeneous plasma. For a linear gradient in the magnetic field, the orbits can be obtained analytically in terms of Fresnel integrals which also describe the shift in frequency and gyroradius of particle motion perpendicular to the magnetic field. With these analytic forms for the particle orbits, the integral equation (6 may be solved by quadrature.

It is convenient to consider the Fourier transform of the fields in the spatial plane perpendicular to the magnetic field and in time.

$$\mathbf{E}(\mathbf{x}, t) \equiv \int\int dk_\perp dt \exp(i(k_\perp x - \omega t))\mathbf{E}(k_\perp, z, \omega) \tag{15}$$

$$\mathbf{B}(\mathbf{x}, t) \equiv \int\int dk_\perp dt \exp(i(k_\perp x - \omega t))\mathbf{B}(k_\perp, z, \omega) \tag{16}$$

Then we find

$$\delta f_j(k_\perp, z, \mathbf{v}, \omega) = -\frac{q_j}{m_j}\int_{-\infty}^{0} d\tau \exp(i(k_\perp(x' - x) - \omega\tau)\int_\Lambda dz'\delta(z' - z - v_z\tau)$$
$$\left(\mathbf{E}(k_\perp, z', \omega) + \frac{\mathbf{v}'(\mathbf{x}, \mathbf{v}, \tau)}{c} \times \mathbf{B}(k_\perp, z', \omega)\right) \cdot \frac{\partial}{\partial \mathbf{v}'} f_0(\epsilon', \mu', \mathbf{x}_{gc}) \tag{17}$$

where Λ is a one-dimensional domain with boundary conditions in the fields imposed at the endpoints.

Inserting the particle orbits we find

$$\delta f_j(k_\perp, z, \mathbf{v}, \omega) = -\frac{q_j}{m_j}\int_{-\infty}^{0} d\tau \int_\Lambda dz'\delta(z' - z - v_z\tau) \exp\left(ik_\perp v_\perp \int_0^\tau d\tau' \sin S(\tau') - i\omega\tau\right)$$
$$\left(\mathbf{E}(k_\perp, z') + \frac{\mathbf{v}'(\mathbf{x}, \mathbf{v}, \tau)}{c} \times \mathbf{B}(k_\perp, z')\right) \cdot \frac{\partial}{\partial \mathbf{v}'} f_0(\epsilon', \mu', \mathbf{x}_{gc}) \tag{18}$$

For the remainder of this discussion we will suppress the dependence of the fields and perturbed distribution on $k_\perp$ and ω.

Now if we examine Equation (6) we see that the response at a location, $\mathbf{x}$, involves all of the field information sampled over the past history of the particle motion and therefore is fully nonlocal. However, if particle motion in the background fields remains somewhat localized in space on the timescale of interest, it is possible to reduce the amount of information that is propagated in order to perform the integral. Essentially, this means that the perturbed distribution at a particular location will only depend on nearby fields. This property is manifested by the fact that the integrand oscillates rapidly in time, leaving little contribution to the response from contributions at times much earlier than a few cyclotron periods. Mathematically, the time contour is rotated into the complex plane leading to an exponential decrease in the contribution from times much longer than a few gyroperiods.

We expand the integrand by means of integration by parts with

$$\int_0^\tau \cos S(\tau')d\tau' = \frac{\sin S}{dS/d\tau}\left[1 - \frac{\cos S(d^2S/d\tau^2)}{(dS/d\tau)^2} + \mathcal{O}(\frac{d^3S/d\tau^3}{(dS/d\tau)^3}, \frac{(d^2S/d\tau^2)^2}{(dS/d\tau)^4})\right]_0^\tau \tag{19}$$

This expansion is appropriate if $d^2S/d\tau^2 \ll (dS/d\tau)^2$ which is equivalent to $v_z d\Omega/dz \ll \Omega^2$, $v_{th}/L_B \ll \Omega$. where L_B is the magnetic field gradient scale length. Physically, this means that the time it takes for a particle to detune from the resonance due to streaming along the field line is much greater than the cyclotron period. For particles in the auroral zone where $\rho_i << 1$. For 1keV hydrogen at 4000 km above the Earth's surface this criterion is marginally satisfied. For most ion heating events that are observed, the criterion is satisfied.

Taking note that $S(0) = \phi$, $S_\tau(0) = \Omega(z)$, and $S_\tau(\tau) = \Omega(z')$, we find

$$\exp\left(ik_\perp v_\perp \int_0^\tau d\tau' \cos S(\tau') - i\omega\tau\right) = \exp\left(ik_\perp v_\perp \sin S(\tau)/S_\tau(\tau) - ik_\perp v_\perp \sin S(0)/S_\tau(0) - i\omega\tau\right) \tag{20}$$

so that

$$\begin{pmatrix} 1 \\ \cos(S) \\ \sin(S) \end{pmatrix} \exp\left(ik_\perp v_\perp \int_0^\tau d\tau' \cos S(\tau') - i\omega\tau\right) =$$

$$= \sum_n \sum_m \begin{pmatrix} J_m(\lambda) J_n(\lambda') \\ n J_m(\lambda) J_n(\lambda')/\lambda' \\ -i J_m(\lambda) dJ_n(\lambda')/d\lambda \end{pmatrix} \exp\left(-i(\omega\tau - n\int_0^\tau d\tau' \Omega(z + v_z\tau')\right) e^{i(n-m)\phi} \tag{21}$$

where $\lambda = k_\perp v_\perp/\Omega(z)$ and $\lambda' = k_\perp v_\perp/\Omega(z')$.

If the magnetic field were constant, then integration over τ will involve $\int_{-\infty}^0 d\tau e^{-i(\omega-\Omega)\tau)}$ which will give the usual resonant denominator. The integral is aympototically determined by the orbit near $\tau = 0$ and because the integral falls off exponentially as the contour is deformed in the complex plane. In other words, the response is local.

Because of the background gradient in the field, the response is nonlocal. We are considering a response on the timescale of the cyclotron frequency. We are considering the response in the perturbed distribution to the entire electromagnetic field structure. As such, we expect that the response should be spatially limited to the distance an orbit could traverse on that timescale. In other words, we expect that $z' - z = v_z\tau$ or $v_z/L_B \ll \Omega$ as assumed previously.

Then it is appropriate to expand

$$\Omega(z') \approx \Omega(z) + \frac{d\Omega}{dz} v_z \tau \tag{22}$$

and to note that

$$\frac{d\Omega}{dz} \approx \frac{\Omega(z') - \Omega(z)}{(z' - z)} \tag{23}$$

so that

$$\int_{-\infty}^0 d\tau \delta(z' - z - v_z\tau) \exp\left(-i(\omega\tau - n\int_0^\tau d\tau' \Omega(z + v_z\tau')\right)$$
$$= \Theta\left(\frac{z - z'}{v_z}\right) \frac{1}{|v_z|} \exp\left(-i\left(\omega - n\frac{\Omega(z) + \Omega(z')}{2}\right)\frac{(z' - z)}{v_z}\right) \tag{24}$$

We then find that

$$\delta f_j(k_\perp, z, \mathbf{v}, \omega) = -\frac{q_j}{m_j} \int_\Lambda dz' \sum_n \sum_m J_n\left(\frac{k_\perp \mathbf{v}_\perp}{\Omega(z')}\right) J_m\left(\frac{k_\perp \mathbf{v}_\perp}{\Omega(z)}\right) \Theta\left(\frac{z - z'}{v_z}\right) \frac{1}{|v_z|}$$

$$\exp\left(-i\left(\omega-\frac{\Omega(z)+\Omega(z')}{2}\right)\frac{(z'-z)}{v_z}\right)e^{(n-m)\phi}$$

$$\left(\mathbf{E}(k_\perp,z')+\frac{\mathbf{v}'(\mathbf{x},\mathbf{v},\tau)}{c}\times\mathbf{B}(k_\perp,z')\right)\cdot\frac{\partial}{\partial\mathbf{v}'}f_0(\epsilon',\mu',\mathbf{x}_{gc})$$

$$=-\frac{q}{m}\int_\Lambda dz'\Theta\left(\frac{z-z'}{v_z}\right)\frac{1}{|v_z|}\sum_n\sum_m J_m(\lambda)e^{i(n-m)\phi}\exp\left(-i\left(\omega-n\frac{\Omega(z)+\Omega(z')}{2}\right)\frac{(z'-z)}{v_z}\right)$$

$$\left\{\left(\frac{nJ_n(\lambda')}{\lambda'}\delta E_x(k_\perp,z',\omega)+iJ_n'(\lambda')\delta E_y(k_\perp,z',\omega)\right)\frac{\partial F}{\partial v_\perp}\right.$$

$$+\frac{i}{\omega}\left(\frac{nJ_n(\lambda')}{\lambda'}\frac{d\delta E_x'(k_\perp,z',\omega)}{dz'}+iJ_n'(\lambda')\frac{d\delta E_y(k_\perp,z',\omega)}{dz'}\right)\left[v_z\frac{\partial F}{\partial v_\perp}-v_\perp\frac{\partial F}{\partial v_z}\right]$$

$$\left.+\delta E_z J_n(\lambda')\left[\frac{\partial}{\partial v_z}+\frac{n\Omega(z')}{\omega}\left(\frac{v_z}{v_\perp}\frac{\partial}{\partial v_\perp}-\frac{\partial}{\partial v_z}\right)\right]F\right\}$$

Isotropic Plasma

For an isotropic plasma $F=(2\pi)^{3/2}\exp(-v^2/2v_t^2)$ where $v_t^2=T/m$. Then the perturbed current is of the form

$$\delta J(k_\perp,z,\omega)=\int_\Lambda dz'\sigma(z,z')\cdot\delta\mathbf{E} \tag{25}$$

where

$$\sigma(z,z')\cdot\delta\mathbf{E}=\frac{1}{4\pi}\sum_j\omega_{pj}^2\int d^3v\mathbf{v}\Theta\left(\frac{z-z'}{v_z}\right)\frac{1}{|v_z|}\sum_n\sum_m J_m(\lambda)e^{i(n-m)\phi}\exp\left(-i\left(\omega-n\frac{\Omega(z)+\Omega(z')}{2}\right)\frac{(z'-z)}{v_z}\right.$$

$$\left(\frac{nJ_n(\lambda')}{\lambda'}\delta E_x(k_\perp,z',\omega)+iJ_n'(\lambda')\delta E_y(k_\perp,z',\omega)+\delta E_z J_n(\lambda')\right)\frac{v_\perp F}{v_{tj}^2}$$

$$\sigma_{xx}(k_\perp,\omega,z,z') == -\frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_\perp^2}{|v_z|}g(z,z')\sum_n \frac{n^2 J_n(\lambda)J_n(\lambda')}{\lambda\lambda'} \tag{26}$$

$$\sigma_{xy}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_\perp^2}{|v_z|}g(z,z')\sum_n \frac{nJ_n(\lambda)J_n'(\lambda')}{\lambda} \tag{27}$$

$$\sigma_{yx}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_\perp^2}{|v_z|}g(z,z')\sum_n \frac{nJ_n(\lambda')J_n'(\lambda)}{\lambda'} \tag{28}$$

$$\sigma_{xz}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_\perp v_z}{|v_z|}g(z,z')\sum_n \frac{nJ_n(\lambda)J_n(\lambda')}{\lambda} \tag{29}$$

$$\sigma_{zx}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_z v_\perp}{|v_z|}g(z,z')\sum_n \frac{nJ_n(\lambda')J_n(\lambda)}{\lambda'} \tag{30}$$

$$\sigma_{yy}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_\perp^2}{|v_z|}g(z,z')\sum_n J_n'(\lambda)J_n'(\lambda') \tag{31}$$

$$\sigma_{yz}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_\perp v_z}{|v_z|}g(z,z')\sum_n J_n(\lambda)J_n'(\lambda') \tag{32}$$

$$\sigma_{zy}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_z v_\perp}{|v_z|}g(z,z')\sum_n J_n(\lambda')J_n'(\lambda) \tag{33}$$

$$\sigma_{zz}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \frac{\omega_{pj}^2}{v_{tj}^2}\int d^3v F(v)\Theta\left(\frac{z-z'}{v_z}\right)\frac{v_z^2}{|v_z|}g(z,z')\sum_n J_n(\lambda)J_n(\lambda') \tag{34}$$

where

$$g(z,z') = \exp\left(-i\left(\omega - n\frac{\Omega(z)+\Omega(z')}{2}\right)\frac{(z'-z)}{v_z}\right) \tag{35}$$

Performing the integration over velocity space, we obtain

$$\sigma_{xx}(k_\perp,\omega,z,z') = \frac{i}{4\pi}\sum_j \omega_{pj}^2 \sum_n \frac{n^2\Gamma_n(b_j,b_j')}{b_j b_j'}\mathcal{I}_0(z,z') \tag{36}$$

$$\sigma_{xy}(k_\perp,\omega,z,z') = \frac{1}{4\pi}\sum_j \omega_{pj}^2 \sum_n \frac{n}{b_j}\frac{\partial\Gamma_n(b_j,)}{\partial b_j'}\mathcal{I}_0(z,z') \tag{37}$$

$$\sigma_{y,x}(k_\perp,\omega,z,z') = -\sigma_{y,x}(k_\perp,\omega,z',z) \tag{38}$$

$$\sigma_{xz}(k_\perp,\omega,z,z') = \frac{i}{4\pi}\sum_j \omega_{pj}^2 \sum_n \frac{n\Gamma_n(b_j,b_j')}{b_j}\mathcal{I}_1(z,z') \tag{39}$$

$$\sigma_{zx}(k_\perp,\omega,z,z') = \frac{i}{4\pi}\sum_j \omega_{pj}^2 \sum_n \frac{n\Gamma_n(b_j,b_j')}{b_j'}\mathcal{I}_1(z,z') \tag{40}$$

$$\sigma_{yy}(k_\perp,\omega,z,z') = \frac{i}{4\pi}\sum_j \omega_{pj}^2 \sum_n \frac{\partial^2\Gamma_n(b_j,b_j')}{\partial b_j \partial b_j'}\mathcal{I}_0(z,z') \tag{41}$$

$$\sigma_{yz}(k_\perp,\omega,z,z') = \frac{i}{4\pi}\sum_j \omega_{pj}^2 \sum_n \frac{\partial\Gamma_n(b_j,b_j')}{\partial b_j'}\mathcal{I}_1(z,z') \tag{42}$$

$$\sigma_{zy}(k_\perp,\omega,z,z') = \frac{i}{4\pi}\sum_j \omega_{pj}^2 \sum_n \frac{\partial\Gamma_n(b_j,b_j')}{\partial b_j}\mathcal{I}_1(z,z') \tag{43}$$

$$\sigma_{zz}(k_\perp,\omega,z,z') = \frac{i}{4\pi}\sum_j \omega_{pj}^2 \sum_n \Gamma_n(b_j,b_j')\mathcal{I}_2(z,z') \tag{44}$$

$$\tag{45}$$

where

$$\mathcal{I}_q(z,z') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dv_z \exp(-\frac{v_z^2}{2v_{tj}^2})\frac{v_z^q}{|v_z|}\Theta\left(\frac{z-z'}{v_z}\right) \tag{46}$$

and

$$\Gamma_n(\alpha,\beta) = \exp\left(-\frac{\alpha^2+\beta^2}{2}\right) I_n(\alpha,\beta) \tag{47}$$

which are evaluated at $b \equiv k_\perp^2 v_{t\perp}^2/(\Omega(z)$ and $b' \equiv k_\perp^2 v_{tj}^2/\Omega(z'))$

It is useful to consider

$$\int_{-\infty}^{\infty} dv_z f_0(v_z)\frac{1}{|v_z|}\Theta\left(\frac{z-z'}{v_z}\right)\exp\left(-i(\omega-\bar{\Omega})\frac{(z'-z)}{v_z}\right) = \int_0^{\infty}\frac{dv_z}{v_z}f_0(v_z)\exp\left(i(\omega-\bar{\Omega})\frac{|z'-z|}{v_z}\right)$$

With the change of variables $p = \sqrt{2}v_t/v_z$ and a Maxwellian distribution in the z direction, $f_0(v_z) = \exp(-v_z^2/2v_t^2)/\sqrt{2\pi}v_t$

$$\int_0^{\infty}\frac{dv_z}{v_z}f_0(v_z)\exp\left(i(\omega-\bar{\Omega})\frac{|z'-z|}{v_z}\right) = \frac{1}{\sqrt{2}v_t}f(\Delta)$$

where

$$f(\Delta) = \frac{-i}{\sqrt{\pi}} \int_0^\infty \frac{dp}{p} \exp(ip\Delta - p^{-2})$$

where

$$\Delta = \frac{\omega - \bar{\Omega}}{\sqrt{2} v_t} |z - z'|$$

It is useful to recognize certain properties of $f(\Delta)$:

$$\int_{-\infty}^{\infty} f(|\Delta|) = 1$$

$$\lim_{\Delta \to 0} f(\Delta) = \frac{\sqrt{\pi}}{2} - i\mathrm{Ei}(\Delta)$$

where Ei is the exponential integral.

$$f(-\Delta) = -f(\Delta)^*$$

To tabulate f, numerically, it is useful to integrate along the path of steepest descent. Examining the integrand, the path should begin along the positive real axis for small k to ensure the exponential kills the singularity at $k = 0$ and then rotate along the imaginary axis for rapid convergence. A saddle point is found at $k = (2/\Delta)^{1/3} e^{i\pi/3}$. An appropriate contour to take advantage of the steep descent is to integrate along the real axis to $(2/\Delta)^{1/3}$ and then integrate along a circular contour to the imaginary axis where the integrand becomes small rapidly. For $\Delta \ll 1$ it is convenient to deform the contour at $k = 1$.

Assume $f_0(-v_z) = f_0(v_z)$ and consider the general integral

$$\int_{-\infty}^{\infty} dv_z f_0(v_z) \frac{v_z^\alpha}{|v_z|} \Theta\left(\frac{z - z'}{v_z}\right) \exp\left(-i(\omega - \bar{\Omega}) \frac{(z' - z)}{v_z}\right)$$

$$= [\Theta(z - z') + (-1)^\alpha \Theta(z' - z)] \int_0^\infty dv_z v_z^{\alpha - 1} f_0(v_z) \exp\left(i(\omega - \bar{\Omega}) \frac{|z' - z|}{v_z}\right)$$

It is useful to tabulate the integral

$$\int_0^\infty \frac{dv_z}{v_z} f_0(v_z) \exp\left(i(\omega - \bar{\Omega}) \frac{|z' - z|}{v_z}\right) = \sqrt{2} {v_t}^{\alpha - 1} f_\alpha(\Delta)$$

where

$$f_\alpha(\Delta) = \frac{-i}{\sqrt{\pi}} \int_0^\infty \frac{dp}{p^{\alpha+1}} \exp(ip\Delta - p^{-2})$$

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References

Johnson, J. R., T. S. Chang, and G. B. Crew, A study of mode conversion in an oxygen-hydrogen plasma, *Phys. Plasmas*, *2*, 1274–1284, 1995.

Perkins, F. W., Heating tokamaks via the ion-cyclotron and ion-ion hybrid resonances, *Nucl. Fusion*, *17*, 1197–1224, 1977.

Swanson, D. G., Radio frequency heating in the ion-cyclotron range of frequencies, *Phys. Fluids*, *28*, 2645–2677, 1985.

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